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A fresh look at Cauchy's Convergence Criterion: Some variations and generalizations

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ABSTRACT. We present several variations, generalizations and extensions of Cauchy's Convergence Criterion for real sequences, including some unusual 2-dimensional versions.

1. INTRODUCTION

A sequence of real numbers whose elements become arbitrarily close to each other is called a *Cauchy sequence*, after Augustin-Louis Cauchy, who first introduced the idea in his textbook *Cours d'Analyse* published in 1821. More precisely, a sequence of real numbers $\{a_n\}$ is called a Cauchy sequence, if for any number $\varepsilon > 0$ there exists a positive integer N such that if $m, n \geq N$, then $|a_m - a_n| < \varepsilon$. Let us recall Cauchy's well-known theorem (see [8], for example), which is equivalent to the fact that the set $\mathbb{R}$ of all real numbers, equipped with the usual (Euclidean) metric $d(x, y) = |x - y|$, is a complete metric space:

Theorem. (Cauchy's Convergence Criterion for Real Sequences) *A sequence of real numbers is convergent if and only if it is a Cauchy sequence.*
There are numerous publications in the literature proposing extensions and alternate versions of Cauchy's convergence criterion (see for example [2], [6], [7]). In this paper we present several less known variations, generalizations, and extensions of this important theorem, and also prove some new ones, including our main result, Theorem 4.10.

2. THREE VARIATIONS OF CAUCHY'S CRITERION

Throughout this paper we assume that k, m, n, i, j, N and M denote positive integers. Below we will state three different versions of Cauchy's Criterion which might be easier to use in practice.

Theorem 2.1. (Cauchy's Criterion - Reformulation I) *Let $\{a_n\}$ be a sequence of real numbers. Then $\{a_n\}$ is convergent if and only if for any $\varepsilon > 0$ there exists a positive integer N such that*

$$\text{if } m \geq N \text{ then } |a_m - a_N| < \varepsilon. \tag{2.1}$$

This says that, in fact, one of the numbers in the definition of a Cauchy sequence can be fixed.

Proof. First assume $a_n \to a$, and let $\varepsilon > 0$. Then there exists a positive integer N such that $|a_m - a| < \varepsilon/2$ if $m \geq N$. Thus if $m \geq N$, $|a_m - a_N| \leq |a_m - a| + |a_N - a| < \varepsilon$.
To prove the converse, suppose that for every $\varepsilon > 0$ there exists an N such that (2.1) is satisfied. Let $\varepsilon > 0$. Then, we can find a positive integer N, such that if $m \geq N$ then

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$|a_m - a_N| < \varepsilon/2$. If $m, n \geq N$ we have $|a_m - a_n| \leq |a_m - a_N| + |a_n - a_N| < \varepsilon$, thus $\{a_n\}$ is Cauchy, so convergent. □

Example 2.1. (see [3]) Consider the sequence

$$a_n = \sum_{k=1}^{n} \frac{\sin kx}{2^k},\ n \geq 1,\ \text{where } x \in \mathbb{R}.$$

We will use Theorem 2.1 to show that this sequence is convergent for any real number x. For any $m > n$,

$$|a_m - a_n| = \left| \frac{\sin(n+1)x}{2^{n+1}} + \frac{\sin(n+2)x}{2^{n+2}} + \cdots + \frac{\sin mx}{2^m} \right| \leq$$
$$\leq \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} + \cdots + \frac{1}{2^m} = \frac{1}{2^{n+1}} \cdot \frac{1 - \frac{1}{2^{m-n}}}{1 - \frac{1}{2}} =$$
$$= \frac{1}{2^n}\left(1 - \frac{1}{2^{m-n}}\right) < \frac{1}{2^n}.$$

For any $\varepsilon > 0$, choose N such that $\frac{1}{2^N} < \varepsilon$. Then, for $m \geq N$, $|a_m - a_N| < \frac{1}{2^N} < \varepsilon$, so the sequence $\{a_n\}$ is convergent by Theorem 2.1.

