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Perturbed approximations of fixed points of nonexpansive mappings in $\mathrm{CAT_p}(0)$ spaces

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ABSTRACT. Using the concept of the $\mathrm{CAT_p}(0)$ spaces proposed by Khamsi et al. [Khamsi, M. A. and Shukri, S. A., *Generalized* $\mathrm{CAT}(0)$ *spaces*. Bull. Belg. Math. Soc. Simon Stevin, **24** (2017), No. 3, 417–426], we establish Δ-convergence and strong convergence of a perturbed variant of Agarwal et al. S-iteration process for nonexpansive mapping. Finally, from them we **deduce** results valid for α-nonexpansive mappings.

1. INTRODUCTION

In this section, we introduce the basic notations and terminologies which we will use throughout this work. A metric space (X, d) is said to be a *length space* if any two points of X can be joined by a rectifiable path (that is, a path of finite length) where the distance between any two points of X is taken to be the infimum of the lengths of all rectifiable paths joining them. In this case, d is said to be a *length metric* (otherwise known as an *inner metric* or *intrinsic metric*). In case no rectifiable path joins two points of the space, the distance between them is taken to be ∞.

A *geodesic path* joining $x \in X$ to $y \in X$ (or, more briefly, a *geodesic* from x to y) is a map c from closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(l) = y$, and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$. The image γ of c is called a geodesic (or metric) *segment* joining x and y. The space (X, d) is said to be a *geodesic space* if every two points of X are joined by a geodesic, and is said to be *uniquely geodesic* if there is exactly one geodesic joining x and y for each $x, y \in X$, which we denote by $[x, y]$, called the segment joining x to y.

The most fundamental examples of geodesic metric spaces are normed vector spaces, As nonlinear examples, one can consider the $CAT(0)$ spaces [6].

A *geodesic triangle* $\Delta(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points in X (called *vertices* of Δ) and a geodesic segment between each pair of vertices (the *edges* of Δ). A *comparison triangle* for geodesic triangle $\Delta(x_1, x_2, x_3)$ in (X, d) is a triangle $\overline{\Delta}(x_1, x_2, x_3) := \Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\bar{x}_i, \bar{x}_j) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$. Such a triangle always exists [6].

Let Δ be a geodesic triangle in X and $\overline{\Delta}$ its comparison triangle in $\mathbb{R}^2$. Then Δ is said to satisfy $CAT(0)$ inequality if for all $x, y \in \Delta$ and all comparison points $\overline{x}, \overline{y} \in \overline{\Delta}$, $d(x, y) \leq d_{\mathbb{R}^2}(x, y)$.

A geodesic metric space X is called a $CAT(0)$ space if all geodesic triangles satisfy the above comparison axiom (i.e. $CAT(0)$ inequality). Some well known examples of $CAT(0)$

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spaces are complete and simply connected Riemannian manifold having non-positive sectional curvature, pre-Hilbert spaces, R-trees, Euclidean buildings and the complex Hilbert ball with a hyperbolic metric.

Complete $\mathrm{CAT}(0)$ spaces are often called *Hadamard spaces*.

Definition 1.1. [19] A sequence $\{x_n\}$ in X is said to Δ-converge to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case we write Δ- $\lim_{n\to\infty} x_n = x$ and we call x the $\Delta - \lim_{n\to\infty} x_n = x$.

Let $T\colon K \to K$ be a mapping. The set of fixed points of T is denoted by $F(T)$. A mapping T is said to be *quasi-nonexpansive* if $F(T) \neq \emptyset$ and $d(Tx, p) \leq d(x, p)$ for all $x \in K$ and $p \in F(T)$; T is said to be *nonexpansive* if $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in K$.

Lemma 1.1. [19]

i. *Every bounded sequence in X has Δ-convergence subsequence.*
ii. *If K is a closed convex subset of X and if $\{x_n\}$ is a bounded sequence in K, then the asymptotic center of $\{x_n\}$ is in K.*
iii. *If K is a closed convex subset of X and if $f : K \to X$ is a nonexpansive mapping, then the conditions, $\{x_n\}$ Δ-converges to x and $d(x_n, f(x_n)) \to 0$, imply $x \in K$ and $f(x) = x$*

Naor and Silberman [27] extended the notion of p-uniformly convexity to the setting of geodesic spaces in the following way.

Definition 1.2. Fix $1 < p < \infty$. A metric space (X, d) is called *p-uniformly convex* with parameter $c > 0$ iff (X, d) is a geodesic space and for any three points $x, y, z \in X$ and all $t \in [0, 1]$,

$$d^p((1-t)x \oplus ty, z) \leq (1-t)\, d^p(x, z) + t\, d^p(y, z) - \frac{c}{2}\, t(1-t)\, d^p(x, y). \tag{1.1}$$

Note that inequality (1.1) guarantees that the space X is uniquely geodesic (see [28, Lemma 2.2] for a proof in the case $p = 2$). Also, any closed convex subset of a p-uniformly convex space is again a p-uniformly convex space with the same parameter.

Moreover, $CAT(0)$ spaces are 2-uniformly convex metric spaces with parameter $c = 2$ and $CAT(\kappa)$ spaces $(\kappa > 0)$ with $diam(X) < \frac{\pi}{2\sqrt{k}}$ are 2-uniformly convex metric spaces with parameter $c = (\pi - 2\sqrt{k}\epsilon)\tan(\sqrt{k}\epsilon)$ for any $\epsilon \leq \frac{\pi}{2\sqrt{k}} - diam(X)$ (see [2]).

It is easy to see that in a complete p-uniformly convex space, any bounded sequence has a Δ-convergent subsequence. In the setting of $\mathrm{CAT}(0)$ spaces, Δ-convergence is equivalent to another concept of weak convergence that uses projections on geodesic segments (see [12]). Reasoning as in [12], one can see that this equivalence also holds for p-uniformly convex spaces.

