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Dedicated to Prof. Hong-Kun Xu on the occasion of his 60^{th} anniversary

A generalization of the (CN) inequality and its applications

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ABSTRACT. We extend the (CN) inequality of Bruhat and Tits in CAT(0) spaces to the general setting of uniformly convex hyperbolic spaces. We also show that, under some appropriate conditions, the sequence of Ishikawa iteration defined by Panyanak converges to a strict fixed point of a multi-valued Suzuki mapping.

1. INTRODUCTION AND PRELIMINARIES

Throughout this paper, $\mathbb{N}$ stands for the set of natural numbers and $\mathbb{R}$ stands for the set of real numbers.

Let (X, d) be a metric space, $x, y \in X$ and $l := d(x, y)$. A *geodesic* joining x to y is a mapping $c : [0, l] \to X$ such that $c(0) = x, c(l) = y$, and $d(c(t), c(s)) = |t - s|$ for all $t, s \in [0, l]$. The image of c is called a *geodesic segment* joining x and y. The space X is said to be a *geodesic space* (resp. $D-$*geodesic space*) if every two points of X (resp. every two points of distance smaller than D) are joined by a geodesic. A subset E of X is said to be *convex* if E includes every geodesic segment joining any two of its points. The set E is said to be *bounded* if

$$\operatorname{diam}(E) := \sup\{d(x, y) : x, y \in E\} < \infty.$$

We denote by $\langle \cdot, \cdot \rangle$ the Euclidean scalar product in $\mathbb{R}^3$. By $\mathbb{S}^2$ we denote the unit sphere in $\mathbb{R}^3$, that is the set $\left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\right\}$. The *spherical distance* on $\mathbb{S}^2$ is defined by

$$d_{\mathbb{S}^2}(x, y) := \arccos\langle x, y\rangle \text{ for all } x, y \in \mathbb{S}^2.$$

Definition 1.1. ([3]) Given $\kappa \geq 0$, we denote by M_κ^2 the following metric spaces:
(i) if $\kappa = 0$ then M_κ^2 is the Euclidean space $\mathbb{E}^2$;
(ii) if $\kappa > 0$ then M_κ^2 is obtained from the spherical space $\mathbb{S}^2$ by multiplying the distance function by $1/\sqrt{\kappa}$.

A *geodesic triangle* $\triangle(x, y, z)$ in a geodesic space (X, d) consists of three points x, y, z in X (the *vertices* of $\triangle$) and three geodesic segments between each pair of vertices (the *edges* of $\triangle$). A *comparison triangle* for a geodesic triangle $\triangle(x, y, z)$ in (X, d) is a triangle $\overline{\triangle}(\bar{x}, \bar{y}, \bar{z})$ in M_κ^2 such that

$$d(x, y) = d_{M_\kappa^2}(\bar{x}, \bar{y}),\ d(y, z) = d_{M_\kappa^2}(\bar{y}, \bar{z}), \text{ and } d(z, x) = d_{M_\kappa^2}(\bar{z}, \bar{x}).$$

It is well known that such a comparison triangle exists if $d(x, y) + d(y, z) + d(z, x) < 2D_\kappa$, where $D_\kappa = \pi/\sqrt{\kappa}$ for $\kappa > 0$ and $D_0 = \infty$. Notice also that the comparison triangle

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is unique up to isometry. A point $\bar{u} \in [\bar{x}, \bar{y}]$ is called a *comparison point* for $u \in [x, y]$ if $d(x, u) = d_{M_\kappa^2}(\bar{x}, \bar{u})$.

A metric space (X, d) is said to be a *CAT(κ) space* if it is $D_\kappa-$geodesic and for each two points u, v of any geodesic triangle $\triangle(x, y, z)$ in X with $d(x, y) + d(y, z) + d(z, x) < 2D_\kappa$ and for their comparison points $\bar{u}, \bar{v}$ in $\overline{\triangle}(\bar{x}, \bar{y}, \bar{z})$, one has

$$d(u, v) \leq d_{M_\kappa^2}(\bar{u}, \bar{v}).$$

Notice that if (X, d) is a CAT(0) space, then for each $x, y, z \in X$ we have

$$\text{(CN)} \qquad d^2(x, m) \leq \frac{1}{2}d^2(x, y) + \frac{1}{2}d^2(x, z) - \frac{1}{4}d^2(y, z),$$

where m is the midpoint of y and z. This is the (CN) inequality of Bruhat and Tits [4] which has been used to prove many results in metric fixed point theory.