Theorem 2.2. (Cauchy's Criterion - Reformulation II) *Let $\{a_n\}$ be a sequence of real numbers. Then $\{a_n\}$ is convergent if and only if for any $\varepsilon > 0$ there exist positive integers N and M such that*

$$\text{(2.2)} \qquad \text{if } m \geq M \text{ then } |a_m - a_N| < \varepsilon.$$

Proof. We will show that the above condition is equivalent to condition (2.1) from the reformulation of Cauchy's Criterion given in Theorem 2.1. Let $\varepsilon > 0$. If (2.2) holds, then there exist positive integers N and M such that if $m \geq M$ then $|a_m - a_N| < \varepsilon/2$, and in particular $|a_M - a_N| < \varepsilon/2$. Then if $m \geq M$ we have $|a_m - a_M| \leq |a_m - a_N| + |a_M - a_N| < \varepsilon$, so (2.1) holds. Conversely, if (2.1) holds, there exists an integer N such that if $m \geq N$, then $|a_m - a_N| < \varepsilon$. Thus (2.2) holds by choosing $M = N$. □

We also prove a more general result involving several terms of the sequence.

Theorem 2.3. (A Generalized Cauchy Criterion) *Let $\{a_n\}$ be a sequence of real numbers, and let $x_1, \ldots, x_k$ be non-zero real numbers. Then $\{a_n\}$ is convergent if and only if there exists a real number L with the property that for every $\varepsilon > 0$ there exists a positive integer N such that*

$$\text{(2.3)} \qquad \text{if } n_1, n_2, \ldots, n_k \geq N \text{ then } |x_1 a_{n_1} + \cdots + x_k a_{n_k} - L| < \varepsilon.$$

Furthermore, if $x_1 + \cdots + x_k = 0$ then $L = 0$, and if $x_1 + \cdots + x_k \neq 0$ then $\{a_n\}$ must converge to $L/(x_1 + \cdots + x_k)$.

Proof. First assume that $a_n \to a$. We can find positive integers N_i such that $|a_{n_i} - a| < \frac{\varepsilon}{k|x_i|}$ for $n_i \geq N_i$, $i = 1, 2, \ldots, k$. Let $N = \max\{N_1, N_2, \ldots, N_k\}$ and let $L := a(x_1 + \cdots + x_k)$. Then if $n_1, \ldots, n_k \geq N$ we have

$$|x_1 a_{n_1} + \cdots + x_k a_{n_k} - L| \leq \sum_{i=1}^{k} |x_i||a_{n_i} - a| < \sum_{i=1}^{k} |x_i| \frac{\varepsilon}{k|x_i|} = \varepsilon.$$

Conversely, let us assume that there exists a real number L with the property that for any $\varepsilon > 0$, we can find a positive integer N such that condition (2.3) is satisfied. Let $\varepsilon > 0$. Then there exists a positive integer N such that if $m \geq N$, then $|x_1 a_N + x_2 a_N \cdots + x_{k-1} a_N +$

$x_k a_m - L| < \frac{\varepsilon|x_k|}{2}$. Similarly, if $n \geq N$ then $|x_1 a_N + x_2 a_N + \cdots + x_{k-1} a_N + x_k a_n - L| < \frac{\varepsilon|x_k|}{2}$. If $m, n \geq N$ we have

$$|a_m - a_n| = \frac{1}{|x_k|}|x_k a_m - x_k a_n| = \frac{1}{|x_k|}|(x_1 a_N + x_2 a_N + \cdots + x_{k-1} a_N + x_k a_m - L) -$$

$$-(x_1 a_N + x_2 a_N + \cdots + x_{k-1} a_N + x_k a_n - L)| < \frac{1}{|x_k|}\left(\frac{\varepsilon|x_k|}{2} + \frac{\varepsilon|x_k|}{2}\right) = \varepsilon,$$

which shows that $\{a_n\}$ is a Cauchy sequence, so convergent.
To prove the additional statements in the theorem, let $n_1, n_2, \ldots, n_k \to \infty$ in equation (2.3). Assuming $a_n \to a$, we obtain $|x_1 a + \cdots + x_k a - L| = 0$, and so $L = a(x_1 + \cdots + x_k)$. Both statements follow. □

3. Related convergence criteria

We note that if x and y are real numbers such that $|x - y|$ is small compared to $|x|$, then the ratio x/y is close to 1 thus the following result should not come as a surprise.