Definition 1.3. [20] Let (X, d) be a geodesic metric space. X is said to be a $\mathrm{CAT_p}(0)$ space, for $p > 2$, if for any geodesic triangle $\triangle$ in X, there exists a comparison triangle $\triangle$ in l_p such that the comparison axiom is satisfied, i.e., for all $x, y \in \triangle$ and all comparison points $\bar{x}, \bar{y} \in \triangle$, we have

$$d(x, y) \leq \|\bar{x} - \bar{y}\|.$$

We will need some known facts about ℓ_p, for $p \geq 2$. The following technical lemma was discovered initially by Lim [23] (see also [32]):

Lemma 1.2. [23] *Let $\gamma, \beta \in [0, 1]$ be such that $\gamma + \beta = 1$. For any $x, y \in \ell_p$, we have*

$$\|\gamma\, x + \beta\, y\|^p + g(\gamma)\, \|x - y\|^p \leq \gamma\, \|x\|^p + \beta\, \|y\|^p,$$

where

$$g(\gamma) = \gamma\,\beta\,\frac{1+[x(\gamma\wedge\beta)]^{p-1}}{[1+x(\gamma\wedge\beta)]^{p-1}},$$

where $x(\gamma)$, for $\gamma \in [0,1/2]$ is the unique solution to

$$(1-\gamma)x^{p-1} - \gamma - ((1-\gamma)x-\gamma)^{p-1} = 0, \quad x \in \left[\frac{\gamma}{1-\gamma}, 1\right].$$

In particular, we have

$$g(\gamma) \geq \gamma\,\beta\frac{1}{2^{p-1}},$$

which implies

$$\|\gamma\,x + \beta\,y\|^p + \frac{\gamma\,\beta}{2^{p-1}}\,\|x-y\|^p \leq \gamma\,\|x\|^p + \beta\,\|y\|^p,$$

Let $x_i \in X$ and $t_i \in [0,1]$ for $i = 1,2,...,n$ such that $\sum\limits_{i=1}^{n} t_i = 1$. Following the definition of unique point $(1-\gamma)x\oplus\gamma y$ on a geodesic segment $[x,y]$, we build the following notations: $\bigoplus_{i=1}^{2} t_i x_i = \frac{t_1}{t_1+t_2}x_1 \oplus \frac{t_2}{t_1+t_2}x_2$. By induction, we can write

$$\bigoplus_{i=1}^{n} t_i x_i = (1-t_n)\left(\frac{t_1}{1-t_n}x_1 \oplus \frac{t_2}{1-t_n}x_2 \oplus \cdots \oplus \frac{t_{n-1}}{1-t_n}x_{n-1}\right) \oplus t_n x_n.$$

Using Lemma 1.2, we get the following:

Lemma 1.3. *Let X be a $\mathrm{CAT_p}(0)$ space, with $p \geq 2$, with $x, x_i \in X$ and $t_i \in [0,1]$ for $i = 1,2,...,n$ ($n \geq 2$) such that $\sum\limits_{i=1}^{n} t_i = 1$. Then*

(i) $d\left(\bigoplus_{i=1}^{n} t_i x_i, x\right) \leq \sum\limits_{i=1}^{n} t_i d(x_i, x)$;

(ii) $d^p\left(\bigoplus_{i=1}^{n} t_i x_i, x\right) \leq \sum\limits_{i=1}^{n} t_i d^p(x_i, x) - \frac{1}{2^{p-1}} t_i t_j d^p(x_i, x_j)$ *for $i, j \in \{1,2,\cdots,n\}$.*

Proof. (i) is evident. For (ii) we are going to prove by induction on n. For $n = 2$, let $x_1, x_2, x_3 \in X$, $x = x_3$. Consider the comparison triangle $\Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in ℓ_p such that $d(x_i, x_j) = \|\bar{x}_i - \bar{x}_j\|_p$, for $i, j \in \{1,2,3\}$. We have

(1.2)

$$\begin{aligned} d^p\left((1-t)x_1 \oplus t\,x_2, x_3\right) &\leq \|(1-t)\bar{x}_1 + t\,\bar{x}_2 - \bar{x}_3\|^p \\ &= \|(1-t)(\bar{x}_1 - \bar{x}_3) + t\,(\bar{x}_2 - \bar{x}_3)\|^p \\ &\leq (1-t)\,\|\bar{x}_1 - \bar{x}_3\|^p + t\,\|\bar{x}_2 - \bar{x}_3)\|^p - \frac{t(1-t)}{2^{p-1}}\,\|\bar{x}_1 - \bar{x}_2\|^p \\ &= (1-t)\,d^p(x_1, x_3) + t\,d^p(x_2, x_3) - \frac{t(1-t)}{2^{p-1}}\,d^p(x_1, x_2). \end{aligned}$$

Inductively, suppose that result holds for $n = k-1$, i.e.,

$$d^p\left(\bigoplus_{i=1}^{k-1} t_i x_i, x\right) \leq \sum_{i=1}^{k-1} t_i d^p(x_i, x) - \frac{1}{2^{p-1}} t_i t_j d^p(x_i, x_j),$$

for $i, j \in \{1,2,\cdots,n-1\}$, where $\sum\limits_{i=1}^{k-1} t_i = 1$.

Now choose $\{t_1, t_2, ..., t_k\} \subset [0,1]$ with $t_k \neq 0$ and such that $\sum_{i=1}^{k} t_i = 1$. Select $i, j \in \{1, ..., k\}$ s.t $i \neq k \neq j$ (we can change position of t_i). By (1.2),

$$\begin{aligned} d^p\left(\bigoplus_{i=1}^{k} t_i x_i, x\right) &= d^p\left((1-t_k)\left(\bigoplus_{i=1}^{k-1} \tfrac{t_i}{1-t_k} x_i\right) \oplus t_k x_k, x\right) \\ &\leq (1-t_k) d^p\left(\left(\bigoplus_{i=1}^{k-1} \tfrac{t_i}{1-t_k} x_i\right), x\right) + t_k d^p\left(x_k, x\right) \\ &\leq (1-t_k) \sum_{i=1}^{k-1} \tfrac{t_i}{1-t_k} d^p(x_i, x) + t_k d^p\left(x_k, x\right) - \tfrac{1-t_k}{2^{p-1}} t_i t_j d^p(x_i, x_j) \\ &\leq \sum_{i=1}^{k} \tfrac{t_i}{1-t_k} d^p(x_i, x) - \tfrac{t_i t_j}{2^{p-1}} d^p(x_i, x_j), \end{aligned}$$

$i, j \in \{1, 2, \cdots, n-1\}$.

□

From Lemma 1.3, if $p \geq 2$, inequality (1.1) is satisfied for $c = 2^{p-2}$. Thus any $\mathrm{CAT_p}(0)$ space is p-uniformly convex.

In the proof of the main results we shall use the next Lemma.

Lemma 1.4 ([31]). *Suppose that $\{a_n\}$ and $\{b_n\}$ are sequence of nonnegative real numbers such that $a_{n+1} \leq a_n + b_n$ for all $n \geq N_0$, for a fixed N_0. If $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.*

2. Main results

The main tool for approximation of fixed points of nonexpansive mappings remains iterative technique. Several authors have extensively studied the iterative techniques for nonexpansive in Banach spaces, including the Mann type (one step), the Ishikawa type (two steps), and the three-step iteration processes (see [5]). Agarwal et al. [1] introduced the following iterative process in Banach spaces:

$$\begin{cases} x_{n+1} = (1-\gamma_n)Tx_n + \gamma_n T y_n \\ y_n = (1-\beta_n)x_n + \beta_n T x_n \qquad n = 1, 2, 3, ..., \end{cases}$$

In $\mathrm{CAT}(0)$ spaces Dhompongsa and Panyanak [11] studied the convergence of the Picard, Mann and Ishikawa iterates.

