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Dedicated to Prof. Ioan A. Rus on the occasion of his 85^{th} anniversary

Frum-Ketkov type multivalued operators

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ABSTRACT. Let (M, d) be a metric space, $X \subset M$ be a nonempty closed subset and $K \subset M$ be a nonempty compact subset. By definition, an upper semi-continuous multivalued operator $F : X \to P(X)$ is said to be a strong Frum-Ketkov type operator if there exists $\alpha \in]0,1[$ such that $e_d(F(x), K) \leq \alpha D_d(x, K)$, for every $x \in X$, where e_d is the excess functional generated by d and D_d is the distance from a point to a set. In this paper, we will study the fixed points of strong Frum-Ketkov type multivalued operators.

1. INTRODUCTION

Let $(\mathbb{B}, \|\cdot\|)$ be a Banach space, K be a nonempty compact subset of $\mathbb{B}$ and $f : \tilde{B}(0;1) \to \tilde{B}(0;1)$ be a continuous operator, where $\tilde{B}(0;1)$ is the closed unit ball in $\mathbb{B}$. We denote by $D_{\|\cdot\|}(x, K)$ the distance from a point $x \in \mathbb{B}$ to the set K from $\mathbb{B}$, i.e., $D_{\|\cdot\|}(x, K) := \inf_{u \in K} \|x - u\|$. In the paper [6], the Russian mathematician Frum-Ketkov proved the existence of a fixed point for the mapping f satisfying the following assumption: there exists $\alpha \in [0, 1[$ such that

$$D_{\|\cdot\|}(f(x), K) \leq \alpha D_{\|\cdot\|}(x, K), \text{ for every } x \in \tilde{B}(0;1). \tag{1.1}$$

For nice extensions of Frum-Ketkov result see [3], [12], [26].

In a recent paper (see [17]), the following more general class of operators is considered and studied. Let (M, d) be a metric space, $X \subset M$ be a nonempty closed subset and $K \subset M$ be a nonempty compact subset. We denote by $D_d(x, K)$ the distance from a point $x \in X$ to the set K from M, i.e., $D_d(x, K) := \inf_{u \in K} d(x, u)$. A continuous operator $f : X \to X$ is said to be a Frum-Ketkov (α, K)-operator if $\alpha \in]0, 1[$ and

$$D_d(f(x), K) \leq \alpha D_d(x, K), \text{ for every } x \in X. \tag{1.2}$$

In the above context, $f : X \to X$ is called a weakly Picard operator if, for all $x \in X$, the sequence $\{f^n(x)\}_{n \in \mathbb{N}}$ of successive approximations for f converges and its limit $x^*(x)$ is a fixed point of f. A weakly Picard operator with a unique fixed point is called a Picard operator. See [2] an [24] for details and related results.

In the recent paper [17] sufficient conditions ensuring that a singlevalued Frum-Ketkov operator is weakly Picard were given. The purpose of this paper is to extend the concept of Frum-Ketkov operator to the multivalued setting and to study the properties of the fixed point inclusion with a Frum-Ketkov type multivalued operator.

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2. PRELIMINARIES

Throughout this paper we will use the notations and symbols from [23] and [24]. For the convenience of the reader we recall some of them. Let (M, d) be a metric space. Then, we denote by $P(M)$ the family of all nonempty subsets of M, by $P_{cl}(M)$ the family of all nonempty closed subsets of M and by $P_{cp}(M)$ the family of all nonempty compact subsets of M. For $x_0 \in M$ and $R > 0$, the symbol $B(x_0; R)$ denotes the open ball centered at x_0 with radius R, while $\tilde{B}(x_0; R)$ is the closed ball centered at x_0 with radius R. For $A, B \in P(M)$, we denote by $D_d(A, B) := \inf\{d(a, b) : a \in A, b \in B\}$ the gap functional generated by d and by $V(Y; \epsilon) = \{x \in M : D_d(x, Y) \leq \epsilon\}$ the closed neighborhood of the set $Y \subset M$. Recall also that a set $A \in P(M)$ is called proximinal if $(Pr)_A(x) := \{a \in A : \ d(x, a) = D_d(x, A)\} \neq \emptyset$, for each $x \in M$. If $(\mathbb{B}, \|\cdot\|)$ is a Banach space, then the symbol $P_{cv}(\mathbb{B})$ denotes the family of all nonempty convex subsets of $\mathbb{B}$, while $P_{cp,cv}(\mathbb{B}) := P_{cp}(\mathbb{B}) \cap P_{cv}(\mathbb{B})$ and $P_{cl,cv}(\mathbb{B}) := P_{cl}(\mathbb{B}) \cap P_{cv}(\mathbb{B})$.