Theorem 3.4. (A Ratio Criterion) *Let δ be strictly positive and let $\{a_n\}$ be a sequence such that $|a_n| \geq \delta$ for all n. Then $\{a_n\}$ is convergent if and only if for any $\varepsilon > 0$ there exists a positive integer N such that*

(3.4) $$\text{if } m \geq N \text{ then } |a_m/a_N - 1| < \varepsilon.$$

Proof. First assume that $\{a_n\}$ is convergent. By Theorem 2.1, for any $\varepsilon > 0$ there exists a positive integer N such that if $m \geq N$ we have $|a_m - a_N| < \delta\varepsilon$. Then for $m \geq N$

$$|a_m/a_N - 1| = \frac{|a_m - a_N|}{|a_N|} = \frac{1}{|a_N|}|a_m - a_N| < \frac{1}{\delta}\,\delta\varepsilon = \varepsilon.$$

Conversely, let's assume that for any $\varepsilon > 0$, there exists an N such that (3.4) is satisfied. For $\varepsilon_1 = 1$ there exists a positive integer N_1 such that

(3.5) $$|a_m/a_{N_1} - 1| < 1 \text{ if } m \geq N_1.$$

Take the smallest positive integer with this property and also denote it by N_1. Then, by (3.5), if $m \geq N_1$ we have

$$|a_m| \leq |a_m - a_{N_1}| + |a_{N_1}| < 2|a_{N_1}|.$$

Now let $\varepsilon > 0$. If $\varepsilon < 2|a_{N_1}|$, then by (3.4) we can find N such that $|a_m/a_N - 1| < \frac{\varepsilon}{2|a_{N_1}|}$ if $m \geq N$. Since N_1 was chosen to be the smallest positive integer for which (3.5) holds, and since $\frac{\varepsilon}{2|a_{N_1}|} < \varepsilon_1 = 1$, we see that $N \geq N_1$. Thus for $m \geq N$ we have

$$|a_m - a_N| = \left|\frac{a_m - a_N}{a_N} a_N\right| = \left|\frac{a_m}{a_N} - 1\right| |a_N| < \frac{\varepsilon}{2|a_{N_1}|} 2|a_{N_1}| = \varepsilon,$$

so by Theorem 2.1, $\{a_n\}$ is convergent. If $\varepsilon \geq 2|a_{N_1}|$, then by (3.5),

$$\left|\frac{a_m - a_{N_1}}{a_{N_1}}\right| < 1 \leq \frac{\varepsilon}{2|a_{N_1}|}, \text{ if } m \geq N_1,$$

so $|a_m - a_{N_1}| < \varepsilon$, therefore $\{a_n\}$ is convergent by Theorem 2.1. □

Remark 3.1. The condition $|a_n| \geq \delta$ cannot be dropped. For example, the sequence $a_n = \frac{1}{n!}$ converges to 0; however, it does not satisfy the Ratio Criterion above.

Example 3.2. Let $x \geq 1$ be an irrational number, and let a_n be the n-place decimal expansion of x. We will use Theorem 3.4 to prove that the sequence $\{a_n\}$ is convergent. For any $m > n$,

$$\left|\frac{a_m}{a_n} - 1\right| = \left|\frac{a_m - a_n}{a_n}\right| \leq \frac{10^{-n-1}}{1} = 10^{-n-1}.$$

For $\varepsilon > 0$, choose N such that $10^{-N-1} < \varepsilon$. Then, for $m \geq N$, $\left|\frac{a_m}{a_N} - 1\right| \leq 10^{-N-1} < \varepsilon$, so the sequence $\{a_n\}$ is convergent by Theorem 3.4.

The idea of extending the usual notion of convergence to include sequences divergent to $\pm\infty$ appears in several classical anaylsis books (see [9] or [4], for example).

Definition 3.1. We say that a sequence of real numbers is *convergent in the extended sense* or *x-convergent*, if it either converges or if it diverges to plus or minus infinity. A sequence $\{a_n\}$ is called an *extended* sequence if $a_n \in \mathbb{R}$ or $a_n = \pm\infty$.

One obvious disadvantage of this terminology is that the sum of two x-convergent sequences is not necessarily x-convergent, since $\infty - \infty$ is not well defined. Nevertheless, we now propose two versions of Cauchy's Criterion for x-convergent sequences.

