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Dedicated to Prof. Ioan A. Rus on the occasion of his 85^{th} anniversary

On a Steklov eigenvalue problem associated with the (p, q)-Laplacian

LUMINIŢA BARBU[a] and GHEORGHE MOROŞANU[b,c]

ABSTRACT. Consider in a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$, with smooth boundary $\partial\Omega$, the following eigenvalue problem

$$\mathcal{A}u := -\Delta_p u - \Delta_q u = \lambda a(x) \mid u \mid^{r-2} u \ \text{ in } \Omega,$$
$$\left(\mid \nabla u \mid^{p-2} + \mid \nabla u \mid^{q-2} \right) \frac{\partial u}{\partial \nu} = \lambda b(x) \mid u \mid^{r-2} u \ \text{ on } \partial\Omega,$$

where $1 < r < q < p < \infty$ or $1 < q < p < r < \infty$; $r \in \left(1, \frac{p(N-1)}{N-p}\right)$ if $p < N$ and $r \in (1, \infty)$ if $p \geq N$; $a \in L^\infty(\Omega),\ b \in L^\infty(\partial\Omega)$ are given nonnegative functions satisfying

$$\int_\Omega a\, dx + \int_{\partial\Omega} b\, d\sigma > 0.$$

Under these assumptions we prove that the set of all eigenvalues of the above problem is the interval $[0, \infty)$. Our result complements those previously obtained by Abreu, J. and Madeira, G., [*Generalized eigenvalues of the $(p, 2)-$Laplacian under a parametric boundary condition*, Proc. Edinburgh Math. Soc., **63** (2020), No. 1, 287–303], Barbu, L. and Moroşanu, G., [*Full description of the eigenvalue set of the (p, q)-Laplacian with a Steklov-like boundary condition*, J. Differential Equations, in press], Barbu, L. and Moroşanu, G., [*Eigenvalues of the negative (p, q)- Laplacian under a Steklov-like boundary condition*, Complex Var. Elliptic Equations, **64** (2019), No. 4, 685–700], Fărcăşeanu, M., Mihăilescu M. and Stancu-Dumitru, D., [*On the set of eigen-values of some PDEs with homogeneous Neumann boundary condition*, Nonlinear Anal. Theory Methods Appl., **116** (2015), 19–25], Mihăilescu, M., [*An eigenvalue problem possesing a continuous family of eigenvalues plus an isolated eigenvale*, Commun. Pure Appl. Anal., **10** (2011), 701–708], Mihăilescu, M. and Moroşanu, G., [*Eigenvalues of $-\triangle_p - \triangle_q$ under Neumann boundary condition*, Canadian Math. Bull., **59** (2016), No. 3, 606–616].

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary $\partial\Omega$. Consider the eigenvalue problem

$$\text{(1.1)} \qquad \begin{cases} \mathcal{A}u := -\Delta_p u - \Delta_q u = \lambda a(x) \mid u \mid^{r-2} u \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu_{\mathcal{A}}} := \left(\mid \nabla u \mid^{p-2} + \mid \nabla u \mid^{q-2} \right) \frac{\partial u}{\partial \nu} = \lambda b(x) \mid u \mid^{r-2} u \ \text{ on } \partial\Omega, \end{cases}$$

where ν is the unit outward normal to $\partial\Omega$. As usual, Δ_p denotes the p-Laplacian, i.e., $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$.

Throughout this paper, the following hypotheses will be assumed

(h_{pqr}) $\quad 1 < r < q < p < \infty$ or $1 < q < p < r < \infty$; $r \in \left(1, \frac{p(N-1)}{N-p}\right)$ if $1 < p < N$ and $r \in (1, \infty)$ if $p \geq N$;

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Corresponding author: Gheorghe Moroşanu; morosanu@math.ubbcluj.ro

(h_{ab}) $a \in L^\infty(\Omega)$ and $b \in L^\infty(\partial\Omega)$ are given nonnegative functions satisfying

$$\int_\Omega a(x)\,dx + \int_{\partial\Omega} b(\sigma)\,d\sigma > 0. \tag{1.2}$$

Remark 1.1. Regarding the assumption $r \in \left(1, \frac{p(N-1)}{N-p}\right)$ if $1 < p < N$ and $r \in (1,\infty)$ if $p \geq N$, we point out that this is directly related to the well-known embeddings $W^{1,p}(\Omega) \hookrightarrow L^r(\Omega)$ which hold in the cases: (i) $1 \leq r \leq p^* = pN/(N-p)$, if $1 < p < N$; (j) $p \leq r < \infty$, if $p = N$; (k) $r = \infty$, if $p > N$. Moreover, these embeddings are compact when $1 \leq r < p^*$ in case (i), all r in case (j), and when reinterpreted as $W^{1,p}(\Omega) \hookrightarrow C^1(\overline{\Omega})$ in case (k). We also have trace compact embeddings $W^{1,p}(\Omega) \hookrightarrow L^r(\partial\Omega)$ for all $1 \leq p \leq r < p(N-1)/(N-p)$ if $1 \leq p < N$, and similarly as before in the other ranges of p (see [2], [6, Section 9.3]).

The solution u of (1.1) will be sought in the space $W := W^{1,p}(\Omega)$, so the normal derivative $\frac{\partial u}{\partial \nu_A}$ exists in a trace sense, and the above problem is satisfied in the distribution sense. According to a Green type formula (see [7], p. 71), one can define the eigenvalues of our problems in term of weak solution as follows

Definition 1.1. $\lambda \in \mathbb{R}$ is an eigenvalue of problem (1.1) if there exists $u_\lambda \in W \setminus \{0\}$ such that

$$\begin{aligned} &\int_\Omega \Big(\mid \nabla u_\lambda \mid^{p-2} + \mid \nabla u_\lambda \mid^{q-2} \Big) \nabla u_\lambda \cdot \nabla w \, dx \\ &\quad = \lambda \Big(\int_\Omega a \mid u_\lambda \mid^{q-2} u_\lambda w \, dx + \int_{\partial\Omega} b \mid u_\lambda \mid^{q-2} u_\lambda w \, d\sigma \Big) \; \forall\, w \in W. \end{aligned} \tag{1.3}$$

According to the above remark, all the integral terms in Definition 1.1 make sense.