The concept of uniformly convex hyperbolic spaces which is more general than the concept of CAT(κ) spaces was introduced by Leustean [15] in 2007.

Definition 1.2. A *hyperbolic space* is a metric space (X, d) together with a function $W : X \times X \times [0, 1] \to X$ such that for all $x, y, z, w \in X$ and $t, s \in [0, 1]$, we have
(W1) $d(z, W(x, y, t)) \leq (1 - t)d(z, x) + td(z, y)$;
(W2) $d\left(W(x, y, t), W(x, y, s)\right) = |t - s|d(x, y)$;
(W3) $W(x, y, t) = W(y, x, 1 - t)$;
(W4) $d(W(x, z, t), W(y, w, t)) \leq (1 - t)d(x, y) + td(z, w)$.

If $x, y \in X$ and $t \in [0, 1]$, then we use the notation $(1 - t)x \oplus ty$ for $W(x, y, t)$. It follows from (W1) that

$$d(x, (1 - t)x \oplus ty) = td(x, y) \text{ and } d(y, (1 - t)x \oplus ty) = (1 - t)d(x, y).$$

A nonempty subset E of X is said to be *convex* if $[x, y] \subseteq E$ for all $x, y \in E$, where $[x, y] := \{(1 - t)x \oplus ty : t \in [0, 1]\}$.

Definition 1.3. The hyperbolic space (X, d, W) is called *uniformly convex* if for any $r \in (0, \infty)$ and $\varepsilon \in (0, 2]$ there exists $\delta \in (0, 1]$ such that for all $x, y, z \in X$ with $d(x, z) \leq r$, $d(y, z) \leq r$ and $d(x, y) \geq r\varepsilon$, we have

$$d\left(\frac{1}{2}x \oplus \frac{1}{2}y, z\right) \leq (1 - \delta)r.$$

A function $\eta : (0, \infty) \times (0, 2] \to (0, 1]$ providing such a $\delta := \eta(r, \varepsilon)$ for given $r \in (0, \infty)$ and $\varepsilon \in (0, 2]$ is called *a modulus of uniform convexity.* We call η *monotone* if it is a nonincreasing function of r for every fixed ε.

The concept of $p-$uniform convexity was used extensively by Xu [22]. Its nonlinear version for $p = 2$ was studied by Khan and Khamsi [12]. Now, we give the definition of a 2-uniformly convex hyperbolic space.

Definition 1.4. Let (X, d) be a uniformly convex hyperbolic space. For each $r \in (0, \infty)$ and $\varepsilon \in (0, 2]$, we define

$$\Psi(r, \varepsilon) := \inf\left\{\frac{1}{2}d^2(x, z) + \frac{1}{2}d^2(y, z) - d^2(\frac{1}{2}x \oplus \frac{1}{2}y, z)\right\},$$

where the infimum is taken over all $x, y, z \in X$ such that $d(x, z) \leq r$, $d(y, z) \leq r$, and $d(x, y) \geq r\varepsilon$. We say that (X, d) is *2-uniformly convex* if

$$c_M := \inf\left\{\frac{\Psi(r, \varepsilon)}{r^2\varepsilon^2} : r \in (0, \infty), \varepsilon \in (0, 2]\right\} > 0.$$

From the definition of c_M, we obtain the following inequality:

$$d^2(\frac{1}{2}x \oplus \frac{1}{2}y, z) \leq \frac{1}{2}d^2(x,z) + \frac{1}{2}d^2(y,z) - c_M d^2(x,y), \tag{1.1}$$

for all $x, y, z \in X$.

Remark 1.1. (1) Every uniformly convex Banach space is a 2-uniformly convex hyperbolic space (see [22]).

(2) If X is a CAT(0) space, then it is a 2-uniformly convex hyperbolic space with $c_M = \frac{1}{4}$ (see [12]).

(3) If $\kappa > 0$ and X is a CAT(κ) space with $\operatorname{diam}(X) \leq \frac{\pi/2-\varepsilon}{\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$, then by Lemma 2.3 of [21] we can conclude that

$$\Psi(r, \varepsilon) = \frac{r^2\varepsilon^2 R}{8},$$

where $R = (\pi - 2\varepsilon)\tan(\varepsilon)$. This clearly implies that X is a 2-uniformly convex hyperbolic space with $c_M = \frac{R}{8}$.