Proposition 3.1. (Cauchy's Criterion for Extended Convergence - Version I) *Let $T : \mathbb{R} \to (-1, 1)$ be a strictly increasing, continuous bijection between $\mathbb{R}$ and $(-1, 1)$. Define $T(-\infty) = -1$, $T(\infty) = 1$. Then an extended sequence $\{a_n\} \subset \mathbb{R} \cup \{\pm\infty\}$ is x-convergent if and only if $\{T(a_n)\}$ is a Cauchy sequence.*

Proof. The sequence of real numbers $\{T(a_n)\} \subset [-1, 1]$ is a Cauchy sequence if and only if it converges to a number $L \in [-1, 1]$. But $T(a_n) \to L$ if and only if $a_n \to T^{-1}(L)$, since $T : \mathbb{R} \cup \{\pm\infty\} \to [-1, 1]$ is a a homeomorphism. □

Remark 3.2. In the above proposition, $T(x)$ could be for example $(2\tan^{-1} x)/\pi$, or $x/\sqrt{1+x^2}$. Or, if F is a continuous strictly increasing cumulative distribution function, then $T(x) = 2F(x) - 1$ could also be used.

The following theorem provides a more practical way to handle x-convergent sequences.

Theorem 3.5. (Cauchy's Criterion for Extended Convergence - Version II) *An extended sequence $\{a_n\} \subset \mathbb{R} \cup \{\pm\infty\}$ is x-convergent if and only if for any $\varepsilon > 0$ there exists a positive integer N such that if $m \geq N$, then at least one of the following holds:*
(a) $|a_m - a_N| < \varepsilon$,
(b) $a_m, a_N \geq 1$ and $|1/a_m - 1/a_N| < \varepsilon$,
(c) $a_m, a_N \leq -1$ and $|1/a_m - 1/a_N| < \varepsilon$.

Proof. First asssume that $\{a_n\}$ is convergent in the extended sense. If $a_n \to L$ then (a) holds, by Theorem 2.1; if $a_n \to \infty$ then (b) holds; if $a_n \to -\infty$ then (c) holds.
To prove the converse, let $\varepsilon = 1/n$, for $n = 1, 2, \ldots$. For each n we can find a positive integer N_n such that, if $m \geq N_n$, then at least one of (a), (b) or (c) holds with $\varepsilon = 1/n$. We create a list as follows: for each $n = 1, 2, \ldots$, write A, if (a) holds, write B, if (b) holds, and write C, if (c) holds. If more than one of the conditions hold for a given n, then we just write A (note that(b) and (c) cannot occur at the same time). At least one of the letters A, B, C will appear infinitely many times on this list. If it is A, then $\{a_n\}$ satisfies our first reformulation of Cauchy's Criterion, Theorem 2.1, (since for any $\varepsilon > 0$, we can solve the inequality $1/n < \varepsilon$), and therefore $\{a_n\}$ is convergent. If B appears infinitely many times on the list, then the sequence $\{1/a_n\}$ satisfies the reformulated Cauchy's Criterion of Theorem 2.1, and therefore $\{1/a_n\}$ converges to a real number. If this number is 0,

then, since $\{a_n\}$ satisfies condition (b), the sequence $\{1/a_n\}$ must approach 0 from the right, and therefore $a_n \to +\infty$. Otherwise, $\{a_n\}$ converges to a real number. The case when C appears infinitely many times is similar. □

4. Some two-dimensional extensions

Let us recall the one-point compactification of the plane, $\mathbb{R}^2 \cup \{\infty\}$. In this compactification, the neighborhood basis of any finite point (x, y) is the usual set of open discs centered at (x, y), while a neighborhood basis of ∞ is $\{\{(x, y) : x^2 + y^2 > R^2\} : R > 0\} \cup \{\infty\}$ (see [5] or [10], for example). In this topology, a sequence $\{(x_n, y_n)\}$ converges to ∞ if and only if one of the following equivalent conditions holds:
(i) For every $R > 0$, there exists N such that if $n \geq N$ then $x_n^2 + y_n^2 \geq R^2$.
(ii) $\lim(|x_n| + |y_n|) = \infty$, that is, if n is large, then $|x_n|$ or $|y_n|$ is large.

Below we introduce a different extension of $\mathbb{R}^2$.

Definition 4.2. Let $\mathbb{T} := \mathbb{R}^2 \cup \{\infty\}$ with the following topology: the neighborhood basis of ∞ is defined to be $\{\{(x, y) : \min(x, y) > R\} : R > 0\} \cup \{\infty\}$, while the neighborhood basis of a finite point is the set of open discs centered at the point.

In this topology (which is Hausdorff), a sequence $\{(x_n, y_n)\}$ converges to ∞ if and only if $x_n \to +\infty$ and $y_n \to +\infty$.