On the other hand, Liu [24] introduced Ishikawa and Mann iteration methods with errors (see also [34]). We note that, in applications, there are perturbations always occurring in the iterative processes because the manipulations are inaccurate. It is no doubt that researching the convergent problems of iterative methods with perturbation members is a significant job. Recently, Calderon et al. [7] presented a pertubed iteration in $\mathrm{CAT}(0)$ spaces.

This leads us, in this paper, to introduce and study a new class of two-step iterative scheme with perturbations for solving the fixed point problem for nonexpansive mappings. This iterative scheme can be viewed as an extension for type iterative schemes of Agarwal et al. [1] adapted it into $\mathrm{CAT_p}(0)$. Here we introduce a perturbation in the model. Consider K to be a nonempty closed convex subset of a complete $\mathrm{CAT_p}(0)$ space X and $T : K \to K$ be a nonexpansive mapping. Fix $x_0 \in K$. Consider $\{x_n\}$ defined by the iteration

$$\text{(SM)} \qquad \begin{cases} z_n = T_3 x_n, \\ y_n = (1-\beta_n-\beta'_n)x_n \oplus \beta_n T_2 z_n \oplus \beta'_n \varepsilon'_n, \\ x_{n+1} = (1-\gamma_n-\gamma'_n)T_2 x_n \oplus \gamma_n T_1 y_n \oplus \gamma'_n \varepsilon_n, \end{cases}$$

for $n \geq 1$, where $\{\epsilon_n\}$ and $\{\epsilon'_n\}$ are bounded sequences in K and $\{\gamma_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$, $\{\gamma'_n\}_{n=1}^{\infty}$ and $\{\beta'_n\}_{n=1}^{\infty}$ are appropriate real sequences in $[0,1]$.

Lemma 2.5. *Let (X,d) be a complete $\mathrm{CAT_p}(0)$ space, with $p \geq 2$. Let K be a nonempty closed convex bounded subset of X. Let $T_1, T_2, T_3 : K \to K$ be three nonexpansive mappings such that $\Omega = F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset$. Let $\{x_n\}$ be the sequence defined by (SM) such that*

C1) $\sum_{n=1}^{\infty} \gamma'_n < \infty$ *and* $\sum_{n=1}^{\infty} \beta'_n < \infty$.
C2) $0 < \gamma \leq \beta_n \leq \beta_n + \beta'_n \leq \beta < 1$.
C3) $\gamma \leq \gamma_n + \gamma'_n$.

Then the following hold:

(i) $\lim_{n\to\infty} d(x_n, x^*)$ *exists, for any x^* of Ω.*
(ii) $\lim_{n\to\infty} d(x_n, T_i x_n) = 0$, *for $i = 1,2,3$, i.e., $\{x_n\}$ is an approximate fixed point sequence of T_i, for $i = 1,2,3$.*

Proof. Let $x^* \in \Omega$. Using **Lemma 1.3**, we get

$$d(z_n, x^*) = d(T_3 x_n, x^*) \leq d(x_n, x^*),$$

$$\begin{aligned} d(y_n, x^*) &= d\left((1-\beta_n-\beta'_n)x_n \oplus \beta_n T_2 z_n \oplus \beta'_n \varepsilon'_n, x^*\right) \\ &\leq (1-\beta_n-\beta'_n)d(x_n, x^*) + \beta_n d(T_2 z_n, x^*) + \beta'_n d(\varepsilon'_n, x^*) \\ &\leq (1-\beta'_n)d(x_n, x^*) + \beta'_n d(\varepsilon'_n, x^*) \\ &\leq d(x_n, x^*) + \beta'_n d(\varepsilon'_n, x^*) \end{aligned}$$

and

$$\begin{aligned} d(x_{n+1}, x^*) &= d\left((1-\gamma_n-\gamma'_n)T_2 x_n \oplus \gamma_n T_1 y_n \oplus \gamma'_n \varepsilon_n, x^*\right) \\ &\leq (1-\gamma_n-\gamma'_n)d(T_2 x_n, x^*) + \gamma_n d(T_1 y_n, x^*) + \gamma'_n d(\varepsilon_n, x^*) \\ &\leq (1-\gamma_n-\gamma'_n)d(x_n, x^*) + \gamma_n d(y_n, x^*) + \gamma'_n d(\varepsilon_n, x^*) \\ &\leq (1-\gamma'_n)d(x_n, x^*) + \gamma_n \beta'_n d(\varepsilon'_n, x^*) + \gamma'_n d(\varepsilon_n, x^*) \\ &\leq d(x_n, x^*) + \gamma_n \beta'_n d(\varepsilon'_n, x^*) + \gamma'_n d(\varepsilon_n, x^*), \end{aligned}$$

for any $n \geq 1$. Lemma 1.4 implies the conclusion of (i).

Let us prove (ii). Set $\lim_{n\to\infty} d(x_n, x^*) = m$. It is clear that if $m = 0$, then (ii) holds. Otherwise we assume $m > 0$. Let us show that $\lim_{n\to\infty} d(y_n, x^*) = m$. We have

$$d(x_{n+1}, x^*) \leq (1-\gamma_n-\gamma'_n)d(x_n, x^*) + \gamma_n d(y_n, x^*) + \gamma'_n d(\varepsilon_n, x^*),$$

which implies

$$(\gamma_n + \gamma'_n)d(x_n, x^*) \leq \gamma_n d(y_n, x^*) + d(x_n, x^*) - d(x_{n+1}, x^*) + \gamma'_n d(\varepsilon_n, x^*)$$

or

$$\begin{aligned} d(x_n, x^*) &\leq \frac{\gamma_n}{\gamma_n + \gamma'_n} d(y_n, x^*) + \frac{1}{\gamma_n + \gamma'_n}\left[d(x_n, x^*) - d(x_{n+1}, x^*)\right] + \frac{\gamma'_n}{\gamma_n + \gamma'_n} d(\varepsilon_n, x^*) \\ &\leq d(y_n, x^*) + \frac{1}{\gamma}\left[d(x_n, x^*) - d(x_{n+1}, x^*)\right] + \frac{\gamma'_n}{\gamma} d(\varepsilon_n, x^*), \end{aligned}$$

for any $n \geq 1$. Therefore we have

$$\begin{aligned} m &= \liminf_{n\to\infty} d(x_n . x^*) \\ &\leq \liminf_{n\to\infty} d(y_n, x^*) + \liminf_{n\to\infty} \frac{1}{\gamma}\left[d(x_n, x^*) - d(x_{n+1}, x^*)\right] + \liminf_{n\to\infty} \frac{\gamma'_n}{\gamma} d(\varepsilon_n, x^*) \\ &= \liminf_{n\to\infty} d(y_n, x^*). \end{aligned}$$