Let (M, d) be a metric space and $F : M \to P(M)$ be a multivalued operator. Throughout this paper, the symbol $Fix(F) := \{x \in M | \ x \in F(x)\}$ denotes the fixed point set of F, while $SFix(F) := \{x \in X | \ \{x\} = F(x)\}$ is the strict fixed point set of F. Also $I(F) := \{Y \subset M : F(Y) \subset Y\}$. For $A, B \in P(M)$, we denote by H_d the Pompeiu-Hausdorff functional generated by d, i.e.,

$$H_d(A, B) := \max\{e_d(A, B), e_d(B, A)\},$$

where $e_d(A, B)$ is the excess of the set A over B and it is represented by the formula

$$e_d(A, B) := \sup\{D_d(a, B) | \ a \in A\}.$$

We will avoid the subscript d when no confusion can occur. The multivalued operator $F : M \to P(M)$ is said to be upper semi-continuous on M (briefly u.s.c.) if for each open subset U of M, the set $F^+(U) := \{x \in M : F(x) \subset U\}$ is open in M. On the other hand, the multivalued operator F is said to be lower semi-continuous (briefly l.s.c.) on M if for each open subset U of M the set $F^-(U) := \{x \in M : F(x) \cap U \neq \emptyset\}$ is open in X. In the same context, F is called H-upper semi-continuous (briefly H-u.s.c.) in $x_0 \in M$ if for all $\epsilon > 0$ there exists $\eta > 0$ such that, for all $x \in B(x_0; \eta)$ we have $F(x) \subset V(F(x_0); \epsilon)$. The multivalued operator F is H-u.s.c. on M if it is H-u.s.c. in each point $x_0 \in M$. For a multivalued operator with compact values u.s.c. and H-u.s.c. are equivalent.

We also recall ([11]) that a multivalued operator $F : M \to P_{cl}(M)$ is said to be an α-contraction if $\alpha \in]0, 1[$ and

$$H_d(F(x), F(y)) \leq \alpha d(x, y), \text{ for all } x, y \in M.$$

In the same framework, the sequence $\{x_n\}_{n \in \mathbb{N}}$ from M is called a sequence of successive approximations for F starting from $x_0 \in M$ if $x_{n+1} \in F(x_n)$, for each $n \in \mathbb{N}$.

A multivalued operator $F : M \to P_{cl}(M)$ is said to be asymptotically regular at the point $x_0 \in M$ if for any sequence $\{x_n\}_{n \in \mathbb{N}}$ of successive approximations starting from x_0 we have that $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$. If F is asymptotically regular at any point $x_0 \in M$, then F is said to be asymptotically regular, see [7], [9], [5]. The multivalued operator F is said to be quasi nonexpansive if

$$H(F(x), F(u)) \leq d(x, u), \text{ for any } x \in M \text{ and any } u \in Fix(F).$$

Finally, F is called contractive if

$$H(F(x), F(y)) < d(x, y), \text{ for any } x, y \in M \text{ with } x \neq y.$$

For related concepts see also [8], [27].

Concerning the concept of fixed point structure, we recall the following notion and examples. For other details and results see [22] and [19].