Remark 4.3. A huge disadvantage of this topology is that $\mathbb{T}$ is neither compact, nor sequentially compact. Indeed, consider the system of open discs with radius 1 around all points (x, y), and $\{(x, y) : \min(x, y) > 2\} \cup \{\infty\}$. This is an open covering of $\mathbb{R}^2 \cup \{\infty\}$, from which we cannot select a finite subcovering, showing non-compactness. Also, the sequence $\{(1, n)\}$, $n = 1, 2, \ldots$, for example, does not approach ∞ (or anything else), and it has no convergent subsequence.

In the next theorem we prove that real functions defined on $\mathbb{T}$ behave nicely.

Theorem 4.6. (a) *Let $u, v : \mathbb{T} \to \mathbb{R}$, and let α and β be real numbers. Let P be any point in $\mathbb{T}$. If the limits $\lim_{(x,y)\to P} u(x, y)$ and $\lim_{(x,y)\to P} v(x, y)$ exist and are finite, then*

$$\lim_{(x,y)\to P}[\alpha u(x, y) + \beta v(x, y)] = \alpha \lim_{(x,y)\to P} u(x, y) + \beta \lim_{(x,y)\to P} v(x, y),$$

(b) *Let A be the set of all continuous real functions on $\mathbb{T}$. Then A is an algebra over $\mathbb{R}$, that is, A is a ring with respect to addition and multiplication, a vector space over $\mathbb{R}$ with respect to addition and scalar multiplication, and if $u, v \in A$ and $\alpha \in \mathbb{R}$, then $(\alpha u)v = u(\alpha v) = \alpha(uv)$.*

Proof. (a) First we show that $\lim(u + v) = \lim u + \lim v$. Let $\lim u(x, y) = L_1$ and let $\lim v(x, y) = L_2$. If $P \neq \infty$, for any $\varepsilon > 0$, there exists an open disc D_1 around P such that, if $(x, y) \in D_1$, then $|u(x, y) - L_1| < \varepsilon/2$, and an open disc D_2 around P such that if $(x, y) \in D_2$, then $|v(x, y) - L_2| < \varepsilon/2$. Then, for any $(x, y) \in D := D_1 \cap D_2$, $|(u(x, y) + v(x, y)) - (L_1 + L_2)| \leq |u(x, y) - L_1| + |v(x, y) - L_2| < \varepsilon$, so $\lim(u + v) = \lim u + \lim v$. If $P = \infty$, according to the way we defined open neighborhoods of ∞ in $\mathbb{T}$, for any $\varepsilon > 0$, there exists $R_1 > 0$ such that, if $(x, y) \in \{(x, y) : \min(x, y) > R_1\} \cup \{\infty\}$, then $|u(x, y) - L_1| < \varepsilon/2$, and there exists $R_2 > 0$ such that, if $(x, y) \in \{(x, y) : \min(x, y) > R_2\} \cup \{\infty\}$, then $|v(x, y) - L_2| < \varepsilon/2$. Then, if $R = \max\{R_1, R_2\}$, for $(x, y) \in \{(x, y) : \min(x, y) > R\} \cup \{\infty\}$, we have $|(u(x, y) + v(x, y)) - (L_1 + L_2)| \leq |u(x, y) - L_1| + |v(x, y) - L_2| < \varepsilon$, so $\lim(u + v) = \lim u + \lim v$.
Next, we show that $\lim c u = c \lim u$. Assume that $\lim_{(x,y)\to P} u(x, y) = L$. If $c = 0$, the statement is clearly true, so assume $c \neq 0$. If $P \neq \infty$, for any $\varepsilon > 0$ there exists an open

disc D around P such that, if $(x, y) \in D$, then $|u(x, y) - L| < \varepsilon/|c|$. Then, $|c\,u(x, y) - c\,L| = |c||u(x, y) - L| < |c|\,\varepsilon/|c| = \varepsilon$, so $\lim c\,u = c \lim u$. If $P = \infty$, for any $\varepsilon > 0$ there exists $R > 0$ such that, if $(x, y) \in \{(x, y) : \min(x, y) > R\} \cup \{\infty\}$, then $|u(x, y) - L| < \varepsilon/|c|$. Then, for $(x, y) \in \{(x, y) : \min(x, y) > R\} \cup \{\infty\}$, we also have $|c\,u(x, y) - c\,L| = |c||u(x, y) - L| < |c|\,\varepsilon/|c| = \varepsilon$, so $\lim c\,u = c \lim u$. The two limit properties we just proved imply that (a) holds. The proof of part (b) is a routine verification of the axioms and we leave it to the reader. □