Similarly, we prove $\limsup_{n\to\infty} d(y_n, x^*) \leq m$, which implies

$$\lim_{n\to\infty} d(y_n, x^*) = m.$$

Similarly, we have

$$\begin{aligned} d(y_n, x^*) &= d\left((1-\beta_n-\beta'_n)x_n \oplus \beta_n T_2 z_n \oplus \beta'_n \varepsilon'_n, x^*\right) \\ &\leq (1-\beta_n-\beta'_n)d(x_n, x^*) + \beta_n d(T_2 z_n, x^*) + \beta'_n d(\varepsilon'_n, x^*) \\ &\leq (1-\beta_n-\beta'_n)d(x_n, x^*) + \beta_n d(z_n, x^*) + \beta'_n d(\varepsilon'_n, x^*), \end{aligned}$$

which implies that

$$(\beta_n+\beta'_n)d(x_n, x^*) \leq (d(x_n, x^*) - d(y_n, x^*)) + \beta_n d(z_n, x^*) + \beta'_n d(\varepsilon'_n, x^*), \quad ,$$

or

$$\begin{aligned} d(x_n, x^*) &\leq \frac{1}{\beta_n+\beta'_n}(d(x_n, x^*) - d(y_n, x^*)) + \frac{\beta_n}{\beta_n+\beta'_n} d(z_n, x^*) + \frac{\beta'_n}{\beta_n+\beta'_n} d(\varepsilon'_n, x^*) \\ &\leq \frac{1}{\gamma}(d(x_n, x^*) - d(y_n, x^*)) + d(z_n, x^*) + \frac{\beta'_n}{\gamma} d(\varepsilon'_n, x^*), \end{aligned}$$

and $m = \liminf_{n\to\infty} d(x_n, x^*) \leq \liminf_{n\to\infty} d(z_n, x^*)$. Similary $\limsup_{n\to\infty} d(z_n, x^*) \leq \limsup_{n\to\infty} d(x_n, x^*) = m$. Thus $\lim_{n\to\infty} d(z_n, x^*) = m$. This shows that

$$\lim_{n\to\infty} d(x_n, z_n) = 0. \tag{2.3}$$

On other hand, using Lemma 1.3, we get

$$\begin{aligned} d^p(y_n, x^*) &= d^p\left((1-\beta_n-\beta'_n)x_n \oplus \beta_n T_2 z_n \oplus \beta'_n \varepsilon'_n, x^*\right) \\ &\leq (1-\beta_n-\beta'_n)d^p(x_n, x^*) + \beta_n d^p(T_2 z_n, x^*) + \beta'_n d^p(\varepsilon'_n, x^*) \\ &- \frac{\beta_n(1-\beta_n-\beta'_n)}{2^{p-1}} d^p(x_n, T_2 z_n) \\ &\leq d^p(x_n, x^*) + \beta'_n d^p(\varepsilon'_n, x^*) - \frac{\beta_n(1-\beta_n-\beta'_n)}{2^{p-1}} d^p(x_n, T_2 z_n) \end{aligned}$$

Since $\{\beta_n, \beta_n+\beta'_n\} \subset [\gamma, \beta]$, we get

$$\begin{aligned} \frac{\gamma(1-\beta)}{2^{p-1}} d^p(x_n, T_2 z_n) &\leq \frac{\beta_n(1-\beta_n-\beta'_n)}{2^{p-1}} d^p(x_n, T_2 z_n) \\ &\leq \beta'_n d^p(\varepsilon'_n, x^*) + d^p(x_n, x^*) - d^p(y_n, x^*), \end{aligned}$$

which implies

$$d^p(x_n, T_2 z_n) \leq \frac{2^{p-1}}{\gamma(1-\beta)}\Big[\beta'_n d^p(\varepsilon'_n, x^*) + d^p(x_n, x^*) - d^p(y_n, x^*)\Big],$$

for any $n \geq 1$. Clearly, our previous results will force

$$\lim_{n\to\infty} d(x_n, T_2 z_n) = 0. \tag{2.4}$$

It follows from (2.4 and (2.3) that

$$d(x_n, T_2 x_n) \leq d(x_n, T_2 z_n) + d(T_2 z_n, T_2 x_n) \leq d(x_n, T_2 z_n) + d(z_n, x_n) \to 0.$$

Using similar arguments , $\lim_{n\to\infty} d(x_n, T_1 x_n) = 0$, which completes the proof of Lemma 2.5. □

Remark 2.1. *(1) From* [20]*, we know that if T is nonexpansive, then T has a nonempty fixed point set $F(T)$. If $T_1 = T_2 = T_3 = T$ in Lemma 2.5, then we obtain that $\Omega = F(T)$.*

(2) *We can prove Lemma 2.5 so far in the context of quasi-nonexpansive mappings with suitable changes. Moreover, it is ease prove Lemma 2.5 combining nonexpansive mappings with quasi-nonexpansive mappings.*

Next, we give the main result of our work.

Theorem 2.1. *Let X, K, T_1, T_2, T_3, $\{x_n\}$ satisfy the hypothesis of Lemma 2.5. Then the sequence $\{x_n\}$ defined by (SM) Δ-converges to a point of Ω.*

Proof. By Lemma 2.5, we know that $\lim_{n\to\infty} d(x_n, T_i x_n) = 0$, for $i = 1, 2, 3$ and $\lim_{n\to\infty} d(x_n, x^*)$ exists, for any $x^* \in \Omega$. Hence $\{x_n\}$ is bounded. We first show that $w_\Delta(x_n) \subseteq \Omega$, where $w_\Delta(x_n)$ is the set of the asymptotic centers of all subsequences of $\{x_n\}$. Let $u \in w_\Delta(x_n)$, then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By Lemma 1.1, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta - \lim_n v_n = v \in K$. By Lemma 1.1, $v \in \Omega$. Now, from Lemma 2.5, $\lim d(x_n, v)$ exists. We now claim that $u = v$. Assume on the contrary, that $u \neq v$. Then, by the uniqueness of asymptotic centers we have

$$\begin{aligned}\limsup_{n\to\infty} d(v_n, v) &< \limsup_{n\to\infty} d(v_n, u) &&\leq \limsup_{n\to\infty} d(u_n, u)\\ &< \limsup_{n\to\infty} d(u_n, v) &&= \limsup_{n\to\infty} d(x_n, v)\\ &= \limsup_{n\to\infty} d(v_n, v) &&.\end{aligned}$$