Definition 2.1. A triple $(X, S(X), M^0)$ is a fixed point structure on X (f.p.s.) if:
(i) $S(X) \subset P(X),\ S(X) \neq \emptyset$;
(ii) $M^0 : P(X) \multimap \bigcup_{Y \in P(X)} \mathbb{M}^0(Y),\ Y \multimap M^0(Y) \subset \mathbb{M}^0(Y)$ is an operator such that if $Z \in P(Y)$ then $M^0(Z) \supset \{F|_Z : F \in M^0(Y)$ and $Z \in I(F)\}$;
(iii) every $Y \in S(X)$ has the fixed point property with respect to $M^0(Y)$, i.e., $Y \in S(X)$, $F \in M^0(Y)$ imply $Fix(F) \neq \emptyset$.

Example 2.1. (The f.p.s. of contractive operators) Let (X, d) be a complete metric space, $S(X) := P_{cp}(X)$ and $M^0(Y) := \{F : Y \to P_{cl}(Y)|\ F$ is contractive $\}$.

Example 2.2. (The f.p.s. of Bohnenblust-Karlin) Let X be a Banach space, $S(X) := P_{cp,cv}(X)$ and $M^0(Y) := \{F : Y \to P_{cp,cv}(Y)|\ F$ is u.s.c.$\}$.

Example 2.3. (The fixed point structure of T. X. Wang) Let (X, d) be a complete metric space, $S(X) := P_{cl}(X)$, $M^0(Y) := \{F : Y \to P_{cl}(Y)|\ F$ is such that there exist $a, b \in \mathbb{R}_+$ with $a+2b < 1$ and $e_d(F(x), F(y)) \leq ad(x, y)+b\,(D_d(x, F(x)) + cD_d(y, F(y))), \forall\, x, y \in Y\}$.

3. Frum-Ketkov type multivalued operators in metric spaces

We will present first two concepts of Frum-Ketkov multivalued operators.

Definition 3.2. Let (M, d) be a metric space, $X \in P_{cl}(M)$ and $K \in P_{cp}(M)$. Then, an upper semicontinuous operator $F : X \to P(X)$ is said to be:

(1) a Frum-Ketkov multivalued (α, K)-operator if $\alpha \in]0, 1[$ and

$$D_d(F(x), K) \leq \alpha D_d(x, K),\ \text{for every } x \in X. \tag{3.3}$$

(2) a strong Frum-Ketkov multivalued (α, K)-operator if $\alpha \in]0, 1[$ and

$$e_d(F(x), K) \leq \alpha D_d(x, K),\ \text{for every } x \in X. \tag{3.4}$$

It is obvious that any strong Frum-Ketkov multivalued (α, K)-operator is a Frum-Ketkov multivalued (α, K)-operator, but the reverse implication, in general, does not hold.

We will present first some examples of Frum-Ketkov type multivalued operators.

Remark 3.1. (1) Let (M, d) be a metric space, $X \in P_{cl}(M)$ and $K \in P_{cp}(M)$. If $X \subset K$ then each u.s.c. multi-operator $F : X \to P(X)$ is a Frum-Ketkov multivalued (α, K)-operator, with any $\alpha \in]0, 1[$. In this case, the fixed point theory for Frum-Ketkov multivalued operators reduces to fixed point theory of u.s.c. multivalued operators on compact metric spaces. Thus, as in the single-valued case, the Frum-Ketkov condition is effective if $X \neq K$.

(2) Let (M, d) be a complete metric space, $X \in P_{cl}(M)$ and $F : X \to P_{cl}(X)$ be a multivalued α-contraction. Then, by Nadler's Contraction Principle (see [11], [4]) we know that $Fix(F) \neq \emptyset$. Let x^* be any fixed point of F and $K := \{x^*\}$. Then F is a Frum-Ketkov multivalued (α, K)-operator.

(3) Let (M, d) be a complete metric space, $X \in P_{cl}(M)$ and $F : X \to P_{cp}(X)$ be a multivalued α-contraction. Then, by Nadler's Contraction Principle and Saint Raymond Theorem [25] we have that $Fix(F) \in P_{cp}(X)$. Consider $K := Fix(F)$. Thus, for each $x \in X$ and any $u \in K$ we can write

$$D_d(F(x), K) \leq D_d(u, F(x)) \leq H_d(F(u), F(x)) \leq \alpha d(u, x).$$

Taking $\inf_{u \in K}$ we obtain

$$D_d(F(x), K) \leq \alpha D_d(x, K),\ \text{for each } x \in X,$$

proving that F is a Frum-Ketkov multivalued (α, K)-operator.

