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Tykhonov triples, well-posedness and convergence results

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ABSTRACT. In this paper we present a unified theory of convergence results in the study of abstract problems. To this end we introduce a new mathematical object, the so-called Tykhonov triple $\mathcal{T} = (I, \Omega, \mathcal{C})$, constructed by using a set of parameters I, a multivalued function Ω and a set of sequences $\mathcal{C}$. Given a problem $\mathcal{P}$ and a Tykhonov triple $\mathcal{T}$, we introduce the notion of well-posedness of problem $\mathcal{P}$ with respect to $\mathcal{T}$ and provide several preliminary results, in the framework of metric spaces. Then we show how these abstract results can be used to obtain various convergences in the study of a nonlinear equation in reflexive Banach spaces.

1. INTRODUCTION

The convergence of various sequences to the solution of a given problem is a fundamental topic in both pure and applied mathematics. It arises in functional analysis, partial differential equations theory, optimization theory and numerical analysis, for instance. Some well-known examples are the following: the convergence of the weak solution of a system of partial differential equations with respect to the data and parameters; the convergence of the solution of a penalty problem to the solution of the original problem when the penalty parameter converges to zero; the convergence of the solution of a regularized problem to the solution of a nonsmooth problem when the regularization parameter converges to zero; the convergence of the solution of a discrete problem to the solution of the continuous problem when the time-step or the spatial discretization parameter converges to zero. Results of this type can be found in [1, 3, 6], for instance.

The mathematical literature dedicated to convergence results in various spaces and under different assumptions is extensive. Such results are obtained by using different methods and functional arguments, including arguments of monotonicity, pseudomonotonicity, compactness, convexity and various estimates. Nevertheless, most of the convergence results in the literature are stated in the following functional framework: given a functional space X and a problem $\mathcal{P}$ which has a unique solution $u \in X$, a family of approximating problems $\{\mathcal{P}_\theta\}$ is constructed such that, if $u_\theta \in X$ is a solution of Problem $\mathcal{P}_\theta$, then u_θ converge to u in X, as θ converges. A careful analysis of this description reveals that, in practice, we need to complete the functional framework above by describing the following three ingredients: a) the set I to which the parameter θ belongs; b) the problem $\mathcal{P}_\theta$ or its sets of solutions, denoted by $\Omega(\theta)$, for each $\theta \in I$; c) the meaning we give to the convergence of the parameter θ.

Collecting these three ingredients we arrive in a natural way to a triple $\mathcal{T} = (I, \Omega, \mathcal{C})$, where $\mathcal{C}$ is a set which governs the convergence of θ. Note that such triples have already been used in [12] to prove the well-posedness of a quasistatic contact problem with elasto-viscoplastic materials. Nevertheless, only particular examples have been considered in [12] and no general results have been provided. In this paper we try to fill this gap. To this end we introduce a new abstract mathematical tool that we refer as a Tykhonov triple,

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together with a new concept of well-posedness. The concept we introduce here represents a generalization of the concept of well-posedness in the sense of Tykhonov, introduced in [13] for a minimization problem and extended in [7, 8] and [2] to variational and hemivariational inequalities, respectively. It can be applied in the study of a large class of problems: minimization problems, operator equations, fixed point problems, differential equations, inclusions, optimal control problems, sweeping processes and various classes of inequalities, as well.

The rest of the paper is structured as follows. In Section 2 we introduce the concepts of Tykhonov triple and well-posedness. We then state and prove some abstract results, in the framework of metric spaces. In Section 3 we use these results to study a nonlinear equation in the framework of reflexive Banach spaces. Finally, in Section 4 we present some concluding remarks.

We end this section by introducing some notation we shall use in the rest of this paper. First, for a nonempty set B we use $\mathcal{S}(B)$ for the set of sequences whose elements belong to B and 2^B for the set of nonempty parts of B. In particular we shall use the notation $\mathcal{S}(B)$ for $B = X$ and $B = I$, and notation 2^B for $B = X$. Moreover, for a nonempty subset A of the metric space (X, d), we denote by $\mathrm{diam}(A)$ the diameter of A.

2. Tykhonov triples and well-posedness

We consider an abstract mathematical object $\mathcal{P}$, called generic "problem", associated to a metric space (X, d). Problem $\mathcal{P}$ could be an equation, a minimization problem, a fixed point problem, an inclusion or an inequality problem. We associate to Problem $\mathcal{P}$ the concept of "solution" which follows from the context. We also denote by $\mathcal{S}_P$ the set of solutions to Problem $\mathcal{P}$. The Problem $\mathcal{P}$ has a unique solution iff S_P has a unique element, i.e., $\mathcal{S}_P$ is a singleton. The concept of well-posedness for Problem $\mathcal{P}$ is related to the so-called Tykhonov triple, defined as follows.