Conversely, by virtue of the same Green formula, if λ is an eigenvalue then any eigenfunctions $u_\lambda \in W \setminus \{0\}$ corresponding to it satisfies problem (1.1) in the distribution sense. Our goal is to determine the set of all eigenvalues of problem (1.1).

The main result of this paper is given by the following theorem

Theorem 1.1. *Assume that* (h_{pqr}) *and* (h_{ab}) *above are fulfilled. Then the set of eigenvalues of problem* (1.1) *is* $[0,\infty)$.

Remark 1.2. It is worth mentioning that if $b \equiv 0$ (Neumann boundary condition) and $1 < p < N$, Theorem (1.1) holds if the condition $1 < r < p(N-1)/(N-p)$ is replaced by the weaker condition $1 < r < pN/(N-p)$.

In the case $q = r = 2,\ a \equiv 1,\ b \equiv 0$, the set of eigenvalues for problem (1.1) was completely described by M. Mihăilescu [11] (for $p > 2$) and M. Fărcăşeanu, M. Mihăilescu and D. Stancu-Dumitru [9] (for $p \in (1,2)$). Problem (1.1) with $q = r = 2,\ p \in (1,\infty) \setminus \{2\}$, was studied by J. Abreu and G. Madeira [1]. Note also that problem (1.1) with $p \in (1,\infty),\ r = q \in (2,\infty),\ p \neq q,\ a \equiv 1,\ b \equiv 0$, was investigated by M. Mihăilescu and G. Moroşanu in [12]; also, problem (1.1) with $p, q \in (1,\infty),\ p \neq q, r = q$ was solved by L. Barbu and G. Moroşanu [3, 4].

2. Preliminary results

Choosing $w = u_\lambda$ in (1.3) shows that the eigenvalues of problem (1.1) cannot be negative. It is also obvious that $\lambda_0 = 0$ is an eigenvalue of this problem and the corresponding eigenfunctions are the nonzero constant functions. So any other eigenvalue belongs to $(0,\infty)$.

If we assume that $\lambda > 0$ is an eigenvalue of problem (1.1) and choose $w \equiv 1$ in (1.3) we deduce that every eigenfunction u_λ corresponding to λ satisfies the equation

$$\int_\Omega a \mid u_\lambda \mid^{r-2} u_\lambda \, dx + \int_{\partial\Omega} b \mid u_\lambda \mid^{r-2} u_\lambda \, d\sigma = 0. \tag{2.4}$$

So all eigenfunctions corresponding to positive eigenvalues necessarily belong to the set

$$\mathcal{C}_r := \Big\{ u \in W;\ \int_\Omega a \mid u \mid^{r-2} u \, dx + \int_{\partial\Omega} b \mid u \mid^{r-2} u \, d\sigma = 0 \Big\}. \tag{2.5}$$

This set is a symmetric cone. Moreover, $\mathcal{C}_r$ is a weakly closed subset of $W := W^{1,p}(\Omega)$. Indeed, let $\big(u_n\big)_n \subset \mathcal{C}_r$ such that $u_n \rightharpoonup u_0$ in W. Since $W \hookrightarrow L^r(\Omega)$ and $W \hookrightarrow L^r(\partial\Omega)$ compactly, there exists a subsequence of $\big(u_n\big)_n$, also denoted $\big(u_n\big)_n$, such that

$$u_n \to u_0 \text{ in } L^r(\Omega),\ u_n \to u_0 \text{ in } L^r(\partial\Omega).$$

By Lebesgue's Dominated Convergence Theorem (see also [6, Theorem 4.9]) we obtain $u_0 \in \mathcal{C}_r$. In addition, $\mathcal{C}_r$ has nonzero elements (see [4, Section 2]).

Let $\mathcal{K}_r : W \to \mathbb{R}$ be the C^1-functional defined by

$$\mathcal{K}_r(u) := \int_\Omega a \mid u \mid^r dx + \int_{\partial\Omega} b \mid u \mid^r d\sigma \ \forall\, u \in W. \tag{2.6}$$

Remark 2.3. If for some $\lambda > 0$, $u \in W \setminus \{0\}$ satisfies the equation

$$\int_\Omega \Big(\mid \nabla u \mid^p + \mid \nabla u \mid^q \Big) \, dx = \lambda \mathcal{K}_r(u),$$

then u cannot be a constant function (see assumption (1.2)) and so $\mathcal{K}_r(u) > 0$. Therefore, denoting $\Gamma_1(u) := \{x \in \Omega;\ a(x)u(x) \neq 0\}$, $\Gamma_2(u) := \{x \in \partial\Omega;\ b(x)u(x) \neq 0\}$, we see that either $|\Gamma_1(u)|_N > 0$ or $|\Gamma_2(u)|_{N-1} > 0$.

Obviously u_λ corresponding to any eigenvalue $\lambda > 0$ cannot be a constant function (see (1.3) with $v = u_\lambda$ and (1.2)).

The following lemmas are useful in the proof of Theorem 1.1.

Lemma 2.1. *If hypotheses (h_{ab}) hold and $r \in \Big(1, \frac{p(N-1)}{N-p}\Big)$ for $1 < p < N$ and $r \in (1, \infty)$ for $p \geq N$, then the following norm is equivalent with the usual norm (denoted by $\| \cdot \|_W$) of the Sobolev space $W = W^{1,p}(\Omega)$*

$$\| u \|_r := \| \nabla u \|_{L^p(\Omega)} + \big(\mathcal{K}_r(u)\big)^{\frac{1}{r}} \ \forall\, u \in W. \tag{2.7}$$

Proof. This fact follows from [8, Proposition 3.9.55]. Indeed, $\big(\mathcal{K}_r(u)\big)^{\frac{1}{r}}$ is a seminorm which satisfies the two requirements of that proposition

(j) $\exists d > 0$ such that $\big(\mathcal{K}_r(u)\big)^{\frac{1}{r}} \leq d \| u \|_W \quad \forall u \in W$, and

(jj) if $u =$ constant, then $\big(\mathcal{K}_r(u)\big)^{\frac{1}{r}} = 0$ implies $u \equiv 0$. □