Using the selection theory for multivalued operators we have the following results.

Lemma 3.1. *Let $(\mathbb{B}, \|\cdot\|)$ be a Banach space, $X \in P_{cl}(\mathbb{B})$, $K \in P_{cp}(\mathbb{B})$ and $F : X \to P_{cl,cv}(X)$ be a l.s.c. strong Frum-Ketkov multivalued (α, K)-operator. Then there exists a Frum-Ketkov (α, K)-selection of F.*

Proof. Indeed, by Michael' selection theorem (see [10]) there exists a continuous selection $f : X \to X$ of F, i.e., $f(x) \in F(x)$, for each $x \in X$. We have

$$D_{\|\cdot\|}(f(x), K) \leq e_{\|\cdot\|}(F(x), K) \leq \alpha D_{\|\cdot\|}(x, K), \text{ for all } x \in X.$$

Thus f is a Frum-Ketkov (α, K)-operator. As a conclusion, any lower semi-continuous strong Frum-Ketkov multivalued (α, K)-operator admits a Frum-Ketkov (α, K)-selection. □

Lemma 3.2. *Let $(\mathbb{B}, \|\cdot\|)$ be a Banach space, $X \in P_{cl}(\mathbb{B})$, $K \in P_{cp}(\mathbb{B})$ and $F : X \to P_{cl,cv}(X)$ be a l.s.c. strong Frum-Ketkov multivalued (α, K)-operator. Additionally, we suppose that there exists a continuous operator $\rho : \mathbb{B} \to K$ such that $D_{\|\cdot\|}(x, K) = \|x - \rho(x)\|$, for every $x \in X$. Then, there exists a continuous selection f of F which is ρ-asymptotically regular.*

Proof. By Michael' selection theorem there exists a continuous selection $f : X \to X$ of F. By the additional hypothesis we know that there exists a continuous operator $\rho : \mathbb{B} \to K$ such that $D_{\|\cdot\|}(x, K) = \|x - \rho(x)\|$, for every $x \in X$. Combining the above relation with the fact that $f(x) \in F(x)$, for each $x \in X$, we have

$$\|f(x) - \rho(f(x))\| = D_{\|\cdot\|}(f(x), K) \leq e_{\|\cdot\|}(F(x), K) \leq \alpha D_{\|\cdot\|}(x, K) = \alpha\|x - \rho(x)\|.$$

This implies that

$$\|f^n(x) - \rho(f^n(x))\| \leq \alpha^n \|x - \rho(x)\| \to 0, \text{ as } n \to \infty, \text{ for each } x \in X.$$

As a consequence, the continuous selection f of F is ρ-asymptotically regular. □

Concerning the properties of a Frum-Ketkov multivalued (α, K)-operator we have the following result.

Theorem 3.1. *Let (M, d) be a metric space, $X \in P_{cl}(M)$ and $K \in P_{cp}(M)$. Suppose that $F : X \to P_{cp}(X)$ is a Frum-Ketkov multivalued (α, K)-operator. Then, for every $x_0 \in X$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of successive approximations for F starting from x_0 which has a convergent subsequence.*

Proof. Let $x_0 \in X$ be arbitrary chosen. Since F has compact values, there exists $x_1 \in F(x_0)$ such that $D_d(F(x_0), K) = D_d(x_1, K)$. By this approach, we can obtain a sequence $\{x_n\}_{n\in\mathbb{N}}$ of successive approximations for F starting from x_0 having the property

$$D_d(F(x_n), K) = D_d(x_{n+1}, K), \text{ for } n \in \mathbb{N}.$$

Moreover, we have

$$D_d(x_{n+1}, K) = D_d(F(x_n), K) \leq \alpha D_d(x_n, K) \leq \cdots \leq \alpha^{n+1} D(x_0, K) \to 0, \text{ as } n \to \infty.$$

Using the compactness of K there exists a sequence $\{y_n\}_{n\in\mathbb{N}}$ in K such that

$$D_d(x_n, K) = d(x_n, y_n), \text{ for } n \in \mathbb{N}.$$

Using again the compactness of K, we can find a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ which converges to an element $u^* \in K$ as $n_i \to \infty$. As a consequence, $\{x_{n_i}\}$ also converges to $u^* \in X \cap K$ as $n_i \to \infty$. □

Remark 3.2. It is an open question to obtain that u^* from the above proof is a fixed point for F.

