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Universal centers and composition conditions on the complex plane

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ABSTRACT. We characterize the universal centers of the ordinary differential equations in the complex plane $d\rho/d\theta = \sum_{i=1}^{\infty} a_i(\theta)\rho^{i+1}$, where $a_i(\theta)$ are trigonometric polynomials with complex coefficients, in terms of the composition conditions.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

We consider the differential equation

$$\frac{d\rho}{d\theta} = \sum_{i=1}^{\infty} a_i(\theta)\rho^{i+1} \tag{1.1}$$

on $(\rho, \theta) \in \mathbb{C} \times \mathbb{S}^1$ in a neighborhood of $\rho = 0$ and where $a_i(\theta)$ are trigonometric polynomials in θ with complex coefficients.

Following the definitions in $\mathbb{R}$, we say that equation (1.1) determines a *center* if $\rho(0) = \rho(2\pi)$. The *center problem* consists on finding conditions on the coefficients a_i under which this equation has a center. This problem in $\mathbb{R}^2$ has a close relation with the explicit expression for the first return map of the differential equation (1.1) (see [10, 12]).

The expression of the first return map can be given in terms of the following iterated integrals of order k

$$I_{i_1\cdots i_k}(a) := \underbrace{\int\cdots\int}_{0\le\tau_1\le\cdots\le\tau_k\le 2\pi} a_{i_k}(\tau_k)\cdots a_{i_1}(\tau_1)\, d\tau_k\cdots\, d\tau_1, \tag{1.2}$$

where by convention when $k = 0$ we set it equal to 1. By the Ree formula [14] the linear space generated by all such functions is an algebra which is commutative, associative and regularly graded (see [5, p.150] for a definition). More precisely, let $\rho(\theta; \rho_0; a)$ with $\theta \in [0, 2\pi]$ and $a(\theta) = (a_1(\theta), \ldots)$ be the solution of equation (1.1) so that $\rho(0; \rho_0; a) = \rho_0$. Then the first return map is $P(a)(\rho_0) = \rho(2\pi; \rho_0; a)$ and in [10, 12] it is proved that for a sufficiently small ρ_0 the first return map $P(a)(\rho_0)$ is an absolute convergent power series given by

$$\rho_0 + \sum_{n=1}^{\infty} c_n(a)\rho_0^{n+1}$$

where

$$c_n(a) = \sum_{i_1+\cdots+i_n=n} c_{i_1\cdots i_k} I_{i_1\cdots i_k}(a),$$

and

$$c_{i_1\cdots i_k} = (n - i_1 + 1)(n - i_1 - i_2 + 1)(n - i_1 - i_2 - i_3 + 1)\cdots 1.$$

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Following the definitions in $\mathbb{R}$ (see [11]) we say that the differential equation (1.1) has a *universal center* if for all positive integers $i_1, \ldots, i_k$ with $k \geq 1$ the iterated integral $I_{i_1 \cdots i_k}(a) = 0$ (note that in the trigonometric polynomial case, this set needs not be finite).

On the other hand we say that equation (1.1) satisfies the *composition conditions* if there is a trigonometric polynomial with complex coefficients q and there are polynomials $p_i \in \mathbb{C}[w]$, for $i \geq 1$ such that

$$\tilde{a}_i = p_i \circ q, \quad \text{where} \quad \tilde{a}_i(\theta) = \int_0^\theta a_i(\tau)\, d\tau, \quad i \geq 1. \tag{1.3}$$

There are plenty of results in the case of $\mathbb{R}$ of results regarding universal centers and composition conditions either for polynomials or analytic functions (see for instance [1, 2, 3, 4, 6, 7, 8, 9] and the references therein), but there are almost no results in the case of $\mathbb{C}$. Brudnyi in [12, Corollary 1.19] proved that equation (1.1) with finitely many a_i's being all trigonometric polynomials has a universal center if and only if it satisfies the composition condition (1.3) for all a_i with $i = 1, 2, \ldots, n$. In the present paper we generalize this result to the differential equation (1.1), that is, with infinitely many a_i's (see Theorem 1.1 below). So we extend to an analytic differential equation in ρ as in (1.1). This is done in this paper for the first time

Theorem 1.1. *Every center of* (1.1) *is universal if and only if* (1.1) *satisfies the composition condition.*

The proof of Theorem 1.1 is given in Section 3. In the case of $\mathbb{R}$ this theorem was proved in [13].