Lemma 2.2. *If hypotheses (h_{ab}) hold and $r \in \Big(1, \frac{p(N-1)}{N-p}\Big)$ for $1 < p < N$ and $r \in (1, \infty)$ for $p \geq N$, then there exists a positive constant C which depends on p, r, N and Ω, such that for every $u \in \mathcal{C}_r$*

$$\big(\mathcal{K}_r(u)\big)^{\frac{1}{r}} \leq C \| \nabla u \|_{L^p(\Omega)} . \tag{2.8}$$

Proof. Suppose that (2.8) is not true. Then we can find a sequence $\big(u_n\big)_n \subset \mathcal{C}_r \subset W$ such that $\mathcal{K}_r(u_n) = 1$ and

$$\| \nabla u_n \|_{L^p(\Omega)} \leq \frac{1}{n} \ \forall\, n \geq 1. \tag{2.9}$$

Clearly, from Lemma 2.1 and (2.9), the sequence $\big(u_n\big)_n$ is bounded in W, thus, by passing to a subsequence if necessary, we may assume that there exists $u_0 \in W$ such that $u_n \rightharpoonup u$ as $n \to \infty$. Since W is embedded compactly in $L^r(\Omega)$ and $L^r(\partial\Omega)$ we have that

$$u_n \to u_0 \text{ in } L^r(\Omega),\ u_n \to u_0 \text{ in } L^r(\partial\Omega).$$

As $\mathcal{K}_r(u_n) = 1\ \forall\ n \geq 1$ and $\big(u_n\big)_n \subset \mathcal{C}_r$ we have $\mathcal{K}_r(u_0) = 1$ and $u_0 \in \mathcal{C}_r$. On the other hand, from (2.9), the sequence $\big(\parallel \nabla(u_n) \parallel_{L^p(\Omega)}\big)_n$ tends to 0. Therefore $\nabla(u_0) \equiv 0$, so u_0 is constant and belongs to $\mathcal{C}_r$, hence $u_0 \equiv 0$. This contradicts the fact that $\mathcal{K}_r(u_0) = 1$. □

3. Proof of Theorem 1.1

We have already stated that $\lambda_0 = 0$ is an eigenvalue of problem (1.1) and any other eigenvalue of this problem belongs to $(0, \infty)$.

In what follows we fix $\lambda > 0$ and define $\mathcal{J}_\lambda : W \to \mathbb{R}$,

$$\mathcal{J}_\lambda(u) = \frac{1}{p}\int_\Omega \mid \nabla u \mid^p dx + \frac{1}{q}\int_\Omega \mid \nabla u \mid^q dx - \frac{\lambda}{r}\mathcal{K}_r(u), \tag{3.10}$$

which is a C^1 functional whose derivative is given by

$$\begin{aligned} \langle \mathcal{J}_\lambda'(u), w\rangle = &\int_\Omega \mid \nabla u \mid^{p-2} \nabla u \cdot \nabla w\, dx + \int_\Omega \mid \nabla u \mid^{q-2} \nabla u \cdot \nabla w\, dx \\ &- \lambda\Big(\int_\Omega a \mid u \mid^{r-2} uw\, dx + \int_{\partial\Omega} b \mid u \mid^{r-2} uw\, d\sigma\Big)\ \ \forall u, w \in W. \end{aligned} \tag{3.11}$$

So, according to Definition 1.1, $\lambda > 0$ is an eigenvalue of problem (1.1) if and only if there exists a critical point $u_\lambda \in W \setminus \{0\}$ of $\mathcal{J}_\lambda$, i. e. $\mathcal{J}_\lambda'(u_\lambda) = 0$.

The proof of Theorem 1.1 will follow as a consequence of several intermediate results.

We shall discuss two distinct cases.

Case 1: (h_{pqr}) **with** $1 < r < q < p < \infty$ **and** (h_{ab})

The following result shows that, for every $\lambda > 0$, the functional defined in (3.11), restricted to the subset $\mathcal{C}_r \subset W$, is coercive.

Lemma 3.3. *If hypotheses* (h_{pqr}) *with* $1 < r < p < \infty$ *and* (h_{ab}) *hold, then for every* $\lambda > 0$, *we have* $\lim\limits_{\|u\|_W \to\infty, u\in\mathcal{C}_r} \mathcal{J}_\lambda(u) = \infty$.

Proof. We know from Lemma 2.2 (for $p = r$) that there exists a positive constant C such that (2.8) holds. Using Hölder's inequality we have,

$$\mathcal{K}_r(u) \leq C^r \parallel \nabla u \parallel^r_{L^r(\Omega)} \leq C^r \mid \Omega \mid_N^{\frac{p-r}{p}} \parallel \nabla u \parallel^{\frac{r}{p}}_{L^p(\Omega)}\ \ \forall\, u \in \mathcal{C}_r. \tag{3.12}$$

Here by $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$. So, we obtain from (3.12) that

$$\mathcal{J}_\lambda(u) \geq \frac{1}{p} \parallel \nabla u \parallel^p_{L^p(\Omega)} - \frac{\lambda}{r} C^r \mid \Omega \mid_N^{\frac{p-r}{p}} \parallel \nabla u \parallel^{\frac{r}{p}}_{L^p(\Omega)}\ \ \forall\, u \in \mathcal{C}_r. \tag{3.13}$$

Taking into account Lemma 2.1, Lemma 2.2 and (3.12), we can see that $\parallel u \parallel_W \to \infty,\ u \in \mathcal{C}_r$ if and only if $\parallel \nabla u \parallel_{L^p(\Omega)} \to \infty$. Since $r < p$, we derive from (3.13) that $\mathcal{J}_\lambda(u) \to \infty$ if $\parallel u \parallel_W \to \infty,\ u \in \mathcal{C}_r$, therefore $\mathcal{J}_\lambda$ is indeed coercive on $\mathcal{C}_r$. □

Proposition 3.1. *In Case 1, every number* $\lambda > 0$ *is an eigenvalue of problem* (1.1).