We will consider now the case of a strong Frum-Ketkov multivalued (α, K)-operator.

Theorem 3.2. *Let (M, d) be a metric space, $X \in P_{cl}(M)$ and $K \in P_{cp}(M)$. Suppose that $F : X \to P_{cl}(X)$ is a strong Frum-Ketkov multivalued (α, K)-operator. Then, the following conclusions hold:*

(a) every sequence $\{x_n\}_{n\in\mathbb{N}}$ of successive approximations for F starting from arbitrary $x_0 \in X$ has a convergent subsequence;

(b) $Fix(F) \subset X \cap K$;

(c) $X \cap K$ is invariant with respect to F, i.e., $F(X \cap K) \subset X \cap K$;

(d) if, additionally, F is asymptotically regular, then $Fix(F) \neq \emptyset$; Moreover, if F has proximinal values and it is quasi nonexpansive, then for any $x_0 \in X$ there exists a sequence of successive approximations for F starting from x_0 which converges to a fixed point of F.

Proof. (a) Let $x_0 \in X$ be arbitrary chosen. Then, for $x_1 \in F(x_0)$ we have

$$D_d(x_1, K) \leq e_d(F(x_0), K) \leq \alpha D_d(x_0, K).$$

Thus, for every sequence $\{x_n\}$ of successive approximations for F starting from arbitrary x_0 we obtain, for every $n \in \mathbb{N}$, that

$$D_d(x_n, K) \leq e_d(F(x_{n-1}), K) \leq \alpha D(x_{n-1}, K) \leq \cdots \leq \alpha^n D_d(x_0, K) \to 0, \text{ as } n \to \infty.$$

Since K is compact, there exists a sequence $\{y_n\}_{n\in\mathbb{N}}$ in K such that

$$D_d(x_n, K) = d(x_n, y_n), \text{ for } n \in \mathbb{N}.$$

Using again the compactness of K, we can find a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ which converges to an element $u^* \in K$ as $n_i \to \infty$. By the above relations, we get that $\{x_{n_i}\}$ also converges to $u^* \in X \cap K$ as $n_i \to \infty$.

(b) Let $u \in Fix(F)$. Then

$$D_d(u, K) \leq e_d(F(u), K) \leq \alpha D_d(u, K).$$

Thus $D_d(u, K) = 0$, which implies that $u \in K$. Hence $u \in X \cap K$.

(c) Let $x \in X \cap K$. Then $e_d(F(x), K) \leq \alpha D_d(x, K) = 0$. Thus we get that $F(x) \subset \overline{K} = K$. As a conclusion, $F(x) \subset X \cap K$.

(d) By (a) we know that any sequence $\{x_n\}$ of successive approximations for F has a convergent subsequence. Say that $\{x_{n_i}\}$ converges to $u^* \in X \cap K$ as $n_i \to \infty$. Since F is asymptotically regular, for any sequence $\{x_n\}$ of successive approximations for F starting from arbitrary x_0, we have that $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$. Then, using the upper semi-continuity assumption on F, we get that

$$D(u^*, F(u^*)) \leq d(u^*, x_{n_i}) + d(x_{n_i}, x_{n_i+1}) + e_d(F(x_{n_i}), F(u^*)) \to 0 \text{ as } n_i \to \infty.$$

Thus $u^* \in Fix(F)$.