In 2013, Ibn Dehaish et al. [10] obtained the following result and applied it to prove the convergence of Mann iteration process for asymptotic pointwise nonexpansive mappings in 2-uniformly convex hyperbolic spaces.

Theorem 1.1. *Let (X, d) be a 2-uniformly convex hyperbolic space. Then*

$$d^2((1-t)x \oplus ty, z) \leq (1-t)d^2(x,z) + td^2(y,z) - 4c_M \min\{t^2, (1-t)^2\}d^2(x,y), \tag{1.2}$$

for all $x, y, z \in X$ and $t \in [0, 1]$.

It is well known that the Ishikawa iteration process [11], which involves two sequences of scalars, is a generalization of the Mann iteration process [18], which involves one sequence of scalars. In this paper, we generalize Theorem 1.1 by replacing the number $\min\{t^2, (1-t)^2\}$ with $t(1-t)$ and show that our result can be used to prove the convergence of Ishikawa iteration process for multi-valued Suzuki mappings. Our method provides an efficient way of extending fixed point theorems in uniformly convex Banach spaces or even in CAT(κ) spaces to the general setting of uniformly convex hyperbolic spaces.

2. Main result

Theorem 2.2. *Let (X, d) be a 2-uniformly convex hyperbolic space. Then*

$$d^2((1-t)x \oplus ty, z) \leq (1-t)d^2(x,z) + td^2(y,z) - 4c_M t(1-t)d^2(x,y), \tag{2.3}$$

for all $x, y, z \in X$ and $t \in [0, 1]$.

Proof. This proof is patterned after the proof of Lemma 2.5 in [6]. We first prove the result for $t = \frac{k}{2^n}$, where $k, n \in \mathbb{N}$ are such that $k \leq 2^n$. We use induction on n. If $n = 1$, then $t \in \{\frac{1}{2}, 1\}$. By (1.1) we can conclude that (2.3) is true for $t = \frac{1}{2}$. If $t = 1$, then (2.3) is true if and only if $d(y, z) \leq d(y, z)$. Therefore, (2.3) is true for $n = 1$. Now, suppose that (2.3) is true for $t = \frac{k}{2^n}$. Hence,

$$d^2((1-\frac{k}{2^n})x \oplus \frac{k}{2^n}y, z) \leq (1-\frac{k}{2^n})d^2(x,z) + \frac{k}{2^n}d^2(y,z) - 4c_M\frac{k}{2^n}(1-\frac{k}{2^n})d^2(x,y), \tag{2.4}$$

for all $k \in \mathbb{N}, k \leq 2^n$ and $x, y, z \in X$.

We have to prove (2.3) for $t = \frac{k}{2^{n+1}}$, where $k \in \mathbb{N}, k \leq 2^{n+1}$. If we denote $u := (1 - \frac{k}{2^{n+1}})x \oplus (\frac{k}{2^{n+1}})y$, then we have to prove

$$d^2(u,z) \leq (1 - \frac{k}{2^{n+1}})d^2(x,z) + \frac{k}{2^{n+1}}d^2(y,z) - 4c_M \frac{k}{2^{n+1}}(1 - \frac{k}{2^{n+1}})d^2(x,y). \tag{2.5}$$

First, we show (2.5) for $k \leq 2^n$, that is, $\frac{k}{2^n} \in [0,1]$. Let $\alpha := (1 - \frac{k}{2^n})x \oplus (\frac{k}{2^n})y$ and $\beta := \frac{1}{2}x \oplus \frac{1}{2}\alpha$. Then $d(x,\beta) = \frac{1}{2}d(x,\alpha) = \frac{k}{2^{n+1}}d(x,y) = d(x,u)$. Since $u, \beta \in [x,y]$, by (W2) we can conclude that $u = \beta$. Applying (1.1) and the induction hypothesis, we obtain

$$\begin{aligned}
d^2(u,z) &= d^2(\frac{1}{2}x \oplus \frac{1}{2}\alpha, z) \\
&\leq \frac{1}{2}d^2(x,z) + \frac{1}{2}d^2(\alpha,z) - c_M d^2(x,\alpha) \\
&\leq \frac{1}{2}d^2(x,z) + \frac{1}{2}[(1 - \frac{k}{2^n})d^2(x,z) + \frac{k}{2^n}d^2(y,z) - 4c_M \frac{k}{2^n}(1 - \frac{k}{2^n})d^2(x,y)] \\
&\quad - c_M[\frac{k}{2^n}d(x,y)]^2 \\
&= (1 - \frac{k}{2^{n+1}})d^2(x,z) + \frac{k}{2^{n+1}}d^2(y,z) - 4c_M \frac{k}{2^{n+1}}(1 - \frac{k}{2^{n+1}})d^2(x,y).
\end{aligned}$$