Definition 2.1. A Tykhonov triple is a mathematical object of the form $\mathcal{T} = (I, \Omega, \mathcal{C})$ where I is a given nonempty set, $\Omega : I \to 2^X$ and $\mathcal{C}$ is a nonempty subset of the set $\mathcal{S}(I)$.

We refer to I as the set of parameters; the family of sets $\{\Omega(\theta)\}_{\theta \in I}$ represents the family of approximating sets; moreover, we say that $\mathcal{C}$ defines the criterion of convergence. Next, inspired by our previous paper [11], we consider the following definitions.

Definition 2.2. Given a Tykhonov triple $\mathcal{T} = (I, \Omega, \mathcal{C})$, a sequence $\{u_n\} \in \mathcal{S}(X)$ is called a $\mathcal{T}$-approximating sequence if there exists a sequence $\{\theta_n\} \in \mathcal{C}$ such that $u_n \in \Omega(\theta_n)$ for each $n \in \mathbb{N}$.

Definition 2.3. Given a Tykhonov triple $\mathcal{T} = (I, \Omega, \mathcal{C})$, Problem $\mathcal{P}$ is said to be well-posed if it has a unique solution and every approximating sequence converges in X to this solution.

We remark that approximating sequences always exist since, by assumption, $\mathcal{C} \neq \emptyset$ and, moreover, for any sequence $\{\theta_n\} \in \mathcal{C}$ and any $n \in \mathbb{N}$, the set $\Omega(\theta_n)$ is not empty. In addition, the concept of approximating sequence depends on the Tykhonov triple $\mathcal{T}$ and, for this reason, we use the terminology "$\mathcal{T}$-approximating sequence". As a consequence, the concept of well-posedness depends on the Tykhonov triple $\mathcal{T}$ and, therefore, we refer to it as "well-posedness with respect to $\mathcal{T}$" or "$\mathcal{T}$-well-posedness", for short. Finally, we note that the definition of approximating sequence and well-posedness for Problem $\mathcal{P}$ introduced in [11] can be recovered by Definitions 2.2 and 2.3 in the particular case when $I = (0, +\infty)$ and $\mathcal{C} = \{\,\{\theta_n\} \in \mathcal{S}(I) : \theta_n \to 0\,\}$. We conclude from here that the results below represent a nontrivial extension of our previous results in [11].

In what follows we assume that $\mathcal{S}_P$ is a singleton and we denote by u the solution of the Problem $\mathcal{P}$, that is $\mathcal{S}_P = \{u\}$. Let $\mathcal{T} = (I, \Omega, \mathcal{C})$ be a Tykhonov triple. We denote $\widetilde{\mathcal{S}}_P$ the set of sequences of X which converge to u and by $\widetilde{\mathcal{S}}_T$ the set of $\mathcal{T}$-approximating sequence, that is,

$$\widetilde{\mathcal{S}}_P = \{\, \{u_n\} \in \mathcal{S}(X) : \ u_n \to u \ \text{ in } \ X \,\}, \tag{2.1}$$

$$\widetilde{\mathcal{S}}_T = \{\, \{u_n\} \in \mathcal{S}(X) : \{u_n\} \text{ is a } \mathcal{T}\text{-approximating sequence} \,\}. \tag{2.2}$$

The example below shows that no particular inclusion holds between these sets.

Example 2.1. Let $(X, \|\cdot\|_X)$ be a normed space, $a,\, b \in X$, $a \neq b$, $J(v) = \|v - a\|_X$ for all $v \in X$ and consider the following minimization problem:

$$(\mathcal{P}) \qquad \text{Find } u \in X \text{ such that } \ J(u) \le J(v) \ \ \forall\, v \in X.$$

Moreover, consider two Tykhonov triples $\mathcal{T}_i = (I_i, \Omega_i, \mathcal{C}_i)$, $i = 1,\, 2$, defined by

$$I_1 = [0, +\infty),\ \Omega_1(\theta) = \{\, \widetilde{u} \in X \ : \ \|\widetilde{u} - a\|_X \ge \theta \,\}\ \forall\, \theta \ge 0,\ \mathcal{C}_1 = \{\, \{\theta_n\} \in \mathcal{S}(I_1) \ : \ \theta_n \to 1 \,\},$$

$$I_2 = [0, +\infty),\ \Omega_2(\theta) = \{\, \widetilde{u} \in X \ : \ \|\widetilde{u} - b\|_X \le \theta \,\}\ \forall\, \theta \ge 0,\ \mathcal{C}_2 = \{\, \{\theta_n\} \in \mathcal{S}(I_2) : \theta_n \to 0 \,\}.$$