Which is a contradiction. Thus, $u = v \in \Omega$ and hence $w_\Delta(x_n) \subseteq \Omega$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$. By Lemma 1.1, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta - \lim_n v_n = v \in K$. Let $A(\{u_n\}) = \{u\}$ and $A(\{x_n\}) = \{x\}$. We have already seen that $u = v$ and $v \in \Omega$. Finally, we claim that $x = v$. If not, then the existence of $\lim_{n\to\infty} d(x_n, v)$ and uniqueness of asymptotic centers imply that

$$\begin{aligned}\limsup_{n\to\infty} d(v_n, v) &< \limsup_{n\to\infty} d(v_n, x) &&\leq \limsup_{n\to\infty} d(x_n, x)\\ &< \limsup_{n\to\infty} d(x_n, v) &&= \limsup_{n\to\infty} d(v_n, v).\end{aligned}$$

Again, it is a contradiction, hence $x = v \in \Omega$. Therefore, $w_\Delta(x_n) = \{x\}$. □

Theorem 2.2. *Let X, K, T_1, T_2, T_3, $\{x_n\}$ satisfy the hypothesis of Lemma 2.5. Then the sequence $\{x_n\}$ defined by (SM)*

i) *Δ-converges to a point of Ω.*
ii) *converges strongly to a fixed point of T if and only if $\liminf_{n\to\infty} d(x_n, \Omega) = 0$, where $d(x, \Omega) = \inf\{d(x, x^*) : x^* \in \Omega\}$.*

Proof. The necessity is obvius. To prove the converse, suppose that

$$\liminf_{n\to\infty} d(x_n, \Omega) = 0.$$

Thus by hypothesis $\lim_{n\to\infty} d(x_n, \Omega) = 0$. Next, we show that $\{x_n\}$ is Cauchy sequence in K. Let $\epsilon > 0$ be arbitrarily chosen. Since $\lim_{n\to\infty} d(x_n, \Omega) = 0$, there exists a positive integer n_0 such that

$$d(x_n, \Omega) < \frac{\epsilon}{4}, \forall n \geq n_0.$$

In particular, $\inf\{d(x_{n_0}, x^*) : x^* \in \Omega\} < \frac{\epsilon}{4}$. Thus there must exist $x^{**} \in \Omega$ such that

$$d(x_{n_0}, x^{**}) < \frac{\epsilon}{2}.$$

Now, for all $m, n \geq n_0$, we have

$$\begin{aligned} d(x_{n+m}, x_n) \leq & d(x_{n+m}, x^{**}) + d(x^{**}, x_n) \\ \leq & 2d(x_{n_0}, x^{**}) \\ \leq & 2\left(\frac{\epsilon}{2}\right) = \epsilon \end{aligned}$$

This proves that $\{x_n\}$ is a Cauchy sequence in K. Thus, the completeness of X implies that $\{x_n\}$ must be convergence. Since $\lim_{n\to\infty} d(x_n, \Omega) = 0$, we get $d(x^*, \Omega) = 0$, closedness of Ω gives that $x^* \in \Omega$. Thus $\{x_n\}$ converges strongly to a point in Ω. This completes the proof. □

3. Applications

3.1. **Proximal Point Algorithm in $CAT(0)$ spaces.** Recall that a function $f : C \to (-\infty, \infty]$ defined on a convex subset C of a $CAT(0)$ space is convex if, for any geodesic $\gamma : [a, b] \to C$, the function $f\gamma$ is convex. Some important examples can be found in [6]. We say that a function f defined on C is *lower semi-continuous at a point* $x \in C$ if $f(x) \leq \liminf_{n\to\infty} f(x_n)$, for each sequence $x_n \to x$. A function f is said to be *lower semi-continuous* on C if it is lower semi-continuous at any point in C.

For any $\lambda > 0$, define the *Moreau-Yosida* resolvent of f in $CAT(0)$ spaces as

$$J_\lambda(x) = argmin_{y\in X}\left[f(y) + \frac{1}{2\lambda}d^2(y, x)\right],$$

for all $x \in X$. Originally, this definition is due to Moreau [26]. The mapping J_λ is well-defined for every $\lambda > 0$ (see [14, 25]).

Let $f : X \to (-\infty, \infty]$ be be a proper convex and lower semi-continuous function. It was shown in [3] that the set $F(J_\lambda)$ of fixed points of the resolvent associated with f coincides with the set $argmin_{y\in X} f(y)$ of minimizers of f.

Lemma 3.6. [14] *Let (X, d) be a complete $CAT(0)$ space and $f : X \to (-\infty, \infty]$ be proper convex and lower semi-continuous. For any $\lambda > 0$, the resolvent J_λ of f is nonexpansive.*

Taking $T_3 = J_{\lambda_n}$ in (SM) iteration, we obtain the next result in $CAT(0)$ spaces as a direct consequence of Theorem 2.1.

Theorem 3.3. *Let (X, d) be a complete* CAT(0) *space. Let K be a nonempty closed convex bounded subset of X. Let $f : X \to (-\infty, \infty]$ be proper convex and lower semi-continuous. Let $T_1, T_2 : K \to K$ be two nonexpansive mappings such that $\Omega = F(T_1) \cap F(T_2) \cap argmin_{y\in X} f(y) \neq \emptyset$. Let $\{x_n\}$ be the sequence defined by*

$$\text{(SM0)} \qquad \begin{cases} z_n = J_{\lambda_n} x_n, \\ y_n = (1 - \beta_n - \beta'_n)x_n \oplus \beta_n T_2 z_n \oplus \beta'_n \varepsilon'_n, \\ x_{n+1} = (1 - \gamma_n - \gamma'_n)T_2 x_n \oplus \gamma_n T_1 y_n \oplus \gamma'_n \varepsilon_n, \end{cases}$$

for $n \geq 1$, where $\{\epsilon_n\}$ and $\{\epsilon'_n\}$ are bounded sequences in K and $\{\gamma_n\}_{n=1}^\infty$, $\{\beta_n\}_{n=1}^\infty$, $\{\gamma'_n\}_{n=1}^\infty$ and $\{\beta'_n\}_{n=1}^\infty$ are appropriate real sequences in $[0, 1]$ and $\{\lambda_n\}_{n=1}^\infty$ is a sequence in $\mathbb{R}$ such that

C1) $\sum_{n=1}^{\infty} \gamma'_n < \infty$ *and* $\sum_{n=1}^{\infty} \beta'_n < \infty$.
C2) $0 < \gamma \leq \beta_n \leq \beta_n + \beta'_n \leq \beta < 1$.
C3) $\gamma \leq \gamma_n + \gamma'_n$.
C4) $\lambda_n \geq \lambda > 0$, *for some* λ.