Definition 4.3. Consider the following subset of $\mathbb{R}^2 \cup \{\infty\}$:

$$\mathbb{S} := \mathbb{N} \times \mathbb{N} \cup \{\infty\} \subset \mathbb{T}$$

where $\mathbb{N}$ is the set of all positive integers. We define the topology on $\mathbb{S}$ the same way it was defined above for $\mathbb{T}$ in Definition 4.2: the neighborhood basis of ∞ in $\mathbb{S}$ consists of the collection of sets $\{\{(m, n) : \min\{m, n\} > N\} : N \in \mathbb{N}\} \cup \{\infty\}$. Just like in $\mathbb{T}$, a sequence $\{(m_k, n_k)\} \subset \mathbb{S}$ converges to ∞ if and only if $m_k \to +\infty$ and $n_k \to +\infty$.

Theorem 4.7. (Cauchy's Criterion - the $\mathbb{S}$ Version) *Let $\{a_n\}$ be a sequence of real numbers. Then $\{a_n\}$ is convergent if and only if $a_{m_k} - a_{n_k} \to 0$ whenever $(m_k, n_k) \to \infty$ in $\mathbb{S}$.*

Proof. First assume $\lim a_n = a$. For any positive integers m and n, define $u(m, n) := a_m$ and $v(m, n) := a_n$. If $(m_k, n_k) \to \infty$ in $\mathbb{S}$, then $m_k \to +\infty$ and $n_k \to +\infty$, hence, using the linearity of the limit provided by Theorem 4.6, $\lim(a_{m_k} - a_{n_k}) = \lim u(m_k, n_k) - \lim v(m_k, n_k) = \lim a_{m_k} - \lim a_{n_k} = a - a = 0$.
Conversely, assume that $a_{m_k} - a_{n_k} \to 0$, whenever $(m_k, n_k) \to \infty$. By contradiction, assume that $\{a_n\}$ is not convergent. Then by Theorem 2.1, there exists an $\varepsilon > 0$ such that for any positive integer N there is an $m \geq N$ such that $|a_m - a_N| \geq \varepsilon$. Thus, for any positive integer k, there exists a positive integer $m_k \geq k$ such that $|a_{m_k} - a_k| \geq \varepsilon$. But this is a contradiction, since $(m_k, k) \to \infty$ in $\mathbb{S}$, so, by hypothesis, $a_{m_k} - a_k$ should converge to 0. □

Since working with infinity could be hard to visualize, we are using the following inversion to bring ∞ to $(0, 0)$ (see [1], for example).

Definition 4.4. Let $I : \mathbb{R}^2 \cup \{\infty\} \to \mathbb{R}^2 \cup \{\infty\}$ by

$$I(x, y) := \Big(\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2}\Big), \text{ if } (x, y) \neq (0, 0),$$
$$I(0, 0) := \infty,\ I(\infty) := (0, 0).$$

Definition 4.5. Let $\bar{\mathbb{S}} := I(\mathbb{S})$. Equip $\bar{\mathbb{S}}$ with the following topology: the neighborhood basis of any $I(i, j)$ is just $\{I(i, j)\}$, and the neighborhood basis of $(0, 0)$ consists of the collection of sets $\{\{I(i, j) : i, j > N\} : N \geq 1\} \cup \{(0, 0)\}$.

Note that under inversion I, the image of the set $\{(x, y) \in \mathbb{R}^2 : x, y > R\}$ is a region D_R in the first quadrant, containing the origin, and bounded by the curves $(\frac{R}{R^2+y^2}, \frac{y}{R^2+y^2}), y \geq R$ and $(\frac{x}{R^2+x^2}, \frac{R}{R^2+x^2}), x \geq R$. Also, $D_{R_1} \supset D_{R_2}$, if $R_1 < R_2$ (see Figure 1).
For any positive integer N, let $B_N := D_N \cap \bar{\mathbb{S}} = \{I(m, n)|\ m, n$ positive integers and $m, n > N\}$. In this notation, a neighborhood basis of $(0, 0)$ in $\bar{\mathbb{S}}$ is the collection of sets $\{B_N \cup \{(0, 0)\} : N \geq 1\}$. Note that the sets B_N satisfy $B_1 \supset B_2 \supset \cdots$.