Let $\{u_n^1\} \subset X$ be the sequence defined by $u_n^1 = a + \frac{1}{n}\, b$ for all $n \in \mathbb{N}$. Since $u_n^1 \to a$ in X and a is the unique solution to Problem $\mathcal{P}$, it follows that $\{u_n^1\} \in \widetilde{\mathcal{S}}_P$. Nevertheless, $\{u_n^1\} \notin \widetilde{\mathcal{S}}_{T_1}$ and, therefore, $\widetilde{\mathcal{S}}_P \not\subset \widetilde{\mathcal{S}}_{T_1}$. Let $\{u_n^2\} \subset X$ be the sequence defined by $u_n^2 = b$ for all $n \in \mathbb{N}$. It follows that $\{u_n^2\}$ is a $\mathcal{T}_2$-approximating sequence which does not converge to the solution of Problem $\mathcal{P}$. Therefore, $\widetilde{\mathcal{S}}_{T_2} \not\subset \widetilde{\mathcal{S}}_P$. □

Next, we use Definition 2.3 and equalities (2.1), (2.2) to see that

$$\text{Problem } \mathcal{P} \text{ is well-posed with } \mathcal{T} \text{ if and only if } \widetilde{\mathcal{S}}_T \subset \widetilde{\mathcal{S}}_P. \tag{2.3}$$

Moreover, notation (2.2) suggests us to introduce the following definition.

Definition 2.4. Given two Tykhonov triples $\mathcal{T}_1 = (I_1, \Omega_1, \mathcal{C}_1)$ and $\mathcal{T}_2 = (I_2, \Omega_2, \mathcal{C}_2)$, we say that $\mathcal{T}_1$ and $\mathcal{T}_2$ are equivalent if their sets of approximating sequences are the same, i.e., $\widetilde{\mathcal{S}}_{T_1} = \widetilde{\mathcal{S}}_{T_2}$. In this case we write $\mathcal{T}_1 \approx \mathcal{T}_2$.

It is easy to see that "$\approx$" represents an equivalence relation on the set of Tykhonov triples. Moreover, using (2.3) we deduce that the following statement holds.

$$\left\{ \begin{array}{l} \text{If } \mathcal{T}_1 \approx \mathcal{T}_2 \text{ then Problem } \mathcal{P} \text{ is well-posed with } \mathcal{T}_1 \\ \text{if and only if it is well-posed with } \mathcal{T}_2. \end{array} \right. \tag{2.4}$$

In what follows we provide necessary and sufficient conditions which guarantee the well-posedness of Problem $\mathcal{P}$ with a given Tykhonov triple $\mathcal{T}$.

Theorem 2.1. *Let $\mathcal{T} = (I, \Omega, \mathcal{C})$ be a Tykhonov triple and consider the following statements.*

(i) Problem $\mathcal{P}$ is well-posed with $\mathcal{T}$.
(ii) $\mathrm{diam}(\Omega(\theta_n)) \to 0$ for any sequence $\{\theta_n\} \in \mathcal{C}$.
(iii) $\mathcal{S}_P \neq \emptyset$ and, for any $\{\theta_n\} \in \mathcal{C}$ and $n \in \mathbb{N}$, the inclusion $\mathcal{S}_P \subset \Omega(\theta_n)$ holds.

Then, the following two implications hold: $\Big(\text{(i)} \implies \text{(ii)}\Big)$ *and* $\Big(\text{(iii)}, \text{(ii)} \implies \text{(i)}\Big)$.

Proof. Assume that (i) holds, i.e., Problem $\mathcal{P}$ is well-posed with $\mathcal{T}$. This implies that $\mathcal{S}_P$ is a singleton. Arguing by contradiction, assume in what follows that there exists a sequence $\{\theta_n\} \in \mathcal{C}$ such that $\mathrm{diam}\,(\Omega(\theta_n)) \not\to 0$. Then, there exist $\delta_0 \ge 0$ and two sequences $\{u_n\}$, $\{v_n\} \subset \Omega(\theta_n)$ such that

$$d(u_n, v_n) \ge \frac{\delta_0}{2} \qquad \forall\, n \in \mathbb{N}. \tag{2.5}$$

Now, since both $\{u_n\}$ and $\{v_n\}$ are $\mathcal{T}$-approximating sequences for Problem $\mathcal{P}$, the well-posedness of $\mathcal{P}$ implies that $u_n \to u$ and $v_n \to u$ in X where u denotes the unique element of $\mathcal{S}_P$. This is in contradiction with (2.5). We conclude from here that condition (ii) holds.