Now, suppose that $2^n < k \leq 2^{n+1}$ and let $p := 2^{n+1} - k$. Then $p \leq 2^n$, by applying (2.5) for p, we get that

$$\begin{aligned}
d^2(u,z) &= d^2(\frac{p}{2^{n+1}}x \oplus (1 - \frac{p}{2^{n+1}})y, z) \\
&= d^2((1 - \frac{p}{2^{n+1}})y \oplus \frac{p}{2^{n+1}}x, z) \\
&\leq (1 - \frac{p}{2^{n+1}})d^2(y,z) + \frac{p}{2^{n+1}}d^2(x,z) - 4c_M \frac{p}{2^{n+1}}(1 - \frac{p}{2^{n+1}})d^2(x,y) \\
&= (1 - \frac{k}{2^{n+1}})d^2(x,z) + \frac{k}{2^{n+1}}d^2(y,z) - 4c_M \frac{k}{2^{n+1}}(1 - \frac{k}{2^{n+1}})d^2(x,y).
\end{aligned}$$

Let $D := \{k/2^n : k, n \in \mathbb{N}, k \leq 2^n\}$. Then D is a dense subset of $[0,1]$. For each $t \in [0,1]$, there exists a sequence $\{t_k\}$ in D such that $\lim_{k\to\infty} t_k = t$. Now, we have

$$d^2((1-t_k)x \oplus t_k y, z) \leq (1-t_k)d^2(x,z) + t_k d^2(y,z) - 4c_M t_k(1-t_k)d^2(x,y). \tag{2.6}$$

Notice from (W2) that the function $f : [0,1] \to [x,y]$ defined by $f(t) := (1-t)x \oplus ty$ is continuous. Letting $k \to \infty$ from (2.6), we get (2.3). □

3. Applications

In this section, we apply Theorem 2.2 to prove Δ and strong convergence theorems for the Ishikawa iteration process defined by Panyanak [19]. From now on, X stands for a complete 2-uniformly convex hyperbolic space with monotone modulus of uniform convexity.

Let $x \in X$ and E be a nonempty subset of X. The *distance* from x to E is defined by

$$\text{dist}(x,E) := \inf\{d(x,y) : y \in E\}.$$

The *radius* of E relative to x is defined by

$$R(x,E) := \sup\{d(x,y) : y \in E\}.$$

We denote by $K(E)$ the family of nonempty compact subsets of E. The *Pompeiu-Hausdorff distance* on $K(E)$ is defined by

$$H(A,B) := \max\left\{\sup_{a\in A}\operatorname{dist}(a,B), \sup_{b\in B}\operatorname{dist}(b,A)\right\} \text{ for all } A,B\in K(E).$$

Definition 3.5. ([9]) Let E be a nonempty subset of X. A multi-valued mapping $T : E \to K(E)$ is said to be *Suzuki* if for each $x, y \in E$,

$$\frac{1}{2}\operatorname{dist}(x,T(x)) \le d(x,y) \text{ implies } H(T(x),T(y)) \le d(x,y).$$

Let $\mu \ge 1$. The mapping T is said to satisfy *condition (E_μ)* if for each $x, y \in E$, we have

$$\operatorname{dist}(x,T(y)) \le \mu \operatorname{dist}(x,T(x)) + d(x,y).$$

It is known from Lemma 3.2 of [7] that every Suzuki mapping satisfies condition (E_3).

An element x in E is called a *fixed point* of T if $x \in T(x)$. Moreover, if $\{x\} = T(x)$, then x is called a *strict fixed point* (or an *endpoint*) of T. It is denoted by $Fix(T)$ the set of all fixed points of T and by $SFix(T)$ the set of all strict fixed points of T. Using these notations, for any mapping $T : E \to K(E)$, we have the following statements:

- $SFix(T) \subseteq Fix(T)$.
- $x \in Fix(T)$ if and only if $\operatorname{dist}(x,T(x)) = 0$.
- $x \in SFix(T)$ if and only if $R(x,T(x)) = 0$.