If $\{p_k\} := \{(m_k, n_k)\}_{k \geq 1}$ is a sequence in $\mathbb{S}$, then $p_k \to \infty$ if and only if $I(p_k) \to I(\infty) = (0, 0)$. This means that for every positive integer N, there exists an index k_0, such that $I(p_k) \in B_N$ if $k \geq k_0$. This is different than the usual convergence to $(0, 0)$, which only means that the distance from $(0, 0)$ approaches zero.

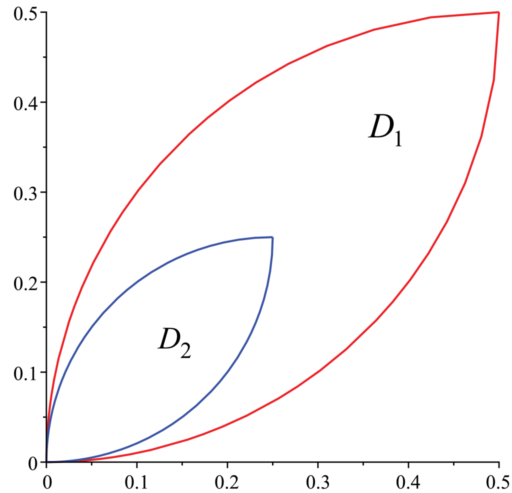

FIGURE 1. Graph of region D_R for $R = 1$ and $R = 2$.

We can now state the following:

Theorem 4.8. (Cauchy's Criterion - the $\bar{\mathbb{S}}$ Version) *Let $\{a_n\}$ be a sequence of real numbers. Then $\{a_n\}$ is convergent if and only if $a_{m_k} - a_{n_k} \to 0$ whenever $I(m_k, n_k) \to (0, 0)$ in $\bar{\mathbb{S}}$.*

We omit the proof since it is similar to the proof of Theorem 4.7.

In the last part of our article we prove two Cauchy-like criteria, based on a class of functions we call *lower α-Lipschitz continuous*.

Definition 4.6. Let $f, g : \mathbb{R}^2 \to \mathbb{R}$ such that $f(x, x) = g(x, x)$, for all $x \in \mathbb{R}$. We say that a sequence $\{a_n\}$ satisfies the *two variable Cauchy property* with respect to f and g, if for any $\varepsilon > 0$ there exists a positive integer N such that if $n \geq N$ then

$$|f(a_n, a_N) - g(a_n, a_N)| < \varepsilon. \tag{4.6}$$

Definition 4.7. (a) A function $f : D \subseteq \mathbb{R} \to \mathbb{R}$ is called *lower α-Lipschitz continuous* if there exist positive constants M and α such that

$$|x - y| \leq M|f(x) - f(y)|^\alpha, \text{ for all } x, y \in D.$$

(b) Two functions $f, g : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ are called *mutually lower α-Lipschitz continuous*, if there exist positive constants M and α such that

$$|x - y| \leq M|f(x, y) - g(x, y)|^\alpha, \text{ for all } (x, y) \in D.$$

Remark 4.4. If a function f has an inverse f^{-1}, and if the inverse is α-Lipschitz continuous, then f is certainly lower α-Lipschitz continuous. Indeed, if

$$|f^{-1}(u) - f^{-1}(v)| \leq M|u - v|^\alpha,$$

then for $x := f^{-1}(u)$ and $y := f^{-1}(v)$ we have

$$|x - y| = |f^{-1}(u) - f^{-1}(v)| = |f^{-1}(f(x)) - f^{-1}(f(y))| \leq M|f(x) - f(y)|^\alpha.$$

Theorem 4.9. (Two Variable Cauchy Criterion - Version I) *Let $\{a_n\}$ be a sequence of real numbers, and let $f, g : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ be continuous and mutually lower α-Lipschitz continuous such that $f(x,x) = g(x,x)$, for all $x \in D$. Then $\{a_n\}$ is convergent if and only if $\{a_n\}$ satisfies the two variable Cauchy property with respect to f and g.*