Proof. Note that $\mathcal{C}_r$ is a weakly closed subset of the reflexive Banach space W, and functional $\mathcal{J}_\lambda$ is coercive (see Lemma 3.3) and weakly lower semicontinuous on $\mathcal{C}_r$ with respect to the norm of W. Standard results in the calculus of variations (see, e.g., [13, Theorem 1.2]) ensures the existence of a global minimizer $u_* \in \mathcal{C}_r$ for $\mathcal{J}_\lambda$, i.e., $\mathcal{J}_\lambda(u_*) = \min_{\mathcal{C}_r} \mathcal{J}_\lambda$.

Next, we are going to prove that $u_* \not\equiv 0$.

Let us choose $u_0 \in \mathcal{C}_r \setminus \{0\}$ such that $\mathcal{K}_r(u_0) > 0$ (see [4, Section 2] for the construction of such a function). Note that the function

$$t \mapsto \mathcal{J}_\lambda(tu_0) = t^r\Big(\frac{t^{p-r}}{p}\int_\Omega |\nabla u_0|^p\ dx + \frac{t^{q-r}}{q}\int_\Omega |\nabla u_0|^q\ dx - \frac{\lambda}{r}\mathcal{K}_r(u_0)\Big),$$

is negative for $t = t_0 > 0$ small enough. Therefore, as $tu_0 \in \mathcal{C}_r \setminus \{0\}$, we have $\mathcal{J}_\lambda(u_*) < 0$, so $u_* \not\equiv 0$.

Next, we are going to show that the global minimizer u_* for $\mathcal{J}_\lambda$ restricted to $\mathcal{C}_r$ is a critical point of $\mathcal{J}_\lambda$ considered on the whole space W, i. e., $\mathcal{J}'_\lambda(u_*) = 0$, in other words, u_* is an eigenfunction of problem (1.1) corresponding to λ.

In order to show this we make use of an argument similar to that used in [5] and [3, Lemma 3]. In this respect, we fix $v \in \mathrm{Lip}(\Omega)$ arbitrarily. For each $n \in \mathbb{N}^*$ define $f_n : \mathbb{R} \to \mathbb{R}$,

$$f_n(s) := \mathcal{K}_r\Big(u_* + \frac{1}{n}v + s\Big) = \int_\Omega a\Big|u_* + \frac{1}{n}v + s\Big|^r dx + \int_{\partial\Omega} b\Big|u_* + \frac{1}{n}v + s\Big|^r d\sigma.$$

It is easily seen that f_n is coercive, since we have

$$f_n(s) \geq \frac{|s|^r}{2^r}\Big(\int_\Omega a\,dx + \int_{\partial\Omega} b\,d\sigma\Big) - \int_\Omega a\Big|u_* + \frac{1}{n}v\Big|^r dx - \int_{\partial\Omega} b\Big|u_* + \frac{1}{n}v\Big|^r d\sigma.$$

We have used the inequality

$$|x|^r \leq (|x+y| + |y|)^r \leq 2^r(|x+y|^r + |y|^r)\ \forall\, x, y \in \mathbb{R},\ r > 1.$$

Moreover, function f_n is continuously differentiable on $\mathbb{R}$ (see [10, Theorem 2.27]) and convex (its derivative is an increasing function). Therefore, for all $n \in \mathbb{N}^*$, f_n admits a minimum point s_n, such that $f'_n(s_n) = 0$, that is

$$\begin{aligned}\int_\Omega a\Big|u_* + \frac{1}{n}v + s_n\Big|^{r-2}\Big(u_* + \frac{1}{n}v + s_n\Big)\,dx \\ + \int_{\partial\Omega} b\Big|u_* + \frac{1}{n}v + s_n\Big|^{r-2}\Big(u_* + \frac{1}{n}v + s_n\Big)\,d\sigma = 0.\end{aligned} \tag{3.14}$$

We denote

$$u_n := u_* + \frac{1}{n}v + s_n\ \forall\, n \in \mathbb{N}^*. \tag{3.15}$$

According to (3.14), $\big(u_n\big)_n \subset \mathcal{C}_r$.

Next, we claim that the sequence $\big(ns_n\big)_n$ is bounded. Arguing by contradiction, let us assume that, up to a sequence, $ns_n \to \infty$ or $ns_n \to -\infty$ as $n \to \infty$. Taking into account that $v \in \mathrm{Lip}(\Omega)$ there exists N_1 large enough such that we have either

$$v(\cdot) + ns_n > 0 \text{ in } \Omega,\ \text{ or } v(\cdot) + ns_n < 0 \text{ in } \Omega\ \forall\, n \geq N_1.$$

Since the function $\tau \mapsto |u_* + \tau|^{r-2}(u_* + \tau)$ is strictly increasing on $\mathbb{R}$, we get

$$\begin{aligned}0 &= \int_\Omega a\,|u_n|^{r-2}\,u_n\,dx + \int_{\partial\Omega} b\,|u_n|^{r-2}\,u_n\,d\sigma \\ &> \int_\Omega a\,|u_*|^{r-2}\,u_*\,dx + \int_{\partial\Omega} b\,|u_*|^{r-2}\,u_*\,d\sigma = 0\ \forall n \geq N_1,\end{aligned} \tag{3.16}$$

if $v(\cdot) + ns_n > 0$ in Ω, or the reverse inequality in the later case, when $v(\cdot) + ns_n < 0$ in Ω. In both cases we get a contradiction.

We point out that the inequality in (3.16) is strict. Indeed, (1.2) implies that either $|\{x \in \Omega;\ a(x) > 0\}|_N > 0$ or $a = 0$ a.e. in Ω and $|\{x \in \partial\Omega;\ b(x) > 0\}|_{N-1} > 0$, hence we can not have equality above, instead of ″>″.