For the second conclusion of this item, since F has proximinal values, for $x_0 \in X$ there exists $x_1 \in F(x_0)$ such that $d(x_1, u^*) = D(F(x_0), u^*)$. Inductively we obtain a sequence $\{x_n\}$ of successive approximations for F with $d(x_{n+1}, u^*) = D(F(x_n), u^*)$, $n \in \mathbb{N}$. Then, by the quasi nonexpansivity of F, we get that

$$d(x_{n+1}, u^*) = D(F(x_n), u^*) \leq H_d(F(x_n), F(u^*)) \leq d(x_n, u^*), n \in \mathbb{N}.$$

This shows that the sequence $\{d(x_n, u^*)\}$ is decreasing and, hence, convergent. Since the subsequence $\{d(x_{n_i}, u^*)\}$ is convergent to 0, the whole sequence $\{d(x_n, u^*)\}$ converges also to 0. Hence, $x_n \to u^*$ and thus there exists a sequence of successive approximations which converges to $u^* \in Fix(F)$. □

Remark 3.3. It is an open question to obtain strict fixed point theorems for strong Frum-Ketkov multivalued (α, K)-operators. For strict fixed point theorems see [1], [15], [16], [18], [20].

4. Frum-Ketkov operators in terms of a fixed point structure

Let $F : X \to P_{cl}(X)$ be a strong Frum-Ketkov multivalued operator. By Theorem 3.2 we have that $X \cap K \neq \emptyset$, $F(X \cap K) \subset X \cap K$ and $Fix(F) \subset X \cap K$.

Let $(M, \mathbb{S}(M), M^0)$ be a fixed point structure. If $X \cap K \in \mathbb{S}(M)$ and the restriction of F to $X \cap K$ belongs to $M^0(X \cap K)$, then $Fix(F) \neq \emptyset$. Thus, we can prove the following result.

Theorem 4.3. *Let (M, d) be a complete metric space, $X \in P_{cl}(M)$ and $K \in P_{cp}(M)$. Suppose that $F : X \to P_{cl}(X)$ is contractive and there exists $\alpha \in]0, 1[$ such that*

$$e_d(F(x), K) \leq \alpha D_d(x, K), \text{ for every } x \in X. \tag{4.5}$$

Then $Fix(F) \neq \emptyset$.

Proof. We consider on M the fixed point structure $(M, P_{cp}(M), M^0)$, where for $Y \in P_{cp}(M)$ we define

$$M^0(Y) := \{G : Y \to P_{cl}(Y) : \text{ G is contractive } \}.$$

Since $X \cap M \in P_{cp}(M)$ and $F|_{X\cap K} \in M^0(X \cap M)$, the conclusion follows by the definition of the fixed point structure of Smithson, see [22]. □

A similar result can be obtained by using the fixed point structure of Bohnenblust-Karlin, see [22].

Theorem 4.4. *Let $(\mathbb{B}, \|\cdot\|)$ be a Banach space, $X \in P_{cl,cv}(\mathbb{B})$ and $K \in P_{cp,cv}(\mathbb{B})$. Suppose that $F : X \to P_{cl,cv}(X)$ is a strong Frum-Ketkov multivalued operator. Then $Fix(F) \neq \emptyset$.*

5. Generalized Frum-Ketkov multivalued operators

We present first the notion of generalized Frum-Ketkov multivalued operator.

Definition 5.3. Let (M, d) be a metric space, $X \in P_{cl}(M)$ and $K \in P_{cp}(M)$. Then, by definition, $F : X \to P(X)$ is a generalized Frum-Ketkov multivalued operator if F is u.s.c. and there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ of successive approximations for F starting from any $x_0 \in X$ such that $D_d(x_n, K) \to 0$ as $n \to \infty$.

Example 5.4. Let (M, d) be a metric space, $X \in P_{cl}(M)$, $K \in P_{cp}(M)$. Let $F : X \to P_{cp}(X)$ be an u.s.c. multivalued operator. If $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a comparison function (i.e., φ is increasing and the sequence $\{\varphi^n(t)\}_{n\in\mathbb{N}}$ converges to 0 as $n \to \infty$, for every $t > 0$) and

$$D_d(F(x), K) \leq \varphi(D_d(x, K)), \text{ for every } x \in X,$$

then F is a generalized Frum-Ketkov multivalued operator. In this case F is called a Frum-Ketkov multivalued φ-operator.