The existence of fixed points and strict fixed points for multi-valued Suzuki mappings was widely studied by many authors, see [1, 7, 8, 14, 17] and the references therein. In 2018, Kudtha and Panyanak [14] proved that a Suzuki mapping T on a nonempty bounded closed convex subset E of X has a strict fixed point if and only if $\inf\{R(x,T(x)) : x \in E\} = 0$. Notice also that if E is closed in X and $T : E \to K(E)$ is Suzuki, then $SFix(T)$ is also closed in X (see [5]).

A multi-valued mapping $T : E \to K(E)$ is said to be *nonexpansive* if $H(T(x),T(y)) \le d(x,y)$ for all $x, y \in E$. The mapping T is said to be *quasi-nonexpansive* if for each $x \in E$ and $y \in Fix(T)$, one has $H(T(x),T(y)) \le d(x,y)$.

Proposition 3.1. *Let E be a nonempty subset of X and $T : E \to K(E)$ be a multi-valued mapping. Then the following statements hold.*
(1) If T is nonexpansive, then T is Suzuki.
(2) If T is Suzuki and $Fix(T) \neq \emptyset$, then T is quasi-nonexpansive.

The following example shows that the converse of (1) in Proposition 3.1 is not true.

Example 3.1. Let $E = [0,3]$ and $T : E \to K(E)$ be defined by

$$T(x) = \begin{cases} \{0\} & \text{if } x \neq 3, \\ [0.9, 1] & \text{if } x = 3. \end{cases}$$

If $x < y$ and $(x,y) \in (E \times E) - ((2,3) \times \{3\})$, then $H(T(x),T(y)) \le d(x,y)$. If $x \in (2,3)$ and $y = 3$, then

$$\frac{1}{2}\operatorname{dist}(x,T(x)) = \frac{x}{2} > 1 > d(x,y) \text{ and } \frac{1}{2}\operatorname{dist}(y,T(y)) = 1 > d(x,y).$$

This implies that T is Suzuki. However, if $x = 2.5$ and $y = 3$, then $d(x,y) = 0.5$ and $H(T(x),T(y)) = 1$. This shows that T is not nonexpansive.

Let E be a nonempty subset of X and $\{x_n\}$ be a bounded sequence in X. The *asymptotic radius* of $\{x_n\}$ relative to E is defined by

$$r(E,\{x_n\}) = \inf\big\{\limsup_{n\to\infty} d(x_n,x) : x \in E\big\}.$$

The *asymptotic center* of $\{x_n\}$ relative to E is defined by

$$A(E,\{x_n\}) = \{x \in E : \limsup_{n\to\infty} d(x_n, x) = r(E,\{x_n\})\}.$$

It is known from Proposition 3.3 of [16] that if E is a nonempty closed convex subset of X, then $A(E,\{x_n\})$ consists of exactly one point. The following lemma was proved by Dhompongsa and Panyanak [6].

Lemma 3.1. *Let E be a nonempty closed convex subset of X and $\{x_n\}$ be a bounded sequence in X. If $A(E,\{x_n\}) = \{x\}$ and $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(E,\{u_n\}) = \{u\}$ and the sequence $\{d(x_n, u)\}$ converges, then $x = u$.*

Now, we give the concept of $\Delta-$convergence and collect some of its basic properties.

Definition 3.6. Let E be a nonempty closed convex subset of X and $x \in E$. Let $\{x_n\}$ be a bounded sequence in X. We say that $\{x_n\}$ $\Delta-$*converges* to x if $A(E,\{u_n\}) = \{x\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case we write $\Delta - \lim_{n\to\infty} x_n = x$ and call x the $\Delta-$*limit* of $\{x_n\}$.

It is known from [13] that every bounded sequence in X has a $\Delta-$convergent subsequence. The following fact can be found in [5].