Assume now that (iii) and (ii) hold. We claim that the set $\mathcal{S}_P$ is a singleton. Let $\{\theta_n\} \in \mathcal{C}$ and assume that $u,\ u' \in \mathcal{S}_P$. Then using the inclusion $\mathcal{S}_P \subset \Omega(\theta_n)$ we deduce that $u,\ u' \in \Omega(\theta_n)$ for any $n \in \mathbb{N}$. Therefore condition (ii) shows that $d(u,u') \le \operatorname{diam}(\Omega(\theta_n)) \to 0$, which implies that $u = u'$ and proves the claim. We conclude from here that Problem $\mathcal{P}$ has a unique solution, denoted in what follows by u.

Let now $\{u_n\} \subset X$ be a $\mathcal{T}$-approximating sequence for Problem $\mathcal{P}$. Then there exists a sequence $\{\theta_n\} \in \mathcal{C}$ such that $u_n \in \Omega(\theta_n)$ for each $n \in \mathbb{N}$. We use the inclusion $\mathcal{S}_P \subset \Omega(\theta_n)$ to see that $u \in \Omega(\theta_n)$ for each $n \in \mathbb{N}$ and, therefore, (ii) yields $d(u,u_n) \le \operatorname{diam}(\Omega(\theta_n)) \to 0$. This implies that $u_n \to u$ in X which shows that Problem $\mathcal{P}$ is well-posed with $\mathcal{T}$ and concludes the proof. □

It follows from Theorem 2.1 that condition (ii) represents a necessary condition for the $\mathcal{T}$-well-posedness of Problem $\mathcal{P}$. It is a sufficient condition if, moreover, condition (iii) holds. Nevertheless, Problem $\mathcal{P}$ could be well-posed with a Tykhonov triple $\mathcal{T}$ even if the condition (iii) is not satisfied, as it results from the example below.

Example 2.2. Let $(X, \|\cdot\|_X)$ be a normed space, $a \in X$, and consider the Problem $\mathcal{P}$ in Example 2.1. Moreover, consider the Tykhonov triple $\mathcal{T} = (I, \Omega, \mathcal{C})$ defined by

$$I = (0,+\infty),\ \Omega(\theta) = \{\, \widetilde{u} \in X \,:\, \theta < \|\widetilde{u} - a\|_X \le 2\theta \,\}\ \forall\, \theta > 0,\ \mathcal{C} = \{\, \{\theta_n\} \in \mathcal{S}(I) : \theta_n \to 0 \,\}.$$

Then it is easy to see that Problem $\mathcal{P}$ is well-posed with $\mathcal{T}$. Nevertheless, since $\mathcal{S}_P = \{a\}$ it follows that condition (iii) does not hold. □

3. A NONLINEAR EQUATION

Everywhere in this section X represents a reflexive Banach space endowed with the norm $\|\cdot\|_X$ and $\langle\cdot,\cdot\rangle$ denotes the duality pairing between X and its dual X^*. In addition, we use 0_X and 0_{X^*} for the zero element of X and X^*, respectively, and " $\to$ " and " $\rightharpoonup$ " for the convergence and the weak convergence in X. All the limits, upper limits and lower limits below are considered as $n \to \infty$, even if we do not mention it explicitly. The Problem $\mathcal{P}$ we study in this section can be formulated as follows.

Problem $\mathcal{P}$. *Find $u \in X$ such that $Au = f$.*

Here $f \in X^*$ and $A : X \to X^*$ is an operator assumed to satisfy the following conditions.

$$(3.6)\qquad \begin{cases} \text{(a) } A \text{ is strongly monotone, i.e., there exists } m_A > 0 \text{ such that} \\ \qquad \langle Av_1 - Av_2, v_1 - v_2\rangle \ge m_A \|v_1 - v_2\|_X^2 \quad \forall\, v_1, v_2 \in X. \\ \text{(b) } A \text{ is pseudomonotone, i.e., it is bounded and} \\ \qquad u_n \rightharpoonup u \text{ in } X \text{ with } \limsup \langle Au_n, u_n - u\rangle \le 0 \\ \qquad \text{implies } \liminf \langle Au_n, u_n - v\rangle \ge \langle Au, u - v\rangle \quad \forall\, v \in X. \end{cases}$$

The unique solvability of Problem $\mathcal{P}$ follows from the following well-known result.

Theorem 3.2. Assume that (3.6) holds. Then, for each $f \in X^*$, there exists a unique element $u \in X$ such that $Au = f$.