Example 5.5. Let (M, d) be a metric space, $X \in P_{cl}(M)$ and $K \in P_{cp}(M)$. Let $F : X \to P(X)$ be an u.s.c. multivalued operator. If $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a comparison function and

$$e_d(F(x), K) \leq \varphi(D_d(x, K)), \text{ for every } x \in X,$$

then F is a generalized Frum-Ketkov multivalued operator. In this case F is called a strong Frum-Ketkov multivalued φ-operator.

Example 5.6. Let (M, d) be a metric space, $X \in P_{cl}(M)$, $K \in P_{cp}(M)$. Let $F : X \to P_{cp}(X)$ be an u.s.c. operator such that for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\varepsilon \leq D_d(x, K) < \varepsilon + \delta \ \Rightarrow \ D_d(F(x), K) < \varepsilon.$$

Then, F is a generalized Frum-Ketkov multivalued operator, i.e., there is a sequence $\{x_n\}$ of successive approximations for F starting from arbitrary $x_0 \in X$ having the property $D_d(x_n, K) \to 0$ as $n \to \infty$, for every $x \in X$. Indeed, since $d_n := D_d(x_n, K)$ is decreasing, we can suppose, by contradiction, that $d_n \searrow \varepsilon > 0$ as $n \to \infty$. Assuming, for some $m \in \mathbb{N}^*$, that $d_m < \varepsilon + \delta$, we get, by the definition of F, that $d_{m+1} < \varepsilon$, which gives the desired contradiction.

The main result for this section is the following theorem.

Theorem 5.5. *Let (M, d) be a metric space, $X \in P_{cl}(M)$ and $K \in P_{cp}(M)$. Suppose that $F : X \to P(X)$ is a generalized Frum-Ketkov operator. Then, the following conclusions hold:*

(a) there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ of successive approximations for F starting from arbitrary $x_0 \in X$ which has a convergent subsequence;

(b) if, additionally, F is asymptotically regular, then $Fix(F) \neq \emptyset$.

Proof. (a) By the hypothesis, we know that F is u.s.c. and there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ of successive approximations for F starting from any $x_0 \in X$ such that $D_d(x_n, K) \to 0$ as $n \to \infty$. Since K is compact, there exists a sequence $\{y_n\}_{n\in\mathbb{N}}$ in K such that

$$D_d(x_n, K) = d(x_n, y_n), \text{ for } n \in \mathbb{N}.$$

Using again the compactness of K, we can find a subsequence $\{y_{n_i}\}$ of (y_n) which converges to an element $u^* \in K$ as $n_i \to \infty$. Thus, the subsequence $\{x_{n_i}\}$ also converges to $u^* \in X \cap K$ as $n_i \to \infty$.

(b) By (a) we know that there exists a sequence $\{x_n\}$ of successive approximations for F which has a convergent subsequence. Say that $\{x_{n_i}\}$ converges to $u^* \in X \cap K$ as $n_i \to \infty$. By the asymptotically regularity of F, the sequence $\{x_n\}$ of successive approximations for F starting from arbitrary x_0 has the property that $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$. Then, using the upper semi-continuity assumption on F, we get that

$$D(u^*, F(u^*)) \leq d(u^*, x_{n_i}) + d(x_{n_i}, x_{n_i+1}) + e_d(F(x_{n_i}), F(u^*)) \to 0 \text{ as } n_i \to \infty.$$

Thus $u^* \in Fix(F)$. □

Example 5.7. Let $M := \mathbb{R}^2$, $d := d_{\|\cdot\|_2}$, $X = [0, 1] \times [0, 1]$ and $K := [0, 1] \times \{0\}$. Let $F : X \to P(X)$ be a multivalued operator defined by $F(x_1, x_2) = \{(1 - x_1, \frac{1}{2}x_2), (x_1, \frac{1}{2}x_2)\}$. Then, F is a strong Frum-Ketkov multivalued operator with $Fix(F) = \{(x_1, 0) : x_1 \in [0, 1]\}$ and $SFix(F) = \{(\frac{1}{2}, 0)\}$. Moreover, F is not asymptotically regular in $(x, 0)$, for $x \in [0, 1] \setminus \{\frac{1}{2}\}$.

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