Consequently, $\big(ns_n\big)_n$ should be bounded. This implies that there exists $S \in \mathbb{R}$ such that, up to a subsequence, $ns_n \to S$ as $n \to \infty$. Therefore, on a subsequence, we have

$$n\big(u_n - u_*\big) \to v + S \text{ and } u_n \to u_* \text{ in } W \text{ as } n \to \infty. \tag{3.17}$$

In addition, there exists $N_2 \in \mathbb{N}^*$ such that $u_n \not\equiv 0 \; \forall \, n \geq N_2$. By using the minimality of u_* and the fact that $u_n \in \mathcal{C}_r \setminus \{0\} \; \forall \, n \geq N_2$, we obtain that

$$0 \leq \lim_{n\to\infty} \frac{\mathcal{J}_\lambda(u_n) - \mathcal{J}_\lambda(u_*)}{(1/n)}. \tag{3.18}$$

On the other hand,

$$n\big(\mathcal{J}_\lambda(u_n) - \mathcal{J}_\lambda(u_*)\big) = \langle \mathcal{J}_\lambda'(u_*), n(u_n - u_*)\rangle + o(n; u_*, v), \tag{3.19}$$

where $o(n; u_*, v)$ is a notation for the term which tends to zero in the definition of the Fréchet derivative of $\mathcal{J}_\lambda$ at u_*, that is $o(n; u_*, v) \to 0$ as $n \to \infty$. It follows from (3.17)-(3.19) in combination with $u_* \in \mathcal{C}_r$ that

$$\begin{aligned} 0 &\leq \lim_{n\to\infty} n\big(\mathcal{J}_\lambda(u_n) - \mathcal{J}_\lambda(u_*)\big) = \lim_{n\to\infty} \langle \mathcal{J}_\lambda'(u_*), n(u_n - u_*)\rangle + o(n; u_*, v) \\ &= \langle \mathcal{J}_\lambda'(u_*), v + S\rangle = \langle \mathcal{J}_\lambda'(u_*), v\rangle. \end{aligned}$$

A similar reasoning with $-v$ instead of v and the density of Lipschitz functions in W yield $\mathcal{J}_\lambda'(u_*) = 0$, which concludes the proof. □

Case 2: (h_{pqr}) **with** $1 < q < p < r < \infty$ **and** (h_{ab})

Let $\lambda > 0$ be a fixed number. Under the assumption $1 < q < p < r < \infty$ we cannot expect coercivity of the functional $\mathcal{J}_\lambda$ on $\mathcal{C}_r$. From now on we analyse the action of $\mathcal{J}_\lambda$ on the Nehari type manifold (see [14]) defined by

$$\begin{aligned} \mathcal{N}_\lambda &= \{v \in \mathcal{C}_r \setminus \{0\}; \langle \mathcal{J}_\lambda'(w), w\rangle = 0\} \\ &= \Big\{w \in \mathcal{C}_r \setminus \{0\}; \int_\Omega \big(\mid \nabla w \mid^p + \mid \nabla w \mid^q \big) dx = \lambda \mathcal{K}_r(w)\Big\}. \end{aligned} \tag{3.20}$$

It is natural to consider the restriction of $\mathcal{J}_\lambda$ to $\mathcal{N}_\lambda$ since any possible eigenfunction corresponding to λ belongs to $\mathcal{N}_\lambda$. Note that on $\mathcal{N}_\lambda$ functional $\mathcal{J}_\lambda$ has the form

$$\mathcal{J}_\lambda(u) = \frac{r-p}{pr} \int_\Omega \mid \nabla u \mid^p \, dx + \frac{r-q}{qr} \int_\Omega \mid \nabla u \mid^p \, dx > 0 \tag{3.21}$$

(see also Remark 2.3).

We have

Lemma 3.4. *In Case 2, for every $\lambda > 0$ we have $\mathcal{N}_\lambda \neq \emptyset$.*

Proof. We fix $u_0 \in \mathcal{C}_r \setminus \{0\}$ such that $\mathcal{K}_r(u_0) > 0$. We claim that for a convenient $t > 0$, $tu_0 \in \mathcal{N}_\lambda$. Since $\mathcal{C}_r$ is a cone, $tu_0 \in \mathcal{C}_r$ for all $t \in \mathbb{R}$. So the condition $tu_0 \in \mathcal{N}_\lambda$, $t > 0$, reads

$$h(t) := t^p \int_\Omega \mid \nabla u_0 \mid^p \; dx + t^q \int_\Omega \mid \nabla u_0 \mid^q \; dx - \lambda t^r \mathcal{K}_r(u_0) = 0.$$

Noting that the function $t \mapsto h(t)$ is continuous on $(0, \infty)$ and

$$t^{-q} h(t) \to \int_\Omega \mid \nabla u_0 \mid^q \; dx > 0 \; \text{ as } t \to 0^+,$$

$$t^{-r} h(t) \to -\lambda \mathcal{K}_r(u_0) < 0 \; \text{ as } t \to \infty,$$

we infer that there exists $t_0 \in (0, \infty)$ such that $h(t_0) = 0$, so $t_0 u_0 \in \mathcal{N}_\lambda$. □

Lemma 3.5. *If hypotheses* (h_{pqr}) *with* $1 < q < p < r < \infty$ *and* (h_{ab}) *hold, then there exists a point* $u_* \in \mathcal{N}_\lambda$ *where* $\mathcal{J}_\lambda$ *attains its minimal value,*

$$m_\lambda := \inf_{w\in\mathcal{N}_\lambda} \mathcal{J}_\lambda(w) > 0.$$

Proof. Let $(u_n)_n \subset \mathcal{N}_\lambda$ be a minimizing sequence for $\mathcal{J}_\lambda$. Since $u_n \in \mathcal{N}_\lambda$ for all n, we obtain from (3.21)

$$\mathcal{J}_\lambda(u_n) = \frac{r-p}{pr}\int_\Omega |\nabla u_n|^p\,dx + \frac{r-q}{qr}\int_\Omega |\nabla u_n|^q\,dx \to m_\lambda \geq 0, \text{ as } n\to\infty. \tag{3.22}$$

On the other hand, we have

$$\int_\Omega |\nabla u_n|^p\ dx + \int_\Omega |\nabla u_n|^q\ dx = \lambda\mathcal{K}_r(u_n)\ \forall\, n\geq 1. \tag{3.23}$$

Now, from (3.22) we obtain that $(\ \|\nabla u_n\|_{L^p(\Omega)}\ )_n$ and $(\ \|\nabla u_n\|_{L^q(\Omega)}\ )_n$ are bounded sequences, therefore taking into account (3.23), we can see that $(\mathcal{K}_r(u_n))_n$ is also a bounded sequence and making use of Lemma 2.1 we obtain that $(u_n)_n$ is bounded in W.