Lemma 3.2. *Let E be a nonempty closed convex subset of X and $T : E \to K(E)$ be a Suzuki mapping. Then the following implication holds:*

$$\{x_n\} \subseteq E,\ \Delta - \lim_{n\to\infty} x_n = x,\ \lim_{n\to\infty} R(x_n, T(x_n)) = 0 \implies x \in SFix(T).$$

Lemma 3.3. *Let E be a nonempty closed convex subset of X, and let $T : E \to K(E)$ be a Suzuki mapping. Suppose $\{x_n\}$ is a bounded sequence in E such that $\lim_{n\to\infty} R(x_n, Tx_n) = 0$ and $\{d(x_n, v)\}$ converges for all $v \in SFix(T)$, then $\omega_w(x_n) \subseteq SFix(T)$. Here $\omega_w(x_n) := \bigcup A(E,\{u_n\})$ where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. Moreover, $\omega_w(x_n)$ consists of exactly one point.*

Proof. Let $u \in \omega_w(x_n)$, then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(E,\{u_n\}) = \{u\}$. Since $\{u_n\}$ is bounded, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ and $v \in E$ such that $\Delta - \lim_{n\to\infty} v_n = v$. By Lemmas 3.1 and 3.2, $u = v \in SFix(T)$. This shows that $\omega_w(x_n) \subseteq SFix(T)$. Next, we show that $\omega_w(x_n)$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$ with $A(E,\{u_n\}) = \{u\}$ and let $A(E,\{x_n\}) = \{x\}$. Since $u \in \omega_w(x_n) \subseteq SFix(T)$, $\{d(x_n, u)\}$ converges. By Lemma 3.1, $x = u$. This completes the proof. $\square$

Definition 3.7. ([19]) Let E be a nonempty convex subset of X, and $\{\alpha_n\}$, $\{\beta_n\}$ be sequences in $[0,1]$, and $T : E \to K(E)$ be a multi-valued mapping. *The sequence of Ishikawa iteration* is defined by $x_1 \in E$,

$$y_n = (1-\beta_n)x_n \oplus \beta_n z_n,\ \ n \in \mathbb{N},$$

where $z_n \in T(x_n)$ such that $d(x_n, z_n) = R(x_n, T(x_n))$, and

$$x_{n+1} = (1-\alpha_n)x_n \oplus \alpha_n v_n,\ \ n \in \mathbb{N}, \tag{3.7}$$

where $v_n \in T(y_n)$ such that $d(y_n, v_n) = R(y_n, T(y_n))$.

A sequence $\{x_n\}$ in X is said to be *Fejér monotone* [2] with respect to E if

$$d(x_{n+1}, p) \le d(x_n, p)\ \text{ for all } p \in E \text{ and } n \in \mathbb{N}.$$

The following lemma is crucial.

Lemma 3.4. *Let E be a nonempty closed convex subset of X and $T : E \to K(E)$ be a Suzuki mapping with $SFix(T) \neq \emptyset$. Let $\{x_n\}$ be the sequence of Ishikawa iteration defined by (3.7). Then $\{x_n\}$ is Fejér monotone with respect to $SFix(T)$.*

Proof. Let $p \in SFix(T)$. We note that T is quasi-nonexpansive. For each $n \in \mathbb{N}$, we have

$$\begin{aligned} d(y_n, p) &\leq (1-\beta_n)d(x_n, p) + \beta_n d(z_n, p) \\ &= (1-\beta_n)d(x_n, p) + \beta_n \text{dist}(z_n, T(p)) \\ &\leq (1-\beta_n)d(x_n, p) + \beta_n H(T(x_n), T(p)) \\ &\leq d(x_n, p), \end{aligned}$$

which implies that

$$\begin{aligned} d(x_{n+1}, p) &\leq (1-\alpha_n)d(x_n, p) + \alpha_n d(v_n, p) \\ &= (1-\alpha_n)d(x_n, p) + \alpha_n \text{dist}(v_n, T(p)) \\ &\leq (1-\alpha_n)d(x_n, p) + \alpha_n H(T(y_n), T(p)) \\ &\leq (1-\alpha_n)d(x_n, p) + \alpha_n d(y_n, p) \\ &\leq d(x_n, p). \end{aligned}$$

This shows that $\{x_n\}$ is Fejér monotone with respect to $SFix(T)$. □

The following fact can be found in [5].

Lemma 3.5. *Let E be a nonempty closed subset of X and $\{x_n\}$ be a Fejér monotone sequence with respect to E. Then $\{x_n\}$ converges strongly to an element of E if and only if $\lim_{n\to\infty} dist(x_n, E) = 0$.*

Now, we prove $\Delta-$convergence theorem.