Then the sequence $\{x_n\}$ defined by (SM0) Δ-converges to a point of Ω.

Theorem 3.3 extends that of Cholamjiak et al. [10] in $CAT(0)$ spaces. In fact, we present a new perturbed proximal point algorithm for solving the convex minimization problem as well as the fixed point problem of nonexpansive mappings in $\mathrm{CAT}(0)$ spaces.

3.2. **Proximal Point Algorithm in $CAT(\kappa)$ spaces.** For a real number κ, a $CAT(\kappa)$ space is defined by a geodesic space (X, d) whose geodesic triangle is sufficiently thinner than the corresponding comparison triangle in a model space with curvature κ. The concept of such a space has been investigated by lots of researchers. In 2003, Kirk [17, 18] first proved the existence of fixed points for nonexpansive mappings in a $CAT(\kappa)$ space for $\kappa \geq 0$. In other to be short, for more detailed discussion on them, the reader can consult, for instance, [6, 9, 29] and references therein. Since many results and basic concepts in $CAT(\kappa)$ can be deduced frome those in $CAT(1)$ spaces, we now sufficiently state useful lemmas and definitions on $CAT(1)$ spaces. Throughout this section, we suppose that X is an admisible $CAT(1)$ space, i.e., $d(x, y) < \frac{\pi}{2}$, for all $x, y \in X$.

Let (X, d) be a $CAT(1)$ space. Let $f : X \to (-\infty, \infty]$ be be a proper convex and lower semi-continuous function. For any $\lambda > 0$, define the *Kimura-Kohsaka* resolvent of f (see [15, 16]) in $CAT(1)$ spaces as

$$R_\lambda(x) = argmin_{y \in X} \left[f(y) + \frac{1}{\lambda} \tan d(y, x) \, \sin d(y, x) \right],$$

for all $x \in X$. The mapping R_λ is well defined for all $\lambda > 0$ (see [15, 16]) and the set $F(R_\lambda)$ of fixed points of the resolvent associated with f coincides with the set $argmin_{y \in X} f(y)$ of minimizers of f.

Lemma 3.7. [16] *Let (X, d) be an admissible complete $CAT(1)$ space and $f : X \to (-\infty, \infty]$ be proper convex and lower semi-continuous. For any $\lambda > 0$, the resolvent R_λ of f is quasi-nonexpansive.*

Taking $T_3 = R_{\lambda_n}$ in (SM) iteration, we obtain the next result in $CAT(1)$ spaces as a direct consequence of Theorem 2.1 and Remark 2.1.

Theorem 3.4. *Let (X, d) be an admissible complete $\mathrm{CAT}(1)$ space. Let K be a nonempty closed convex bounded subset of X. Let $f : X \to (-\infty, \infty]$ be proper convex and lower semi-continuous. Let $T_1, T_2 : K \to K$ be two quasi-nonexpansive mappings such that $\Omega = F(T_1) \cap F(T_2) \cap argmin_{y \in X} f(y) \neq \emptyset$. Let $\{x_n\}$ be the sequence defined by*

$$\text{(SM1)} \qquad \begin{cases} z_n = R_{\lambda_n} x_n, \\ y_n = (1 - \beta_n - \beta'_n) x_n \oplus \beta_n T_2 z_n \oplus \beta'_n \varepsilon'_n, \\ x_{n+1} = (1 - \gamma_n - \gamma'_n) T_2 x_n \oplus \gamma_n T_1 y_n \oplus \gamma'_n \varepsilon_n, \end{cases}$$

for $n \geq 1$, where $\{\epsilon_n\}$ and $\{\epsilon'_n\}$ are bounded sequences in K and $\{\gamma_n\}_{n=1}^\infty$, $\{\beta_n\}_{n=1}^\infty$, $\{\gamma'_n\}_{n=1}^\infty$ and $\{\beta'_n\}_{n=1}^\infty$ are appropriate real sequences in $[0, 1]$ and $\{\lambda_n\}_{n=1}^\infty$ is a sequence in $\mathbb{R}$ such that

C1) $\sum_{n=1}^{\infty} \gamma'_n < \infty$ *and* $\sum_{n=1}^{\infty} \beta'_n < \infty$.
C2) $0 < \gamma \leq \beta_n \leq \beta_n + \beta'_n \leq \beta < 1$.
C3) $\gamma \leq \gamma_n + \gamma'_n$.
C4) $\lambda_n \geq \lambda > 0$, *for some* λ.

Then the sequence $\{x_n\}$ defined by (SM1) Δ-converges to a point of Ω.

Theorem 3.4 is also valid for $T_1, T_2 : K \to K$ be two nonexpansive mappings and, in this sense, Theorem 3.4 extends that of Pakkatanang et al. [29, 30] in $CAT(1)$ spaces for quasi-nonexpansive mappings. In fact, we present a new perturbed proximal point algorithm for solving the convex minimization problem as well as the fixed point problem of

nonexpansive mappings in $\mathrm{CAT}(1)$ spaces. A version for $\kappa > 0$ can be deduced following [29].

3.3. **Fixed point results for α-nonexpansive mappings.** Inspired by Goebel and Japón Pineda definiton in Banach space [13], we say that

Definition 3.4. *Let X be a $\mathrm{CAT_p}(0)$ space. $T : X \to X$* is *mean nonexpansive* (or α-nonexpansive) if, for some multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_1, \alpha_n > 0$, $\alpha_j \geq 0$ for all j, and $\alpha_1 + \cdots + \alpha_n = 1$, we have

$$\bigoplus_{\mathbf{j=1}}^{\mathbf{n}} \alpha_j d(T^j x, T^j y) \leq d(x, y)$$

for all $x, y \in X$.

Since the first coefficient $\alpha_1 > 0$, T satisfies the Lipschitz condition

$$d(Tx, Ty) \leq \frac{1}{\alpha_1} d(x, y).$$

Trivially, $T_\alpha : X \to X$ defined by $T_\alpha(x) = \oplus_{i=1}^n \alpha_i T^i x$ is a nonexpansive mapping.

In this section, we shall prove some fixed point theorems for α-nonexpansive mapping in a $\mathrm{CAT_p}(0)$ spaces ($\mathrm{CAT}(0)$ spaces).