Proof. Assume $a_n \to a$. Since f and g are continuous, for any $\varepsilon > 0$ there exists a positive integer M such that if $n, N \geq M$, then $|f(a_n, a_N) - f(a,a)| < \varepsilon/2$ and $|g(a_n, a_N) - g(a,a)| < \varepsilon/2$. Since $f(a,a) = g(a,a)$ we have

$$|f(a_n, a_N) - g(a_n, a_N)| \leq |f(a_n, a_N) - f(a,a)| + |g(a,a) - g(a_n, a_N)| < \varepsilon,$$

so $\{a_n\}$ satisfies the two variable Cauchy property with respect to f and g.
Conversely, assume $\{a_n\}$ satisfies (4.6) and let $\varepsilon > 0$. Then, there exists a positive integer N such that, if $n \geq N$, $|f(a_n, a_N) - g(a_n, a_N)| < \left(\frac{\varepsilon}{M}\right)^{1/\alpha}$. Then, by the mutual lower α-Lipschitz continuity of f and g, for $n \geq N$, $|a_n - a_N| \leq M|f(a_n, a_N) - g(a_n, a_N)|^\alpha < \varepsilon$, hence $\{a_n\}$ is convergent by Theorem 2.1. □

Next we prove our main result, which is a generalization of Theorem 4.7.

Theorem 4.10. (Two Variable Cauchy Criterion - Version II) *Let $\{a_n\}$ be a sequence of real numbers, and assume $h_1, h_2, h_3, h_4 : D \subseteq \mathbb{R} \to \mathbb{R}$ are continuous functions satisfying the following properties:*
(i) There exists $c > 0$ such that $|h_2(x)| \geq c$ and $|h_3(x)| \geq c$, for all $x \in D$.
(ii) The function h_1/h_3 is lower α-Lipschitz continuous.
(iii) $h_1(x)h_2(x) = h_3(x)h_4(x)$, for all $x \in D$.
Then $\{a_n\}$ is convergent if and only if $\{a_n\}$ satisfies the two variable Cauchy property with respect to the functions $f(x,y) := h_1(x)h_2(y)$ and $g(x,y) := h_3(x)h_4(y)$.

Proof. If $\{a_n\}$ is convergent, the proof that $\{a_n\}$ satisfies the two variable Cauchy property with respect to the functions f and g is very similar to the one in the Theorem 4.9, so we omit it.
To prove the converse, assume that $\{a_n\}$ satisfies (4.6) and let $\varepsilon > 0$. Then there exists a positive integer N such that, if $n \geq N$ we have

$$|f(a_n, a_N) - g(a_n, a_N)| = |h_1(a_n)h_2(a_N) - h_3(a_n)h_4(a_N)| < \frac{c^2}{2}\left(\frac{\varepsilon}{M}\right)^{1/\alpha},$$

where M and α are the constants appearing in the lower α-Lipschitz continuity condition satisfied by $h_1(x)/h_3(x)$. Division by $|h_2(a_N)h_3(a_n)|$ yields

$$\left|\frac{h_1(a_n)}{h_3(a_n)} - \frac{h_4(a_N)}{h_2(a_N)}\right| < \frac{1}{2}\left(\frac{\varepsilon}{M}\right)^{1/\alpha}.$$

Now from (ii), if $m, n \geq N$ we have

$$|a_m - a_n| \leq M\left|\frac{h_1(a_m)}{h_3(a_m)} - \frac{h_1(a_n)}{h_3(a_n)}\right|^\alpha \leq$$

$$\leq M\left(\left|\frac{h_1(a_m)}{h_3(a_m)} - \frac{h_4(a_N)}{h_2(a_N)}\right| + \left|\frac{h_4(a_N)}{h_2(a_N)} - \frac{h_1(a_n)}{h_3(a_n)}\right|\right)^\alpha <$$

$$< M\left[\frac{1}{2}\left(\frac{\varepsilon}{M}\right)^{1/\alpha} + \frac{1}{2}\left(\frac{\varepsilon}{M}\right)^{1/\alpha}\right]^\alpha = \varepsilon.$$

Thus $\{a_n\}$ is a Cauchy sequence, so convergent by Cauchy's Criterion. □

Remark 4.5. We note that Theorem 4.7 follows from Theorem 4.10 if we choose $h_1(x) = h_4(x) = x$, and $h_2(x) = h_3(x) = 1$, which satisfy the hypotheses of Theorem 4.10. For this choice of functions,

$$f(a_{m_k}, a_{n_k}) - g(a_{m_k}, a_{n_k}) = h_1(a_{m_k})h_2(a_{n_k}) - h_3(a_{m_k})h_4(a_{n_k}) = a_{m_k} - a_{n_k}.$$

This implies that $a_{m_k} - a_{n_k} \to 0$ as $(m_k, n_k) \to \infty$ in the topology of $\mathbb{S}$, if and only if the sequence $\{a_n\}$ satisfies the two variable Cauchy property with respect to f and g.

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