Theorem 3.3. *Let E be a nonempty closed convex subset of X and $T : E \to K(E)$ be a Suzuki mapping with $SFix(T) \neq \emptyset$. Let $\alpha_n, \beta_n \in [a, b] \subset (0, 1)$ and $\{z_n\}$ be the sequence of Ishikawa iteration defined by (3.7). Then $\{x_n\}$ $\Delta-$converges to a strict fixed point of T.*

Proof. Fix $p \in SFix(T)$. By Theorem 2.2 we have

$$\begin{aligned} d^2(y_n, p) &\leq (1-\beta_n)d^2(x_n, p) + \beta_n d^2(z_n, p) - 4c_M\beta_n(1-\beta_n)d^2(x_n, z_n) \\ &\leq (1-\beta_n)d^2(x_n, p) + \beta_n H^2(T(x_n), T(p)) - 4c_M\beta_n(1-\beta_n)d^2(x_n, z_n) \\ &\leq d^2(x_n, p) - 4c_M\beta_n(1-\beta_n)d^2(x_n, z_n). \end{aligned}$$

This implies that

$$\begin{aligned} d^2(x_{n+1}, p) &\leq (1-\alpha_n)d^2(x_n, p) + \alpha_n d^2(v_n, p) - 4c_M\alpha_n(1-\alpha_n)d^2(x_n, v_n) \\ &\leq (1-\alpha_n)d^2(x_n, p) + \alpha_n H^2(T(y_n), T(p)) - 4c_M\alpha_n(1-\alpha_n)d^2(x_n, v_n) \\ &\leq (1-\alpha_n)d^2(x_n, p) + \alpha_n d^2(y_n, p) \\ &\leq (1-\alpha_n)d^2(x_n, p) + \alpha_n d^2(x_n, p) - 4c_M\alpha_n\beta_n(1-\beta_n)d^2(x_n, z_n) \\ &= d^2(x_n, p) - 4c_M\alpha_n\beta_n(1-\beta_n)d^2(x_n, z_n). \end{aligned}$$

Since $c_M > 0$, it follows that

$$\sum_{n=1}^{\infty} a^2(1-b)d^2(x_n, z_n) \leq \sum_{n=1}^{\infty} \alpha_n\beta_n(1-\beta_n)d^2(x_n, z_n) < \infty. \tag{3.8}$$

Thus $\lim_{n\to\infty} d^2(x_n, z_n) = 0$, and hence

$$\lim_{n\to\infty} R(x_n, T(x_n)) = \lim_{n\to\infty} d(x_n, z_n) = 0. \tag{3.9}$$

By Lemma 3.4, $\{d(x_n, v)\}$ converges for all $v \in SFix(T)$. By Lemma 3.3, $\omega_w(x_n)$ consists of exactly one point and is contained in $SFix(T)$. This shows that $\{x_n\}$ $\Delta-$converges to an element of $SFix(T)$. □

Next, we prove strong convergence theorems. For this, we will add more conditions. Recall that a mapping $T : E \to K(E)$ is said to satisfy *condition (J)* if there exists a non-decreasing function $h : [0, \infty) \to [0, \infty)$ with $h(0) = 0$, $h(r) > 0$ for $r \in (0, \infty)$ such that

$$R(x, T(x)) \geq h(\text{dist}(x, SFix(T))) \text{ for all } x \in E.$$

The mapping T is called *semicompact* if for any sequence $\{x_n\}$ in E such that

$$\lim_{n\to\infty} R(x_n, T(x_n)) = 0,$$

there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $q \in E$ such that $\lim_{k\to\infty} x_{n_k} = q$.

Theorem 3.4. *Let E be a nonempty closed convex subset of X and $T : E \to K(E)$ be a Suzuki mapping with $SFix(T) \neq \emptyset$. Let $\alpha_n, \beta_n \in [a, b] \subset (0, 1)$ and $\{x_n\}$ be the sequence of Ishikawa iteration defined by (3.7). If T satisfies condition (J), then $\{x_n\}$ converges strongly to a strict fixed point of T.*

Proof. Since T satisfies condition (J), by (3.9) we get that $\lim_{n\to\infty} \text{dist}(x_n, SFix(T)) = 0$. By Lemma 3.4, $\{x_n\}$ is Fejér monotone with respect to $SFix(T)$. The conclusion follows from Lemma 3.5. □

Example 3.2. Let E and T be as in Example 3.1. Then T is a Suzuki mapping with $SFix(T) = \{0\}$. Notice that T satisfies condition (J) with the function $h : [0, \infty) \to [0, \infty)$ defined by $h(r) = \frac{r}{2}$ for all $r \in [0, \infty)$. For each $n \in \mathbb{N}$, we let $\alpha_n = \beta_n = \frac{1}{2}$. Then by Theorem 3.4, the sequence of Ishikawa iteration defined by (3.7) converges strongly to 0. However, we cannot directly apply Theorem 3.5 of [19] because, in this situation, T is not nonexpansive.