In Section 2 we introduce some notation and auxiliary results that will be used in the proof of Theorem 1.1 and whose proofs are given in the Appendix.

2. Auxiliary results

We introduce some notation and auxiliary results that will be used in the proof of Theorem 1.1. Given a trigonometric polynomial p we call $\deg(p) = \ell$ the degree of the Fourier series corresponding to p, that is

$$f(\theta) = \sum_{k=-\ell}^{\ell} a_k e^{ki\theta}, \quad a_k \in \mathbb{C}, \quad \text{with } a_\ell, a_{-\ell} \neq 0.$$

To introduce a variant of the Lüroth theorem we introduce some notation. We denote by $\mathbb{C}(x)$ the quotient field of the ring of polynomials $\mathbb{C}[x]$ with coefficients in $\mathbb{C}$ and by $\mathbb{C}(\theta)$ the quotient field of the ring of trigonometric polynomials $\mathbb{C}[\theta]$, also with coefficients in $\mathbb{C}$. It is well-known that $\mathbb{C}(\theta)$ is isomorphic to $\mathbb{C}(x)$ by means of the map $\Phi\colon \mathbb{C}(\theta) \to \mathbb{C}(x)$ defined by

$$\Phi(\sin\theta) = \frac{x^2 - 1}{2ix} \quad \Phi(\cos\theta) = \frac{x^2 + 1}{2x}.$$

Moreover, if we consider the function field $F = \mathbb{C}(x, y)$ with $x^2 + y^2 = 1$ and the the ring of trigonometric polynomials with complex coefficients $T = \mathbb{C}[x, y]$ (again with $x^2 + y^2 = 1$), then the algebraic curve over $\mathbb{C}$ given by the equation $x^2 + y^2 = 1$ has no singularities, and so the ring T is integrally closed. Due to the equation

$$(x + iy)(x - iy) = 1 \quad (\text{with } i^2 = -1),$$

if we set $z = x + iy$ then $F = \mathbb{C}(z)$ and $T = \mathbb{C}[z, z^{-1}]$. In particular, every $t \in T \setminus \mathbb{C}[z]$ has the form f/z^m, for some $m \in \mathbb{N}$ and $f \in \mathbb{C}[z]$ not divisible by z. The first of our auxiliary results is the following.

We first recall Lüroth's theorem which states that every intermediate field E with $\mathbb{C} \subseteq E \subseteq \mathbb{C}(\theta)$ is a simple transcendental extension, that is $E = \mathbb{C}(s(\theta))$ where $s(\theta)$ is a nonconstant quotient of trigonometric polynomials with coefficients in $\mathbb{C}$. For a proof of Lüroth's theorem see [16, page 21]. We will adapt Lüroth's theorem to our purposes and prove the more convenient following theorem.

Theorem 2.2. *An intermediate field E with $\mathbb{C} \subseteq E \subseteq \mathbb{C}(\theta)$ satisfies that $E = \mathbb{C}(r)$ for some non-constant trigonometric polynomial r.*

Let

$$\tilde{a}(\theta) = \int_0^\theta a(s)\,ds$$

Lemma 2.1. *If equation* (1.1) *has a universal center, then $\tilde{a}_i(\theta)$ is a trigonometric polynomial for all $i \geq 1$.*