Theorem 3.5. *Let K be a closed convex subset of a $\mathrm{CAT_p}(0)$ space. Let $T : K \to K$ an α-nonexpansive mapping with $\alpha = (\alpha_1, ..., \alpha_n)$ such that $\alpha_1 > 2^{\frac{1}{1-n}}$. Then the set of fixed points of T, $F(T)$, coincides with the set of fixed point of T_α, $F(T_\alpha)$.*

Proof. Proof is similar to [21, Theorem 3.1] Trivially, $F(T) \subset F(T_\alpha)$. For contrary, we check that

$$\begin{aligned} d(T^j x, Tx) &\leq \frac{1}{\alpha_1} d(T^{j-1}x, x) \leq \frac{1}{\alpha_1} \left(d(T^{j-1}x, Tx) + d(Tx, x) \right) \\ &\leq \frac{1}{\alpha_1^2} d(T^{j-2}x, x) + \frac{1}{\alpha_1} d(Tx, x) \\ &\vdots \\ &\leq \left(\frac{1}{\alpha_1^{j-1}} + ... + \frac{1}{\alpha_1^2} + \frac{1}{\alpha_1} \right) d(Tx, x), \end{aligned} \tag{3.5}$$

for $j = 2, ..., n$.

If $x \in F(T_\alpha)$, then, by Lemma 1.3 and (3.5),

$$\begin{aligned} d(x, Tx) = d(T_\alpha(x), Tx) &\leq \sum_{j=1}^n \alpha_j d(T^j x, Tx) \\ &\leq \sum_{j=2}^n \alpha_j \left(\frac{1}{\alpha_1^{j-1}} + ... + \frac{1}{\alpha_1^2} + \frac{1}{\alpha_1} \right) d(Tx, x) \\ &= \left(\sum_{j=1}^{n-1} \frac{1 - \alpha_1 - \alpha_2 - ... - \alpha_j}{\alpha_1^j} \right) d(Tx, x) \\ &\leq \frac{1 - \alpha_1^{n-1}}{\alpha_1^{n-1}} d(Tx, x) \end{aligned} \tag{3.6}$$

Since $\alpha > 2^{\frac{1}{1-n}}$, $\frac{1-\alpha_1^{n-1}}{\alpha_1^{n-1}} < 1$ and $Tx = x$. □

Lemma 3.8. *In same conditions of Theorem 3.5. Let $\{x_m\}$ be a bounded sequence in K. Then $d(x_m, Tx_m) \to 0$ if and only if $d(x_m, T_\alpha(x_m)) \to 0$ as $m \to \infty$.*

Proof. Suppose $d(x_m, Tx_m) \to 0$ as $m \to \infty$. By (3.5),

$$\begin{aligned} d(T^j x_m, x_m) &\leq d(T^j x_m, Tx_m) + d(Tx_m, x_m) \\ &\leq \left(\frac{1}{\alpha_1^{j-1}} + ... + \frac{1}{\alpha_1^2} + \frac{1}{\alpha_1} + 1 \right) d(Tx_m, x_m) \to 0, \end{aligned} \tag{3.7}$$

as $m \to \infty$, for $j = 2, ..., n$. For $j = 1$ is trivial. Now,

$$d(T_\alpha(x_m), x_m) \leq \sum_{j=1}^{n} \alpha_j d(T^j x_m, x_m) \to 0,$$

as $m \to \infty$.

Conversely, Suppose $d(x_m, T_\alpha(x_m)) \to 0$ as $m \to \infty$. Using a similar procedure to (3.6),

$$\begin{aligned} d(x_m, Tx_m) &\leq d(x_m, T_\alpha(x_m)) + d(Tx_m, T_\alpha(x_m)) \\ &\leq d(x_m, T_\alpha(x_m)) + \frac{1 - \alpha_1^{n-1}}{\alpha_1^{n-1}} d(Tx_m, x_m) \to 0, \end{aligned}$$

i.e., $d(Tx_m, x_m) \leq \left(2 - \frac{1}{\alpha_1^{n-1}}\right) d(x_m, T_\alpha(x_m)) \to 0$, as $m \to \infty$. □

Corollary 3.1. *Let (X, d) be a complete $\mathrm{CAT_p}(0)$ space, with $p \geq 2$. Let K be a nonempty closed convex bounded subset of X. Let $T_1, T_2, T_3 : K \to K$ be three α-nonexpansive mappings with $\alpha = (\alpha_1, ..., \alpha_n)$ such that $\alpha_1 > 2^{\frac{1}{1-n}}$ and such that $\Omega = F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset$. Let $\{x_n\}$ be the sequence defined by (SM) such that*

C1) $\sum_{n=1}^{\infty} \gamma_n' < \infty$ *and* $\sum_{n=1}^{\infty} \beta_n' < \infty$.
C2) $0 < \gamma \leq \beta_n \leq \beta_n + \beta_n' \leq \beta < 1$.
C3) $\gamma \leq \gamma_n + \gamma_n'$.

Then the following hold:

(i) $\lim_{n\to\infty} d(x_n, x^*)$ *exists, for any fixed point x^* of T.*
(ii) $\lim_{n\to\infty} d(x_n, T_i x_n) = 0$, *i.e., $\{x_n\}$ is an approximate fixed point sequence of T_i, for $i = 1, 2, 3$.*

Corollary 3.2. *Let X, K, T_1, T_2, T_3, $\{x_n\}$ satisfy the hypothesis of Corollary 3.1. Then the sequence $\{x_n\}$ defined by (SM) Δ-converges to a point of Ω.*

4. Numerical examples.

In this section, we provide the numerical examples to illustrate its performance and to compare propose iteration with existing methods.

Example 4.1. Consider $X = \mathbb{R}^2$ equipped with the Euclidean norm. Let $x = (x_1, x_2) \in \mathbb{R}^2$, then the squared distance of x from the origin is

$$||x||^2 = x_1^2 + x_2^2.$$

Consider K as the closed unit disk:

$$K = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1\right\}$$

FIGURE 1. Convergence behavior for Ex. 1.