Remark 3.2. In the proofs of Theorems 3.3 and 3.4, one may observe that it is not necessary to use Theorem 2.2 because Theorem 1.1 is sufficient. The following result extends Theorem 3.6 of [19]. We will show that the proof is quite simple when we apply Theorem 2.2.

The following fact is also needed.

Lemma 3.6. ([20]) *Let $\{\alpha_n\}, \{\beta_n\}$ be two real sequences in $[0, 1)$ such that $\beta_n \to 0$ and $\sum \alpha_n\beta_n = \infty$. Let $\{\gamma_n\}$ be a nonnegative real sequence such that $\sum \alpha_n\beta_n(1 - \beta_n)\gamma_n < \infty$. Then $\{\gamma_n\}$ has a subsequence which converges to zero.*

Theorem 3.5. *Let E be a nonempty closed convex subset of X and $T : E \to K(E)$ be a Suzuki mapping with $SFix(T) \neq \emptyset$. Let $\alpha_n, \beta_n \in [0, 1)$ be such that $\beta_n \to 0$ and $\sum \alpha_n\beta_n = \infty$ and let $\{x_n\}$ be the sequence of Ishikawa iteration defined by (3.7). If T is semicompact, then $\{x_n\}$ converges strongly to a strict fixed point of T.*

Proof. From (3.8), we get that

$$\sum_{n=1}^{\infty} \alpha_n\beta_n(1 - \beta_n)d^2(x_n, z_n) < \infty.$$

By Lemma 3.6, there exist subsequences $\{x_{n_k}\}$ and $\{z_{n_k}\}$ of $\{x_n\}$ and $\{z_n\}$ respectively, such that $\lim_{k\to\infty} d^2(x_{n_k}, z_{n_k}) = 0$. Hence

$$\lim_{k\to\infty} R(x_{n_k}, T(x_{n_k})) = \lim_{k\to\infty} d(x_{n_k}, z_{n_k}) = 0. \tag{3.10}$$

Since T is semicompact, by passing to a subsequence, we may assume that $x_{n_k} \to q$ for some $q \in E$. Since T satisfies (E_3),

$$\begin{aligned}\text{dist}(q, T(q)) &\le d(q, x_{n_k}) + \text{dist}(x_{n_k}, T(q)) \\ &\le 2d(q, x_{n_k}) + 3\text{dist}(x_{n_k}, T(x_{n_k})) \to 0 \text{ as } k \to \infty.\end{aligned}$$

Hence $q \in T(q)$. Since T is quasi-nonexpansive, we have

$$H(T(x_{n_k}), T(q)) \le d(x_{n_k}, q) \to 0 \text{ as } k \to \infty. \tag{3.11}$$

We now let $v \in T(q)$ and choose $w_{n_k} \in T(x_{n_k})$ so that $d(v, w_{n_k}) = \text{dist}(v, T(x_{n_k}))$. From (3.10) and (3.11) we have

$$\begin{aligned}d(q, v) &\le d(q, x_{n_k}) + d(x_{n_k}, w_{n_k}) + d(w_{n_k}, v) \\ &\le d(q, x_{n_k}) + R(x_{n_k}, T(x_{n_k})) + H(T(x_{n_k}), T(q)) \to 0 \text{ as } k \to \infty.\end{aligned}$$

Hence $v = q$ for all $v \in T(q)$. Therefore $q \in SFix(T)$. By Lemma 3.4, $\lim_{n\to\infty} d(x_n, q)$ exists and hence q is the strong limit of $\{x_n\}$. □

Finally, we finish the paper by providing an example which shows the efficiency of Theorem 3.5.

Example 3.3. Let E and T be as in Example 3.1. Then T is a Suzuki mapping with $SFix(T) = \{0\}$. Notice also that T is semicompact since E is compact. For each $n \in \mathbb{N}$, we let $\alpha_n = \frac{1}{2}$ and $\beta_n = \frac{1}{n+1}$. Then $\beta_n \to 0$ and $\sum \alpha_n \beta_n = \infty$. By Theorem 3.5, the sequence of Ishikawa iteration defined by (3.7) converges strongly to 0. However, we cannot directly apply Theorem 3.6 of [19] because, in this situation, T is not nonexpansive.

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