Given a k-vector of indexes $i_1 i_2 \cdots i_k$ we define

$$I_{i_1 i_2 \cdots i_k}(\theta) = \underbrace{\int \cdots \int}_{0 \leq \tau_1 \leq \cdots \leq \tau_k \leq 2\pi} a_{i_k}(\tau_k) \cdots a_{i_1}(\tau_1)\, d_{\tau_k} \cdots d_{\tau_1} \tag{2.4}$$

and if we denote by $\vec{i} = i_1 i_2 \cdots i_k$ then by (2.4) we have

$$I_{\vec{i}j}(\theta) = \int_0^\theta I_{\vec{i}}(\tau) a_j(\tau)\, d\tau, \tag{2.5}$$

where by convention we have $I_\emptyset(\theta) = 1$. The Ree's formula (see [14]) establishes a way to write the product of two iterated integrals $I_{\vec{i}}(\theta)$ and $I_{\vec{j}}(\theta)$ as a summation of all the iterated integrals indexed by the shuffle products of the indexes $\vec{i}$ and $\vec{j}$ (we recall that a (r,s)-shuffle is a permutation σ of $r+s$ letters with $\sigma^{-1}(1) < \sigma^{-1}(2) < \sigma^{-1}(r)$ and $\sigma^{-1}(r+1) < \sigma^{-1}(r+2) < \cdots \sigma^{-1}(r+s)$). More precisely it guarantees that

$$I_{\vec{i}}(\theta) I_{\vec{j}}(\theta) = \sum_\sigma I_{\sigma(\vec{i},\vec{j})}(\theta), \tag{2.6}$$

where the sum runs over all $\sigma(\vec{i},\vec{j})$ or (r,s)-shuffles. For instance, Ree's formula gives that

$$\begin{aligned}
I_{i_1}(\theta) I_{i_2}(\theta) &= I_{i_1 i_2}(\theta) + I_{i_2 i_1}(\theta), \\
I_i^m(\theta) &= m!\, I_{\underbrace{i \ldots i}_{m \text{ times}}}(\theta), \\
I_{i_1}(\theta) I_{i_2 i_3}(\theta) &= I_{i_1 i_2 i_3}(\theta) + I_{i_2 i_1 i_3}(\theta) + I_{i_2 i_3 i_1}(\theta), \\
I_{i_1 i_2}(\theta) I_{i_3 i_4}(\theta) &= I_{i_1 i_2 i_3 i_4}(\theta) + I_{i_1 i_3 i_2 i_4}(\theta) + I_{i_1 i_3 i_4 i_2}(\theta) + I_{i_3 i_1 i_2 i_4}(\theta) \\
&\quad + I_{i_3 i_1 i_4 i_2}(\theta) + I_{i_3 i_4 i_1 i_2}(\theta),
\end{aligned}$$

and so on.

As a direct consequence of Ree's formula (2.6) we have the following lemma

Lemma 2.2. *There exist non-negative numbers n_j for $j = 1, 2, \ldots, J$ such that*

$$\tilde{a}_{i_1}^{m_1}(\theta) \tilde{a}_{i_2}^{m_2}(\theta) \cdots \tilde{a}_{i_k}^{m_k}(\theta) = \sum_{j=1}^J n_j I_{\sigma_j(\vec{i})}(\theta), \tag{2.7}$$

where $i_j \geq 1$, $m_j \geq 0$ for $j = 1, \ldots, k$, σ_j runs over all permutations of the vector

$$\vec{i} = \underbrace{i_1 i_1 \cdots i_1}_{m_1 \text{ times}} \underbrace{i_2 i_2 \cdots i_2}_{m_2 \text{ times}} \cdots \underbrace{i_k i_k \cdots i_k}_{m_k \text{ times}}$$

and $J = (m_1 + m_2 + \cdots + m_k)!$.

Given equation (1.1) we denote by $\Gamma(a)$ the minimal field containing all the functions $\tilde{a}_i(\theta)$ and $\mathbb{C}$. We note that $\Gamma(a)$ is the quotient field of the polynomial domain formed by all the linear combinations with coefficients in $\mathbb{C}$ of monomials of the form

$$\tilde{a}_{i_1}^{m_1}(\theta)\tilde{a}_{i_2}^{m_2}(\theta)\cdots\tilde{a}_{i_k}^{m_k}(\theta) \tag{2.8}$$

where $i_j \geq 1$ and $m_j \geq 0$ for $j = 1, 2, \dots, k$. We consider two polynomials $p(\theta)$ and $q(\theta)$ of $\Gamma(a)$ that is, two functions formed by linear combinations of monomials of the form in (2.8) with coefficients in $\mathbb{C}$.