In the study of this problem we consider the Tykhonov triples $\mathcal{T}_1 = (I_1, \Omega_1, \mathcal{C}_1)$, $\mathcal{T}_2 = (I_2, \Omega_2, \mathcal{C}_2)$ defined as follows:

$$I_1 = [0,+\infty),$$
$$\Omega_1(\theta) = \{\, u \in X \,:\, \|Au - f\|_{X^*} \le \theta \,\} \quad \forall\, \theta \in I_1,$$
$$\mathcal{C}_1 = \{\, \{\theta_n\} \in \mathcal{S}(I_1) \,:\, \theta_n \to 0 \,\},$$

$I_2 = \{\, \boldsymbol{\theta} = (f_\theta, \varepsilon_\theta) \;:\; f_\theta \in X^*, \varepsilon_\theta \ge 0 \,\}$,
$\Omega_2(\boldsymbol{\theta}) = \{\, u \in X : \|Au - f_\theta\|_{X^*} \le \varepsilon_\theta \} \quad \forall\, \boldsymbol{\theta} = (f_\theta, \varepsilon_\theta) \in I_2$,
$\mathcal{C}_2 = \{\, \{\boldsymbol{\theta}_n\} \;:\; \boldsymbol{\theta}_n = \{f_n, \varepsilon_n\} \in I_2 \;\; \forall\, n \in \mathbb{N},\; f_n \to f \text{ in } X^*,\; \varepsilon_n \to 0 \,\}$.

Denote by u the solution of equation $Au = f$ obtained in Theorem 3.2. Then $u \in \Omega_1(\theta)$ for each $\theta \in I_1$ which proves that $\Omega_1(\theta) \ne \emptyset$. Similar arguments show that $\Omega_2(\boldsymbol{\theta}) \ne \emptyset$, for each $\boldsymbol{\theta} \in I_2$.

Our first result concerning the well-posedness of Problem $\mathcal{P}$ is the following.

Theorem 3.3. *Assume* (3.6) *and let $f \in X^*$. Then, the following statements hold.*
a) *The Tykhonov triples $\mathcal{T}_1$ and $\mathcal{T}_2$ are equivalent.*
b) *Problem $\mathcal{P}$ is well-posed with both Tykhonov triples $\mathcal{T}_1$ and $\mathcal{T}_2$.*
c) $\mathrm{diam}(\Omega_1(\theta_n)) \to 0$ for any sequence $\{\theta_n\} \in \mathcal{C}_1$.
d) $\mathrm{diam}(\Omega_2(\boldsymbol{\theta}_n)) \to 0$ for any sequence $\{\boldsymbol{\theta}_n\} \in \mathcal{C}_2$.

Proof. a) Let $\{u_n\}$ be a $\mathcal{T}_1$-approximating sequence for Problem $\mathcal{P}$. Then there exists a sequence $\{\theta_n\}$ such that $0 \le \theta_n \to 0$ and $\|Au_n - f\|_{X^*} \le \theta_n$ for each $n \in \mathbb{N}$. This shows that $u_n \in \Omega_2(\boldsymbol{\theta}_n)$ with $\boldsymbol{\theta}_n = (f, \theta_n)$ for each $n \in \mathbb{N}$. It follows from here that $\{u_n\}$ is a $\mathcal{T}_2$-approximating sequence, too. Conversely, assume now that $\{u_n\}$ is a $\mathcal{T}_2$-approximating sequence. Then there exists a sequence $\{\boldsymbol{\theta}_n\}$ such that $\boldsymbol{\theta}_n = (f_n, \varepsilon_n) \in I_2$, $\|Au_n - f_n\|_{X^*} \le \varepsilon_n$ for all $n \in \mathbb{N}$, and, moreover, $f_n \to f$ in X^*, $\varepsilon_n \to 0$. Let $n \in \mathbb{N}$ be fixed. We have

$$\|Au_n - f\|_{X^*} \le \|Au_n - f_n\|_{X^*} + \|f_n - f\|_{X^*} \le \varepsilon_n + \|f_n - f\|_{X^*}$$

This show that $u_n \in \Omega_1(\theta_n)$ with $\theta_n = \varepsilon_n + \|f_n - f\|_{X^*} \ge 0$. It follows from here that $\{u_n\}$ is a $\mathcal{T}_1$-approximating sequence. To conclude, we proved that $\widetilde{\mathcal{S}}_{T_1} = \widetilde{\mathcal{S}}_{T_2}$ and, using Definition 2.4, we deduce that $\mathcal{T}_1 \approx \mathcal{T}_2$.

b) Let $\{u_n\}$ be a $\mathcal{T}_1$-approximating sequence for Problem $\mathcal{P}$. Then there exists a sequence $\{\theta_n\}$ such that $0 \le \theta_n \to 0$ and $\|Au_n - f\|_{X^*} \le \theta_n$ for each $n \in \mathbb{N}$. Using assumption (3.6)(a) and equality $Au = f$ we deduce that

$$\begin{aligned} m_A \|u_n - u\|_X^2 &\le \langle Au_n - Au, u_n - u\rangle = \langle Au_n - f, u_n - u\rangle \\ &\le \|Au_n - f\|_{X^*} \|u_n - u\|_X \le \theta_n \|u_n - u\|_X \end{aligned}$$

and, therefore,

$$\|u_n - u\|_X \le \frac{\theta_n}{m_A}. \tag{3.7}$$

The well-posedness of Problem $\mathcal{P}$ with respect to $\mathcal{T}_1$ is a direct consequence of inequality (3.7) and the convergence $\theta_n \to 0$. The well-posedness of Problem $\mathcal{P}$ with respect to $\mathcal{T}_2$ is now a direct consequence of the equivalence (2.4).