Next, let us prove that $m_\lambda = \inf_{w\in\mathcal{N}_\lambda}\mathcal{J}_\lambda(w) > 0$. Assume that, on the contrary, $m_\lambda = 0$. Let $(u_n)_n \subset \mathcal{N}_\lambda$ be a minimizing sequence for $\mathcal{J}_\lambda$. Note that $\mathcal{K}_r(u_n) > 0$ for all n (see Remark 2.3). We have (see (3.22))

$$\mathcal{J}_\lambda(u_n) = \frac{r-p}{pr}\int_\Omega |\nabla u_n|^p\,dx + \frac{r-q}{qr}\int_\Omega |\nabla u_n|^q\,dx \to 0 \text{ as } n\to\infty. \tag{3.24}$$

We know that $(u_n)_n$ is bounded in W, so there exists $u_0\in W$ such that, on a subsequence denoted again $(u_n)_n$, $u_n \rightharpoonup u_0$ in W (hence also in $W^{1,q}(\Omega)$), and $u_n \to u_0$ in $L^r(\Omega)$, $u_n \to u_0$ in $L^r(\partial\Omega)$. Clearly $u_0 \in \mathcal{C}_r$ and from (3.24) we deduce that u_0 is a constant function, so $u_0 \equiv 0$. Summarizing, we have proved that

$$u_n \rightharpoonup 0 \text{ in } W,\ \mathcal{K}_r(u_n)\to 0 \text{ as } n\to\infty. \tag{3.25}$$

We define $v_n := u_n/\big(\mathcal{K}_r(u_n)\big)^{1/r},\ n\geq 1$. By (3.23) we can see that

$$0 < \int_\Omega |\nabla u_n|^q dx = \lambda\mathcal{K}_r(u_n) - \int_\Omega |\nabla u_n|^p dx\ \forall\, n\geq 1.$$

Dividing this inequality by $\big(\mathcal{K}_r(u_n)\big)^{p/r}$, we obtain

$$\int_\Omega |\nabla v_n|^p dx < \lambda\big(\mathcal{K}_r(u_n)\big)^{\frac{r-p}{r}}\ \forall\, n\geq 1. \tag{3.26}$$

Since $p<r$ and $\big(\mathcal{K}_r(u_n)\big)^{\frac{r-p}{r}} \to 0$ as $n\to\infty$, we get by (3.26) that $(\ \|\nabla v_n\|_{L^p(\Omega)}\ )_n$ is a bounded sequence. In addition, $\mathcal{K}_r(v_n) = 1$ for all $n\geq 1$, thus, $(v_n)_n$ is bounded in W (see Lemma 2.1). As $(v_n)_n$ is a sequence in $\mathcal{C}_r$ which is weakly closed in W, it follows that there exists a $v_0\in\mathcal{C}_r$ such that, on a subsequence, $v_n \rightharpoonup v_0$ in W and $v_n\to v_0$ in $L^r(\Omega)$ as well as in $L^r(\partial\Omega)$.

Next, since $v_n \rightharpoonup v_0$ in W, we infer from (3.26)

$$\int_\Omega |\nabla v_0|^p dx \leq \liminf_{n\to\infty}\int_\Omega |\nabla v_n|^p\ dx = 0.$$

Therefore v_0 is a constant function and in fact $v_0\equiv 0$ since $v_0\in\mathcal{C}_r$. Thus, $v_n\to 0$ in both $L^r(\Omega)$ and $L^r(\partial\Omega)$. But this contradicts the fact that $\mathcal{K}_r(v_n) = 1\ \forall\, n\geq 1$. This contradiction shows that $m_\lambda > 0$.

Finally, we are going to prove that there exists $u_*\in\mathcal{N}_\lambda$ such that $\mathcal{J}_\lambda(u_*) = m_\lambda$.

Let $\big(u_n\big)_n \subset \mathcal{N}_\lambda$ be a minimizing sequence: $\mathcal{J}_\lambda(u_n) \to m_\lambda$. Since $\big(u_n\big)_n$ is bounded in W, on a subsequence, $\big(u_n\big)_n$ converges weakly in W to some $u_* \in W$ and strongly in both $L^r(\Omega)$ and $L^r(\partial\Omega)$ (to the same u_*). Thus,

$$\mathcal{J}_\lambda(u_*) \le \liminf_{n\to\infty} \mathcal{J}_\lambda(u_n) = m_\lambda. \tag{3.27}$$

As $\big(u_n\big)_n \subset \mathcal{N}_\lambda$ we have

$$\int_\Omega \big(\mid \nabla u_n \mid^p + \mid \nabla u_n \mid^q \big) dx = \lambda \mathcal{K}_r(u_n) \; \forall \, n \ge 1. \tag{3.28}$$

It is easily seen that u_* is not the null function. Indeed, assuming that $u_* \equiv 0$, we infer by (3.28) that $\big(u_n\big)_n$ converges strongly to 0 in W and $\| \nabla u_n \|^q_{L^q(\Omega)} \to 0$ as $n \to \infty$. Then (3.22) will give $m_\lambda = 0$ which is a contradiction. Obviously $u_* \in \mathcal{C}_r \setminus \{0\}$. Letting $n \to \infty$ in (3.28) yields

$$\int_\Omega \big(\mid \nabla u_* \mid^p + \mid \nabla u_* \mid^q \big) dx \le \lambda \mathcal{K}_r(u_*). \tag{3.29}$$

If (3.29) holds with equality then we are done. We shall prove that assuming strict inequality in (3.29) leads to a contradiction. Thus, let us assume that

$$\int_\Omega \big(\mid \nabla u_* \mid^p + \mid \nabla u_* \mid^q \big) dx < \lambda \mathcal{K}_r(u_*). \tag{3.30}$$

Now, we can choose $t_0 \in (0,1)$ such that $t_0 u_* \in \mathcal{N}_\lambda$. Indeed if we define $j : (0,\infty) \to \mathbb{R}$,

$$j(t) := t^r \Big(\int_\Omega \big(t^{p-r} \mid \nabla u_* \mid^p + t^{q-r} \mid \nabla u_* \mid^q \big) \, dx - \lambda \mathcal{K}_r(u_*) \Big)$$

we have $j(1) < 0$ (see (3.30)) and $t^{-r} j(t) \to \infty$ as $t \to 0_+$. Therefore, there exists $t_0 \in (\delta_0, 1)$ such that $j(t_0) = 0$, which implies $t_0 u_* \in \mathcal{N}_\lambda$.