Lemma 2.3. *Consider two polynomials* $p_1, p_2 \in \Gamma(a)$*. If equation* (1.1) *has a universal center, then*

$$\int_0^{2\pi} p_1(\theta)p_2'(\theta)\, d\theta = 0.$$

3. Proof of Theorem 1.1

Assume first that equation (1.1) satisfies the composition condition

$$\tilde{a}_i(\theta) = p_i(q(\theta)), \quad i \geq 1 \tag{3.9}$$

and we will show that equation (1.1) has a universal center. We take an iterated integral $I_{\vec{i}}(\theta)$ of order k and by induction over k we will show that there exists a polynomial $P_{\vec{i}}(w) \in \mathbb{C}[w]$ such that $I_{\vec{i}}(\theta) = P_{\vec{i}}(q(\theta))$ and $P_{\vec{i}}(q(0) = 0$.

When $k = 1$, given any index $i \geq 1$ we have

$$I_i(\theta) = \int_0^\theta a_i(\tau)\, d\tau = \tilde{a}_i(\theta).$$

Thus, since the equation satisfies the composition condition, there exists a polynomial $p_i(w) \in \mathbb{C}[w]$ such that $I_i(\theta) = p_i(q(\theta))$. Note that $p_i(q(0)) = 0$.

Now we assume that the statement holds for k and we will prove it for $k+1$. We take $\vec{i} = i_1 i_2 \cdots i_k$. By induction hypothesis we have that there exists a polynomial $P_{\vec{i}} \in \mathbb{C}[w]$ such that $I_{\vec{i}}(\theta) = P_{\vec{i}}(q(\theta))$. We consider any index $j \geq 1$ and by (2.5) we have

$$I_{\vec{i}j}(\theta) = \int_0^\theta I_{\vec{i}}(\tau)a_j(\tau)\, d\tau.$$

Moreover, since $\tilde{a}_j'(\theta) = a_j(\theta)$ and by the composition condition we know that there exists a polynomial $p_j(w) \in \mathbb{C}[w]$ such that $\tilde{a}_j(\theta) = p_j(q(\theta))$ we get that

$$I_{\vec{i}j}(\theta) = \int_0^\theta P_{\vec{i}}(q(\tau))p_j'(q(\tau))q'(\tau)\, d\tau = P_{\vec{i}j}(q(\theta)) - P_{\vec{i}j}(q(0)),$$

where $P_{\vec{i}j}(w)$ is a polynomial that is a primitive of the polynomial $P_{\vec{i}}(w)p_j'(w)$ (that is $P_{\vec{i}j}'(w) = P_{\vec{i}}(w)p_j'(w)$. Without loss of generality we can assume that $P\vec{i}j(w)$ satisfies that $P_{\vec{i}j}(q(0)) = 0$. We recall that equation (1.1) has a universal center if any iterated integral $I_{\vec{i}}(a) = 0$. Given any $\vec{i}$ we have proved that there exists a polynomial $P_{\vec{i}}(w) \in \mathbb{C}[w]$ such that $I_{\vec{i}}(\theta) = P_{\vec{i}}(q(\theta))$ and $P_{\vec{i}}(q(0)) = 0$. Therefore,

$$I_{\vec{i}}(a) = I_{\vec{i}}(2\pi) = P_{\vec{i}}(q(2\pi)).$$

Since $q(\theta)$ is a trigonometric polynomial, we have that $q(2\pi) = q(0)$ and since $P_{\vec{i}}(q(0)) = 0$ we obtain that $I_{\vec{i}}(a) = 0$. In short, equation (1.1) has a universal center, as we wanted to show.

Now we prove the converse, that is, we assume that equation (1.1) has a universal center and we will show that it satisfies the composition condition (3.9). If equation (1.1) has a trivial center, that is $\tilde{a}_i(\theta) \equiv 0$ for all i, then the composition condition trivially holds taking any polynomial $q(\theta)$ and $p_i \equiv 0$ for all i. We thus assume that equation (1.1) has a nontrivial universal center. As before we denote by $\Gamma(a)$ the minimal field containing all the function $\tilde{a}_i(\theta)$ and $\mathbb{C}$.