c), d). The well-posedness of Problem $\mathcal{P}$ with respect to $\mathcal{T}_1$ and $\mathcal{T}_2$, guaranteed by part b) of the current theorem, allows us to use the implication (i) $\Longrightarrow$ (ii) in Theorem 2.1. In this way we deduce that c) and d) hold, which concludes the proof. □

We remark that Theorem 3.3 provides an example of problem which is well-posed with two equivalent Tykhonov triples $\mathcal{T}_1$ and $\mathcal{T}_2$. Note that, in contrast, the history-dependent problem considered in [12] is well-posed with two Tykhonov triples $\mathcal{T}_1$ and $\mathcal{T}_2$ which fail to be equivalent.

We now proceed with the following elementary result which can be interpreted as a consequence of Theorem 3.3.

Corollary 3.1. *Assume* (3.6). *Then, the solution u of Problem $\mathcal{P}$ depends continuously on $f \in X^*$, i.e., if $f_n \in X^*$, $Au_n = f_n$ for all $n \in \mathbb{N}$ and $f_n \to f$ in X^*, then $u_n \to u$ in X.*

Proof. Let $n \in \mathbb{N}$. We use assumption (3.6)(a) and equalities $Au_n = f_n$, $Au = f$ to see that

$$\|u_n - u\|_X \le \frac{1}{m_A}\,\|f_n - f\|_{X^*}. \tag{3.8}$$

It is obvious to see that the convergence $u_n \to u$ in X follows from (3.8), since $f_n \to f$ in X^*. Nevertheless, it can be recovered by Theorem 3.3 in three steps, as follows: a) Obviously, $u_n \in \Omega_1(\theta_n)$ with $\theta_n = \|f_n - f\|_{X^*}$; b) $\{u_n\}$ is a $\mathcal{T}_1$-approximating sequence for Problem $\mathcal{P}$ since $f_n \to f$ in X^* and, therefore, $\theta_n \to 0$; c) Theorem 3.3 b) and Definition 2.2 imply that $u_n \to u$ in X. □

Corollary 3.1 provides a sequence of elements $\{u_n\}$ which converges in X to the solution of Problem $\mathcal{P}$. Nevertheless, additional convergence results to this solution are known in the literature, related either to the internal approximation of the space V (see [1] for details) or the well-posedness concept defined in [4, 10, 14]. Such convergence results can be recovered in a unified way by considered the well-posedness of Problem $\mathcal{P}$ with respect to a Tykhonov triple which contains relevant approximating sequences. Our aim in what follows is to construct such a triple. To this end, we denote by $\mathcal{X}$ the family of subspaces of X and, for a sequence $\{X_n\} \in \mathcal{S}(\mathcal{X})$, we use the following convergence:

$$\begin{cases} X_n \xrightarrow{M} X \ \text{ as } \ n \to \infty \ \text{ if for each } v \in X, \text{there exists a sequence} \\ \{v_n\} \subset X \text{ such that } v_n \in X_n \text{ for each } n \in \mathbb{N} \text{ and } v_n \to v \text{ in } X. \end{cases} \tag{3.9}$$

Note that (3.9) represents the convergence of $\{X_n\} \in \mathcal{S}(\mathcal{X})$ to the space X in the sense of Mosco [9]. In addition, this convergence shows that the $\{X_n\}$ represents an internal approximation of the space X. In practice the spaces X_n are finite-dimensional spaces constructed by using the finite element method.

Consider now the Tykhonov triple $\mathcal{T} = (I, \Omega, \mathcal{C})$ defined as follows.

$$I = \{\,\boldsymbol{\theta} = (X_\theta, f_\theta, \varepsilon_\theta) \ :\ X_\theta \in \mathcal{X},\ f_\theta \in X^*,\ \varepsilon_\theta \ge 0\,\}, \tag{3.10}$$

$$\Omega(\boldsymbol{\theta}) = \{\,u \in X_\theta : \langle Au, v\rangle + \varepsilon_\theta \|v\|_X \ge \langle f_\theta, v\rangle \quad \forall\, v \in X_\theta\}, \quad \forall\,\boldsymbol{\theta} = (X_\theta, f_\theta, \varepsilon_\theta) \in I, \tag{3.11}$$

$$\mathcal{C} = \{\,\{\boldsymbol{\theta}_n\}\ :\ \boldsymbol{\theta}_n = (X_n, f_n, \varepsilon_n) \in I \ \ \forall\, n \in \mathbb{N},\quad X_n \xrightarrow{M} X, \quad f_n \to f \quad \text{in } X^*, \quad \varepsilon_n \to 0\,\}. \tag{3.12}$$

Note that the solvability of the equation $Au = f_\theta$ for each $f_\theta \in X^*$, guaranteed by assumption (3.6), shows that $\Omega(\boldsymbol{\theta}) \neq \emptyset$, for each $\boldsymbol{\theta} \in I$.