Next, using the form of $\mathcal{J}_\lambda$ on the Nehari manifold $\mathcal{N}_\lambda$, we get

$$\mathcal{J}_\lambda(t_0 u_*) = \frac{t_0^p(r-p)}{pr} \int_\Omega \mid \nabla u_* \mid^p dx + \frac{t_0^q(r-q)}{qr} \int_\Omega \mid \nabla u_* \mid^q dx. \tag{3.31}$$

Therefore,

$$\begin{aligned}
0 < m_\lambda \le \mathcal{J}_\lambda(t_0 u_*) &= \frac{t_0^p(r-p)}{pr} \int_\Omega \mid \nabla u_* \mid^p dx + \frac{t_0^q(r-q)}{qr} \int_\Omega \mid \nabla u_* \mid^q dx \\
&< t_0^q \Big(\frac{r-p}{pr} \int_\Omega \mid \nabla u_* \mid^p \; dx + \frac{r-q}{qr} \int_\Omega \mid \nabla u_* \mid^q dx \Big) \\
&\le \frac{r-p}{pr} \liminf_{n\to\infty} \Big(\int_\Omega \mid \nabla u_n \mid^p dx \Big) + \frac{r-q}{qr} \liminf_{n\to\infty} \Big(\int_\Omega \mid \nabla u_n \mid^q dx \Big) \\
&\le \liminf_{n\to\infty} \mathcal{J}_\lambda(u_n) = m_\lambda,
\end{aligned}$$

which is impossible. □

The next result states that the minimizer u_*, given by Lemma 3.5, is a critical point of $\mathcal{J}_\lambda$ considered on the whole space W.

Proposition 3.2. *In Case 2, the minimizer $u_* \in \mathcal{N}_\lambda$ from Lemma 3.5 is an eigenfunction of problem* (1.1) *with corresponding eigenvalue λ.*

Proof. It suffices to prove that $\mathcal{J}'_\lambda(u_*)=0$. So, let $v\in \mathrm{Lip}(\Omega)$ be an arbitrary but fixed function and let $u_*\in\mathcal{N}_\lambda$ be the minimizer of $\mathcal{J}_\lambda$ over $\mathcal{N}_\lambda$. As in the Case 1 (see Proposition 3.1) we are able to obtain a sequence $\big(u_n\big)_n\subset\mathcal{C}_r\setminus\{0\}$,

$$u_n := u_* + \frac{1}{n}v + s_n \;\forall\, n\geq 1. \tag{3.32}$$

The sequence $\big(ns_n\big)_n$ is also bounded, so it converges, on a subsequence, to some $S\in\mathbb{R}$. Therefore, we have

$$n\big(u_n-u_*\big)\to v+S,\; u_n\to u_* \text{ in } W \text{ as } n\to\infty. \tag{3.33}$$

Since $u_*\in\mathcal{K}_r(u_*)$ and $\mathcal{K}_r(u_n)\to\mathcal{K}_r(u_*)>0$, one can assume that $\mathcal{K}_r(u_n)>0$ for all $n\geq 1$.

Using the sequence $\big(u_n\big)_n$, we shall construct a sequence $\big(t_n\big)_n\subset\mathbb{R}\setminus\{0\}$ such that, up to a subsequence, $\big(t_nu_n\big)_n\subset\mathcal{N}_\lambda$, i.e.,

$$t_n^{p-r}\int_\Omega |\nabla u_n|^p\,dx + t_n^{q-r}\int_\Omega |\nabla u_n|^q\,dx = \lambda\mathcal{K}_r(u_n). \tag{3.34}$$

Let us show that for every $n\geq 1$ there exists $t_n>0$ such that (3.34) holds. Define

$$h_n:(0,\infty)\to\mathbb{R},\; h_n(t):=t^{p-r}\int_\Omega |\nabla u_n|^p\,dx + t^{q-r}\int_\Omega |\nabla u_n|^q\,dx - \lambda\mathcal{K}_r(u_n).$$

Obviously, $h_n(t)\to\infty$ as $t\to 0+$ and $h_n(t)\to -\lambda\mathcal{K}_{ab}(u_n)<0$ as $t\to\infty$. So, there exists $t_n>0$ such that $h_n(t_n)=0\;\forall\, n\geq 1$, hence (3.34) holds, as claimed.

In what follows we shall prove that the sequence $\big(n(t_n-1)\big)_n$ is bounded. To this purpose, we rewrite (3.34) in the equivalent form

$$n\big(t_n^{p-r}-1\big)A(u_n)+n\big(t_n^{q-r}-1\big)B(u_n)=n\big(\lambda\mathcal{K}_r(u_n)-A(u_n)-B(u_n)\big), \tag{3.35}$$

where $A(u_n):=\int_\Omega|\nabla u_n|^p\,dx,\; B(u_n):=\int_\Omega|\nabla u_n|^q\,dx$.

We shall prove first that the sequence $\big(n(\lambda\mathcal{K}_r(u_n)-A(u_n)-B(u_n))\big)_n$ is convergent. To this purpose, let us define the C^1-functional $\mathcal{L}_\lambda: W\to\mathbb{R}$,

$$\mathcal{L}_\lambda(u)=-\int_\Omega|\nabla u|^p\,dx-\int_\Omega|\nabla u|^q\,dx+\lambda\mathcal{K}_r(u)\;\forall\, u\in W. \tag{3.36}$$

For all $u,w\in W$

$$\begin{aligned}\langle\mathcal{L}'_\lambda(u),w\rangle &= -p\int_\Omega|\nabla u|^{p-2}\,\nabla u\cdot\nabla w\,dx - q\int_\Omega|\nabla u|^{q-2}\,\nabla u\cdot\nabla w\,dx\\ &\quad+\lambda r\Big(\int_\Omega a\,|u|^{r-2}\,uw\,dx+\int_{\partial\Omega}b\,|u_\lambda|^{r-2}\,uw\,d\sigma\Big).\end{aligned} \tag{3.37}$$

From (3.36) and $u_*\in\mathcal{N}_\lambda$, we infer that $\mathcal{L}_\lambda(u_*)=0$, so we get

$$n\big(\lambda\mathcal{K}_r(u_n)-A(u_n)-B(u_n)\big)=n\big(\mathcal{L}_\lambda(u_n)-\mathcal{L}_\lambda(u_*)\big). \tag{3.38}$$

We have

$$n\big(\mathcal{L}_\lambda(u_n)-\mathcal{L}_\lambda(u_*)\big)\to\langle\mathcal{L}'_\lambda(u_*),v+S\rangle \text{ as } n\to\infty. \tag{3.39}$$

From (3.38) and (3.39) we deduce that the sequence $\big(n(\lambda\mathcal{K}_r(u_n)-A(u_n)-B(u_n)\big)_n$ has a finite limit.