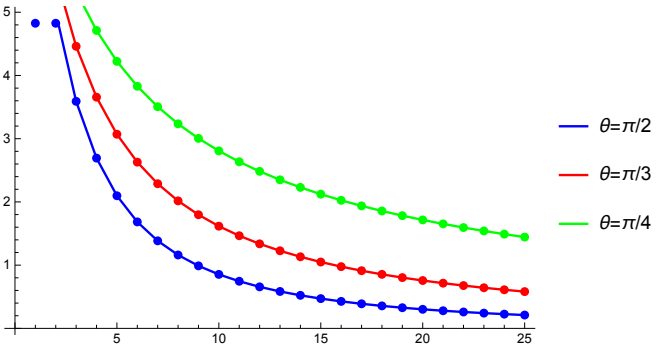


TABLE 1. Error convergence behavior for Ex. 1

Error	$\theta = \pi/2$	$\theta = \pi/3$	$\theta = \pi/4$
$\|x^2_{(1)}\|$	7.07107	7.07107	7.07107
$\|x^2_{(3)}\|$	3.59117	4.46127	5.33126
$\|x^2_{(5)}\|$	2.09783	3.07041	4.2235
$\|x^2_{(7)}\|$	1.38474	2.28779	3.50592
$\|x^2_{(9)}\|$	0.988957	1.79742	3.00499
$\|x^2_{(11)}\|$	0.745293	1.4653	2.63483
$\|x^2_{(13)}\|$	0.583899	1.22732	2.34957
$\|x^2_{(15)}\|$	0.471072	1.04946	2.12257
$\|x^2_{(17)}\|$	0.388866	0.912109	1.93738
$\|x^2_{(19)}\|$	0.326979	0.80324	1.78322
$\|x^2_{(21)}\|$	0.279137	0.715097	1.65276
$\|x^2_{(23)}\|$	0.241334	0.642464	1.54085
$\|x^2_{(25)}\|$	0.210907	0.581713	1.44372

which is bounded, closed, and convex in X. We define mapping $\mathrm{Rot}_\theta : K \to K$ by:

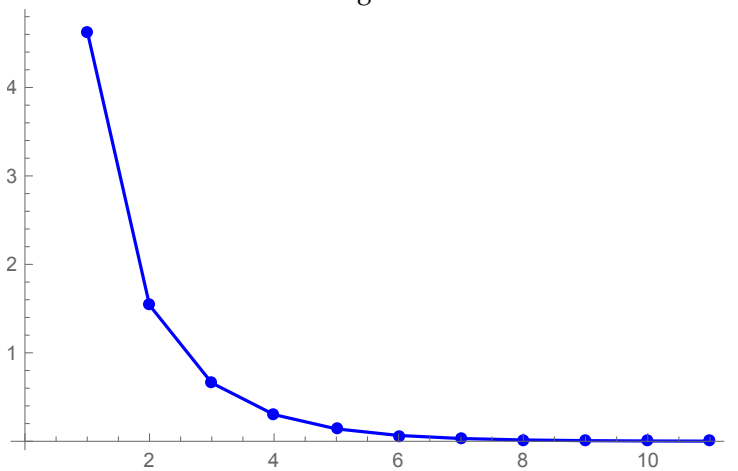
$$\mathrm{Rot}_\theta(x_1, x_2) = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

For $\theta = \frac{\pi}{4}$ and $T_1 = T_2 = T_3 = \mathrm{Rot}_\theta$, our algorithm is the following:

$$\begin{cases} z_{(n)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} x_{(n)} \\ y_{(n)} = \begin{pmatrix} 1-\beta_n & \beta_n \\ -\beta_n & 1-\beta_n \end{pmatrix} x_{(n)} \\ x_{(n+1)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1-2\gamma_n\beta_n & 1 \\ -1 & 1-2\gamma_n\beta_n \end{pmatrix} x_{(n)} \end{cases}$$

It is easy to see that Rot_θ is nonexpansive. Clearly, zero is the only fixed point of the mapping Rot_θ. Let $\gamma_n = 1 - \frac{1}{\sqrt{n+1}}$ and $\beta_n = \frac{1}{n+1}$ for all $n \in \mathbb{N}$ and without perturbations. By using Wolfram Mahtematica 10, we computed the iterates of (SM0) for an initial point $x_{(1)} = (250, 250)$ for $\theta = \frac{\pi}{4}, \frac{\pi}{8}$ and $\frac{\pi}{14}$. The convergence behavior of all iterations for approximating the fixed point $(0, 0)$ are given in Figure 1 and Table 1.

FIGURE 2. Erro convergence behavior for Ex. 2.



Example 4.2. Consider $X = \mathbb{R}^3$ equipped with the Euclidean norm. Define the nonexpansive mappings $T_1, T_2, T : X \to X$ as follows:

$$T_1(x,y,z) = T_2(x,y,z) = T(x,y,z) = \left(\frac{1}{5}(3x-2y-8), \frac{1}{5}(-2x+3y-8), \frac{z}{2}\right).$$

We define $f : \mathbb{R}^3 \to \mathbb{R}$ by

$$f(u) = \frac{1}{2}\|Au-b\|^2, \qquad u \in \mathbb{R}^3,$$

where

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}, \qquad b = (-4,-4,4).$$

The function f is proper convex and lower semi-continuous. Hence for $\lambda > 0$, we have

$$J_\lambda(u) = (I + \lambda A^*A)^{-1}(u + \lambda A^*b).$$

Note that

$$\Omega = Fix(T) \cap argmin_{y \in X} f(y) = \{(x,y,z) \in \mathbb{R}^3 : x + y + 4 = z = 0\}.$$

The algorithm (SM0) becomes:

$$\begin{cases} z_n = (I + \lambda_n A^*A)^{-1}(x_n + \lambda_n A^*b), \\ y_n = (1 - \beta_n - \beta'_n)x_n \oplus \beta_n T_2 z_n \oplus \beta'_n \varepsilon'_n, \\ x_{n+1} = (1 - \gamma_n - \gamma'_n)T_2 x_n \oplus \gamma_n T_1 y_n \oplus \gamma'_n \varepsilon_n, \end{cases} \tag{4.8}$$

We choose $\lambda_n = 0.35$, $\gamma'_n = \beta'_n = 0$ $\gamma_n = 1 - \frac{1}{\sqrt{n+1}}$ and $\beta_n = \frac{1}{n+1}$ for all $n \in \mathbb{N}$. By using Wolfram Mahtematica 10, we computed the iterates of (SM0) for an initial point $x_{(1)} = (-1,2,3)$. The convergence behavior of all iterations for approximating the fixed point $(-3.5,-0.5,0)$ are given in Figure 2 and Table 2.

TABLE 2. Error convergence behavior for Ex. 2

Iteration	x_i	y_i	z_i	Error
0	-1.	2.	3.	4.63681
1	-3.06154	-0.0615429	1.40869	1.53912
2	-3.42143	-0.421434	0.649992	0.65942
3	-3.48523	-0.485235	0.300787	0.301511
4	-3.497	-0.497005	0.140183	0.140247
5	-3.49931	-0.499314	0.0657979	0.0658051
6	-3.49981	-0.499815	0.0310751	0.0310762
7	-3.49994	-0.49994	0.0147528	0.0147531
8	-3.49998	-0.499978	0.00703466	0.00703473
9	-3.49999	-0.499991	0.00336684	0.00336687
10	-3.5	-0.499996	0.00161652	0.00161653
⋮	⋮	⋮	⋮	⋮
20	-3.5	-0.5	1.173×10^{-6}	1.173×10^{-6}

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