We have the following result.

Theorem 3.4. *Assume* (3.6) *and let* $f \in X^*$. *Then, Problem* $\mathcal{P}$ *is well-posed with the Tykhonov triple* (3.10)–(3.12).

Proof. Let $\{\boldsymbol{u}_n\}$ be a $\mathcal{T}$-approximating sequence for Problem $\mathcal{P}$. Then by definition, there exists a sequence $\{\boldsymbol{\theta}_n\}$ such that $\boldsymbol{\theta}_n = (X_n, f_n, \varepsilon_n)$,

$$u_n \in X_n, \qquad \langle Au_n, v\rangle + \varepsilon_n \|v\|_X \ge \langle f_n, v\rangle \qquad \forall\, v \in X_n, \tag{3.13}$$

for all $n \in \mathbb{N}$ and, moreover the convergences in (3.12) hold. We take $v = -u_n$ in (3.13) and use the strong monotonicity of the operator A, (3.6)(a), to deduce that

$$\|u_n\|_X \le \frac{1}{m_A}\big(\|f_n\|_{X^*} + \|A0_X\|_{X^*} + \varepsilon_n\big) \qquad \forall\, n \in \mathbb{N}. \tag{3.14}$$

We combine estimate (3.14) and the convergences in (3.12) to see that the sequence $\{u_n\}$ is bounded in X. This implies that there exists an element $\widetilde{u} \in X$ and a subsequence of $\{u_n\}$, again denoted by $\{u_n\}$, such that

$$u_n \rightharpoonup \widetilde{u} \quad \text{in} \quad X. \tag{3.15}$$

Let $v \in X$ and, using (3.12), consider $\{v_n\} \subset X$ such that $v_n \in X_n$ for each $n \in \mathbb{N}$ and

$$v_n \to v \quad \text{in} \quad X, \tag{3.16}$$

We use (3.13) to see that $\langle Au_n, u_n - v_n\rangle \le \langle f_n, u_n - v_n\rangle + \varepsilon_n\|v_n - u_n\|_X$ and, keeping in mind the convergences (3.12), (3.15), (3.16), we deduce that

$$\limsup\, \langle Au_n, u_n - v_n\rangle \le \langle f, \widetilde{u} - v\rangle. \tag{3.17}$$

On the other hand, we write

$$\langle Au_n, u_n - v\rangle = \langle Au_n, u_n - v_n\rangle + \langle Au_n, v_n - v\rangle$$

and, since the operator A is bounded, the covergences (3.15) and (3.16) imply that

$$\limsup\, \langle Au_n, u_n - v\rangle = \limsup\, \langle Au_n, u_n - v_n\rangle. \tag{3.18}$$

We now use relations (3.17) and (3.18) to deduce that

$$\limsup\, \langle Au_n, u_n - v\rangle \le \langle f, \widetilde{u} - v\rangle \qquad \forall\, v \in X. \tag{3.19}$$

Next, we take $v = \widetilde{u}$ in (3.19) to find that $\limsup\, \langle Au_n, u_n - \widetilde{u}\rangle \le 0$ and, combining this inequality with the convergence (3.15) and assumption (3.6)(b), we find that

$$\liminf\, \langle Au_n, u_n - v\rangle \ge \langle A\widetilde{u}, \widetilde{u} - v\rangle \qquad \forall\, v \in X. \tag{3.20}$$

We now use inequalities (3.19) and (3.20) to see that $\langle A\widetilde{u}, \widetilde{u} - v\rangle \le \langle f, \widetilde{u} - v\rangle$ for all $v \in X$ which shows that $A\widetilde{u} = f$. Therefore, the unique solvability of Problem $\mathcal{P}$ implies that $\widetilde{u} = u$.

