

Dedicated to Prof. Qamrul Hasan Ansari on the occasion of his 60<sup>th</sup> anniversary

## Approximating fixed points of enriched nonexpansive mappings by Krasnoselskij iteration in Hilbert spaces

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### ABSTRACT.

Using the technique of enrichment of contractive type mappings by Krasnoselskij averaging, presented here for the first time, we introduce and study the class of *enriched nonexpansive mappings* in Hilbert spaces. In order to approximate the fixed points of enriched nonexpansive mappings we use the Krasnoselskij iteration for which we prove strong and weak convergence theorems. Examples to illustrate the richness of the new class of contractive mappings are also given.

Our results in this paper extend some classical convergence theorems established by Browder and Petryshyn in [Browder, F. E., Petryshyn, W. V., *Construction of fixed points of nonlinear mappings in Hilbert space*, J. Math. Anal. Appl., **20** (1967), 197–228] from the case of nonexpansive mappings to that of enriched nonexpansive mappings, thus including many other important related results from literature as particular cases.

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