Returning to (3.35), we observe that $\big(A(u_n)\big)_n$, $\big(B(u_n)\big)_n$ are bounded sequences of positive numbers. If we assume the contrary, that the sequence $\big(n(t_n^{p-r}-1)\big)_n$ has an unbounded subsequence converging, e.g., to $+\infty$, then the corresponding subsequence of $\big(n(t_n^{q-r}-1)\big)_n$ will have positive terms (since $q-r<0$ and $p-r<0$), so the sequence defined by the left hand side of (3.35) will be unbounded, thus contradicting the fact that the right hand side defines a convergent sequence. An analogue reasoning works in the case of a subsequence converging to $-\infty$. Therefore, $\big(n(t_n^{p-r}-1)\big)_n$ is a bounded sequence. Hence, there is $K>0$ such that for all $n\geq 1$, $n\mid t_n^{p-r}-1\mid\leq K$, which implies

$$1-\frac{K}{n}\leq t_n^{p-r}\leq 1+\frac{K}{n}\ \forall\, n\geq 1.$$

Since, there exists $N_1\in\mathbb{N}^*$ such that $1-K/n>0\ \forall\, n\geq N_1$, we have

$$n\left(\left(1+\frac{K}{n}\right)^{\frac{1}{p-r}}-1\right)\leq n(t_n-1)\leq n\left(\left(1-\frac{K}{n}\right)^{\frac{1}{p-r}}-1\right)\ \forall\, n\geq N_1. \tag{3.40}$$

Taking into account the relations

$$\lim_{x\to 0}\frac{(1+Kx)^{1/(p-r)}-1}{x}=K/(p-r),\ \lim_{x\to 0}\frac{(1-Kx)^{1/(p-r)}-1}{x}=-K/(p-r),$$

we infer from (3.40) that the sequence $\big(n(t_n-1)\big)_n$ is bounded, thus, by possibly passing to a subsequence, there exists $T\in\mathbb{R}$, such that $n(t_n-1)\to T$ as $n\to\infty$. We define

$$z_n:=t_n\Big(u_*+\frac{1}{n}v+s_n\Big)=t_nu_n\ \forall\, n\geq N_1, \tag{3.41}$$

with $\big(z_n\big)_n\subset\mathcal{N}_\lambda$. In addition, as $\big(n(t_n-1)\big)_n$ is a bounded sequence, we can see that

$$t_n\to 1 \text{ in } \mathbb{R},\ z_n\to u_* \text{ in } W \text{ as } n\to\infty. \tag{3.42}$$

By using the minimality of u_* and the fact that $\big(z_n\big)_n\subset\mathcal{N}_\lambda$ we obtain that

$$0\leq\lim_{n\to\infty}\frac{\mathcal{J}_\lambda(z_n)-\mathcal{J}_\lambda(u_*)}{\frac{1}{n}}. \tag{3.43}$$

Since functional $\mathcal{J}_\lambda\in C^1(W;\mathbb{R})$, we can write

$$n\big(\mathcal{J}_\lambda(z_n)-\mathcal{J}_\lambda(u_*)\big)=\big(\langle\mathcal{J}_\lambda'(u_*),n(z_n-u_*)\rangle+o(n;u_*,v), \tag{3.44}$$

with $o(n;u_*,v)\to 0$ as $n\to\infty$. Taking into account (3.41) and (3.42), we can see that

$$n(z_n-u_*)=n\big(t_n-1\big)u_*+v+ns_n\to Tu_*+v+S \text{ as } n\to\infty \text{ in } W. \tag{3.45}$$

It follows from (3.43) and (3.45) that

$$0\leq\langle\mathcal{J}_\lambda'(u_*),v+S+Tu_*\rangle. \tag{3.46}$$

Since $u_*\in\mathcal{N}_\lambda$, we obtain that $\langle\mathcal{J}_\lambda'(u_*),u_*\rangle=0,\ \langle\mathcal{J}_\lambda'(u_*),S\rangle=0$, hence (3.46) implies

$$0\leq\langle\mathcal{J}_\lambda'(u_*),v\rangle.$$

A similar reasoning with $-v$ instead of v shows that the converse inequality holds, hence $0=\langle\mathcal{J}_\lambda'(u_*),v\rangle$. Finally, using the density of Lipschitz functions in W we obtain that $\mathcal{J}_\lambda'(u_*)=0$, which concludes the proof. □

Therefore, as it has already been pointed out, $\lambda=0$ is an eigenvalue, so the conclusion of Theorem 1.1 follows from Propositions 3.1 and 3.2.

Remark 3.4. As we have already mentioned in Introduction, in the case $p,q\in(1,\infty),\ p\neq q,\ r\in\{p,q\}$, the set of eigenvalues of problem (1.1) has been completely determined in [3]. The case $1<q<r<p<\infty$ remains open.

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[a]OVIDIUS UNIVERSITY
FACULTY OF MATHEMATICS AND COMPUTER SCIENCE
DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE
124 MAMAIA BLVD, 900527, CONSTANŢA, ROMANIA
Email address: lbarbu@univ-ovidius.ro

[b]ACADEMY OF ROMANIAN SCIENTISTS, BUCHAREST, ROMANIA,
[c]BABEŞ-BOLYAI UNIVERSITY
FACULTY OF MATHEMATICS AND COMPUTER SCIENCE
1 M. KOGĂLNICEANU STR., 400084 CLUJ-NAPOCA, ROMANIA
Email address: morosanu@math.ubbcluj.ro