In order to apply Theorem 2.2 we need to show that $\Gamma(a)$ is an intermediate field between $\mathbb{C}$ and $\mathbb{C}(\theta)$. By definition $\mathbb{C} \subset \Gamma(a)$, but if these two fields were equal then we would have that all the function $\tilde{a}_i(\theta)$ would be constant and since $\tilde{a}_i'(\theta) = a_i(\theta)$ we would have a trivial universal center, which we have already discarded. Thus we have that $\mathbb{C} \subsetneq \Gamma(a)$. Moreover, by Lemma 2.1 we have that $\Gamma(a) \subset \mathbb{C}(\theta)$. If these two fields were equal then we would have that the polynomials $p_1(\theta) = e^{i\theta}$ and $p_2(\theta) = e^{-i\theta}$ belong to $\Gamma(a)$. Then

$$\int_0^{2\pi} p_1(\theta)p_2'(\theta)\, d\theta = \int_0^{2\pi} -i\, d\theta = -2\pi i,$$

in contradiction with Lemma 2.3 (this integral should be zero). Therefore, we have $\Gamma(a) \subsetneq \mathbb{C}(\theta)$. It follows from Lüroth's theorem that there exists a nonconstant quotient of trigonometric polynomials $q(\theta)$ such that $\Gamma(a) = \mathbb{C}(q(\theta))$. Since $\Gamma(a)$ contains at least one $\tilde{a}_i(\theta)$ which is a non-constant trigonometric polynomial, by Theorem 2.2 we have that there exists a nonconstant trigonometric polynomial $r(\theta)$ such that $\Gamma(a) = \mathbb{C}(r(\theta))$, which proves that equation (1.1) satisfies the composition condition. In short, Theorem 1.1 is proved.

APPENDIX: PROOF OF THE RESULTS IN SECTION 2

Proof of Theorem 2.2. We first state and prove two auxiliary results.

Lemma 3.4. *A proper subfield $E \neq \mathbb{C}(\theta)$ of $C(\theta)$ is generated by an element $t \in T \setminus \mathbb{C}[z]$ if and only if it is generated by an element of the form*

$$\frac{af + bz^m}{cf + dz^m} \quad \text{with } a, b, c, d \in \mathbb{C},\ ad - bc \neq 0,$$

and where $f \in \mathbb{C}[z]$ is not divisible by z, and $m \in \mathbb{N}$.

Proof of the lemma. Assume first that $E = \mathbb{C}(t)$, $t = f/z^m$ for some $m \in \mathbb{N}$, $f \in \mathbb{C}[z]$ not divisible by z. Every generator of E has the form

$$\frac{at + b}{ct + d} = \frac{a\frac{f}{z^m} + b}{c\frac{f}{z^m} + d} = \frac{af + bz^m}{cf + dz^m}$$

with $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$. For the converse, let $g := \frac{af+bz^m}{cf+dz^m}$ be a generator of E. Then

$$\frac{dg - b}{a - cg} = \frac{f}{z^m} \in T$$

is a generator of E. This concludes the proof of the lemma. □

Lemma 3.5. *A proper subfield $E \neq \mathbb{C}$ of $\mathbb{C}(\theta)$ is generated by an element $t \in T \setminus \mathbb{C}$ if and only if $T \cap E \neq \mathbb{C}$.*

Proof of the Lemma. Let $t \in T$ be an element of $E \setminus \mathbb{C}$. If $t \in \mathbb{C}[z]$, then [15, Theorem 4, Section 1.2] guarantees that E is generated by a nonconstant polynomial in z. If $t \in T \setminus \mathbb{C}[z]$, then let $t = \frac{f}{z^m} \in E \setminus \mathbb{C}$ for some $m \in \mathbb{N}$ and $f \in \mathbb{C}[z]$ not divisible by z. Let also