Now, a careful examination of proof above reveals that any weakly convergent subsequence of the sequence $\{u_n\}$ converges weakly in X to u, as $n \to \infty$. Moreover, recall that the sequence $\{u_n\}$ is bounded. Therefore, a standard argument shows that whole sequence $\{u_n\}$ converges weakly in X to u. Next, we use (3.19), (3.20) with $v = u$ and $\widetilde{u} = u$ to deduce that $0 \le \liminf\, \langle Au_n, u_n - u\rangle \le \limsup\, \langle Au_n, u_n - u\rangle \le 0$, which implies that

$$\langle Au_n, u_n - u\rangle \to 0. \tag{3.21}$$

Finally, we use condition (3.6)(a) to see that $m_A\|u_n - u\|_X^2 \le \langle Au_n, u_n - u\rangle - \langle Au, u_n - u\rangle$. Therefore, (3.15), (3.21) and equality $\widetilde{u} = u$ show that $u_n \to u$ in X which concludes the proof. □

Some direct consequences of Theorem 3.4 is provided by the following result.

Corollary 3.2. *Assume* (3.6) *and denote by u the solution to Problem $\mathcal{P}$ for $f \in X^*$. The following statements hold.*

a) *If $X_n \in \mathcal{X}$ and u_n represents the solution of the variational equation*

$$u_n \in X_n, \qquad \langle Au_n, v\rangle = \langle f, v\rangle \qquad \forall\, v \in X_n, \tag{3.22}$$

for all $n \in \mathbb{N}$, then $X_n \xrightarrow{M} X$ implies that $u_n \to u$ in X.

b) *If $f_n \in X^*$ and u_n represents the solution of the equation*

$$u_n \in X, \qquad Au_n = f_n, \tag{3.23}$$

for all $n \in \mathbb{N}$, then $f_n \to f$ in X^* implies that $u_n \to u$ in X.

c) *If* $\varepsilon_n \geq 0$ *and* u_n *is a solution of the variational inequality*

$$u_n \in X, \qquad \langle Au_n, v\rangle + \varepsilon_n\|v\|_X \geq \langle f, v\rangle \qquad \forall\, v \in X, \tag{3.24}$$

for all $n \in \mathbb{N}$*, then* $\varepsilon_n \to 0$ *implies that* $u_n \to u$ *in* X.

Proof. The convergences in Corollary 3.2 follow by using the well-posedness of Problem $\mathcal{P}$ with respect to the Tykhonov triple (3.10)–(3.12) with an appropriate choice of approximating sequences. The details are presented below.

a) Assume that $X_n \xrightarrow{M} X$. Then, it follows that the sequence $\{\boldsymbol{\theta}_n\}$ defined by $\boldsymbol{\theta}_n = (X_n, f, 0)$ belongs to $\mathcal{C}$. Moreover, using (3.11) and (3.22) we find that $u_n \in \Omega(\boldsymbol{\theta}_n)$ for all $n \in \mathbb{N}$. This shows that $\{u_n\}$ is an approximating sequence with the Tykhonov triple (3.10)–(3.12) and, therefore, Theorem 3.4 implies that $u_n \to u$ in X.

b), c) We use the same argument as above by choosing the sequence $\{\boldsymbol{\theta}_n\}$ defined by $\boldsymbol{\theta}_n = (X, f_n, 0)$ and $\boldsymbol{\theta}_n = (X, f, \varepsilon_n)$ for all $n \in \mathbb{N}$, respectively. □

We proceed with the following comments. First, Corollary 3.2 a) shows the convergence of the solution of the discrete scheme (3.22) to the solution of Problem $\mathcal{P}$, provided that $\{X_n\}$ represents an internal approximation of the space X. Such kind of convergence results are important in the numerical analysis of Problem $\mathcal{P}$. Next, the convergence in Corollary 3.2 b) was already obtained in Corollary 3.1, in a different way. Finally, the convergence in Corollary 3.2 c) is related to a well-posed result obtained in [10] where approximating sequences $\{u_n\}$ defined by using inequalities of the form (3.24) have been considered. Obtaining all these convergence results as a consequence of a unique general result, Theorem 3.4, shows, once more, the importance of the concept of Tykhonov triple.

4. Conclusions

In this paper we introduced the concept of Tykhonov triple and then used it to study the well-posedness of abstract problems in metric spaces. Exploiting the general properties of Tykhonov triples we obtained convergence results for a nonlinear equation in reflexive Banach spaces. These results have been obtained in the following way: first, we chosed a convenient Tykhonov triple $\mathcal{T}$ and proved that the corresponding problem is well-posed with respect to this triple. This implies that all the $\mathcal{T}$-approximating sequences converge to the unique solution of the problem. Then, we selected relevant particular sequences, among the approximating sequences associated to $\mathcal{T}$. The convergence of each particular sequence provided a convergence result for the problem. The general method we presented in this paper can be used in the study of variational and hemivariational inequalities, mixed problems, optimal control problems, fixed point problems and inclusions, among others. Its use will be illustrated in some forthcoming papers. The tools developed in this paper have important applications in Physics, Mechanics and Engineering since, as illustrated in [12], they allow us to establish the link between mathematical models used to describe various physical processes.

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