Dedicated to Prof. Billy E. Rhoades on the occasion of his 90th anniversary

A strong convergence theorem for maximal monotone operators in Banach spaces with applications

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ABSTRACT.

An algorithm is constructed to approximate a zero of a maximal monotone operator in a uniformly convex and uniformly smooth real Banach space. The sequence of the algorithm is proved to converge strongly to a zero of the maximal monotone map. In the case where the Banach space is a real Hilbert space, our theorem complements the celebrated proximal point algorithm of Martinet and Rockafellar. Furthermore, our convergence theorem is applied to approximate a solution of a Hammerstein integral equation in our general setting. Finally, numerical experiments are presented to illustrate the convergence of our algorithm.

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