$E = \mathbb{C}(\frac{p}{q})$ with coprime polynomials $p, q \in \mathbb{C}[z]$. Then there exist coprime polynomials $G, H \in \mathbb{C}[\frac{p}{q}]$ such that

$$\frac{f}{z^m} = \frac{G(\frac{p}{q})}{H(\frac{p}{q})}.$$

Since both polynomials G and H split into linear factors of the form $\frac{p}{q} - \gamma$ for some $\gamma \in \mathbb{C}$ and $\frac{p}{q} - \gamma = \frac{p-\gamma q}{q}$ one gets

$$\frac{f}{z^m} = q^n \frac{\prod_{i=1}^{r}(p - \alpha_i q)^{c_i}}{\prod_{j=1}^{s}(p - \beta_j q)^{d_j}}, \tag{3.10}$$

for some $n \in \mathbb{Z}$, $r, s, c_i, d_j \in \mathbb{N}$ and where α_i, β_j are pairwise distinct.

Note that by assumption the polynomials $p - \alpha_i q$ and $p - \beta_j q$ have no zeros in common with q and that the two polynomials $p - \alpha q$ and $p - \beta q$ with $\alpha \neq \beta$ have no common zeroes. Hence, in equation (3.10) no linear factors can cancel out and since the denominator of the left-hand-side of equation (3.10) has z as its only linear factor, only the following three cases are possible: either the polynomial H is constant, or $s = 1$ (note that if $m = 0$ then H must be constant).

In case H is constant then $n < 0$ and so $q = z^\ell$ for some $\ell \in \mathbb{N}$ and so $\frac{p}{q} \in T$.

In case $s = 1$ we have that $p - \beta_1 q = z^\ell$ for some $\ell \in \mathbb{N}$ and therefore $\frac{p}{q} = \frac{\beta_1 q + z^\ell}{q}$. It follows from Lemma 3.4 that $\frac{q}{z^\ell} \in T$ is a generator of E. This concludes the proof of the lemma. □

The proof of Theorem 2.2 follows directly from Lemma 3.5 □

Proof of Lemma 2.1. Given an index $i \geq 1$ we consider the expansion in Fourier series of the coefficient $a_i(\theta)$ and we denote by ℓ_i the degree of the trigonometric polynomial $a_i(\theta)$. Then

$$a_i(\theta) = \sum_{n=-\ell_i}^{\ell_i} c_{ni} e^{in\theta},$$

where $c_{ni} \in \mathbb{C}$ for all $i \geq 1$. We have that

$$\tilde{a}_i(\theta) = \int_0^\theta a_i(\tau)\, d\tau = c_{0i}\theta + \sum_{n=-\ell_i, n\neq 0}^{\ell_i} \frac{1}{ni} c_{ni}(e^{in\theta} - 1).$$

Since equation (1.1) has a universal center and $\tilde{a}_i(2\pi) = I_i(a)$ we have that $\tilde{a}_i(2\pi) = 0$ and so $c_{0i} = 0$. Hence $\tilde{a}_i(\theta)$ is a trigonometric polynomial. □

Proof of Lemma 2.3. Take $p_1(\theta)$ a monomial of the form (2.8) and $p_2(\theta) = \tilde{a}_k(\theta)$ for some index $k \geq 1$. Then

$$\int_0^\theta p_1(\tau) p_2'(\tau)\, d\tau = \int_0^\theta \tilde{a}_{i_1}^{m_1}(\tau) \tilde{a}_{i_2}^{m_2}(\tau) \cdots \tilde{a}_{i_k}^{m_k}(\tau) a_k(\tau)\, d\tau.$$

By Lemma 2.2 we have that (2.7) holds and so

$$\int_0^\theta p_1(\tau) p_2'(\tau)\, d\tau = \sum_{j=1}^{J} n_j \int_0^\theta I_{\sigma_j(\vec{i})}(\tau) a_k(\tau)\, d\tau.$$

Moreover, by the relation (2.5) we have

$$\int_0^\theta I_{\sigma_j(\vec{i})}(\tau) a_k(\tau)\, d\tau = I_{\sigma_j(\vec{i})k}(\theta).$$

Since equation (1.1) has a universal center we have that $I_{\sigma_j(\vec{i})k}(2\pi) = 0$ and so

$$\int_0^{2\pi} p_1(\tau)p_2'(\tau)\,d\tau = \sum_{j=1}^{J} n_j \cdot 0 = 0,$$

as we wanted to show. □

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