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Invited paper

Set-theoretical aspect of the fixed point theory: some examples

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ABSTRACT. In this paper we revisit some of my own contributions to the fixed point theory, contributions which are related to set-theoretical terms. My intention is to do this in a unitary way. New notions and results are given. Open problems are also formulated.

1. INTRODUCTION

In the fixed point theory there are notions and problems which are set-theoretical (see [55], [74], [103], [106], [107], [109], [111], [121], [126], ...). If X is a structured set (ordered set, group, ring, algebra, L-space, topological space, metric space, generalized metric space, Banach space, Hilbert space, ordered L-space, ...) and $f : X \to X$ is an operator, then these set-theoretical problems are studied in terms of these structures on X. Moreover there are some fixed point results in the theory of category ([80], [14], [77], [24], [146], [81], [59], [103], [108], ...).

In this paper we revisit some of my own contributions, in the fixed point theory, in relation to set-theoretical terms. We shall do this in a unitary way. New notions and results are given, and open problems are formulated. The structure of the paper is as follows:

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6.3. Fixed point theorems in terms of an operator with intersection property: (θ, l)-contractions
7. Cyclic covering of a set with respect to an operator
8. Equivalent fixed point equations
8.1. Admissible perturbations of an operator
8.2. Operator with a weakly Picard operator perturbation
9. Set retractions and fixed points of nonself operators
10. Multivalued operators

Throughout this paper we follow the notation and terminology in [121], [126] and [140].

Regarding the set and category theory see: [58], [80], [14], [25], [24], [69], [76], [81], [82], [108], [59], [70], ...

Regarding the basic fixed point theorems in sets and structured sets see: [55], [74], [140], [54], [63], [70], [73], [106], [121], [48], [103], [105], [107], [109], ...

2. Set-theoretical fixed point results: R-contractions

Before 1988 there were given some set-theoretical fixed point theorems (Chu-Diaz [36], Abian [1], [2], Wisniewski [159], Eilenberg [55], Lim [79], Rus [102], [103], [106], Deaconescu [40]). From these we mention the following two:

Abian-Wisniewski Theorem ([2], [159]). *Let X be a nonempty set and $f : X \to X$ be an operator. Then the following statements are equivalent:*

(i) $F_f := \{x \in X \mid f(x) = x\} = \emptyset$

(ii) *there exists three mutually disjoint subsets such that:*

(a) $X = X_1 \cup X_2 \cup X_3$;

(b) $X_i \cap f(X_i) = \emptyset$, *for each* $i \in \{1, 2, 3\}$.

Eilenberg Theorem ([55], p.19). *Let X be a nonempty set and let $(R_n)_{n\in\mathbb{N}}$ be a sequence of equivalence relations in X such that:*

(1) $X \times X = R_0 \supset R_1 \supset \ldots \supset R_n \supset \ldots$;

(2) $\bigcap\limits_{n\in\mathbb{N}} R_n = \Delta(X) := \{(x, x) \mid x \in X\}$;

(3) *if $(x_n)_{n\in\mathbb{N}}$ is a sequence in X such that $(x_n, x_{n+1}) \in R_n$, for each $n \in \mathbb{N}$, then there exists an $x \in X$ such that $(x_n, x) \in R_n$, for all $n \in \mathbb{N}$.*

If $f : X \to X$ is such that,

$$n \in \mathbb{N},\ (x, y) \in R_n \ \Rightarrow\ (f(x), f(y)) \in R_{n+1},$$

then f has a unique fixed point x^* and $(f^n(x), x^*) \in R_n$, for each $x \in X$ and $n \in \mathbb{N}$.

In our papers [115] and [116], we consider, in the Eilenberg Theorem, instead equivalence relations R_n, the symmetric relations and in [116] we give a new notion: R-contraction. In what follows, we revisit this notion.

Let X be an abstract set and $R := (R_n)_{n\in\mathbb{N}}$ be a sequence of symmetric binary relations in X. By definition R is an admissible sequence iff the following conditions are satisfied:

(C_1) $X \times X = R_0 \supset R_1 \supset \ldots \supset R_n \supset \ldots$;

(C_2) $\bigcap\limits_{n\in\mathbb{N}} R_n = \Delta(X)$;

(C_3) the following implication holds,

$$(x_n)_{n\in\mathbb{N}} \subset X,\ x, y \in X,\ (x_n, x) \in R_n,\ (x_n, y) \in R_n,\ n \in \mathbb{N} \Rightarrow x = y.$$

By definition we call a pair (X, R), where R is an admissible sequence, an R-space.

Remark 2.1. From (C_2) we have that, if (X, R) is an R-space, then the binary relations R_n, are reflexive.

Definition 2.1. *A sequence $(x_n)_{n\in\mathbb{N}}$ in (X,R) is convergent if there exists $x \in X$ such that $(x_n, x) \in R_n$, for all $n \in \mathbb{N}$.*

From (C_3) it follows that the limit of a convergent sequence is unique. We denote a convergent sequence $(x_n)_{n\in\mathbb{N}}$ with the limit x, by $x_n \overset{R}{\to} x$.

By (C_2) it follows that, if $x_n = x$, for all $n \in \mathbb{N}$, then $x_n \overset{R}{\to} x$. Also, from Definition 2.1 and (C_1), each subsequence of a convergent sequence is a convergent sequence with the same limit as $(x_n)_{n\in\mathbb{N}}$.

From the above considerations it follows that an R-space structure on a set X induces an L-space structure of Fréchet type on X (see [49], [69], [43], [122], [140], ...).

Definition 2.2. An R-space (X,R) is complete if the following implication holds:

$$x_n \in X,\ (x_n, x_{n+p}) \in R_n,\ \text{for all } n,p \in \mathbb{N} \ \Rightarrow\ (x_n)_{n\in\mathbb{N}} \text{ is convergent.}$$

Remark 2.2. Let (X,R) be an R-space, where $R = (R_n)_{n\in\mathbb{N}}$ is a sequence of equivalence relations and $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. The following statements are equivalent:

(1) $(x_n, x_{n+1}) \in R_n$, for all $n \in \mathbb{N}$;
(2) $(x_n, x_{n+p}) \in R_n$, for all $n, p \in \mathbb{N}$.

Definition 2.3. Let (X,R) be an R-space. By definition an operator $f : X \to X$ is an R-contraction if the following implication holds:

$$(x,y) \in R_n \ \Rightarrow\ (f(x), f(y)) \in R_{n+1},\ \text{for all } n \in \mathbb{N}.$$

In the above terminology our basic results in [116] take the following form.

Theorem 2.1. *Let (X,R) be a complete R-space and $f : X \to X$ be an R-contraction. Then we have that:*

(i) $F_f = F_{f^n} = \{x^*\}$, for all $n \in \mathbb{N}^* = \mathbb{N} \setminus \{0\}$;
(ii) $f^n(x) \overset{R}{\to} x^*$, for all $x \in X$.

Theorem 2.2. *Let (X,R) be a complete R-space and $f : X \to X$ be an operator. We suppose that there exists $k \in \mathbb{N}^*$ such that f^k is an R-contraction. Then:*

(i) $F_f = F_{f^n} = \{x^*\}$, for all $n \in \mathbb{N}^*$;
(ii) $f^n(x) \overset{R}{\to} x^*$, for all $x \in X$.

Proof. From Theorem 2.1, $F_{f^k} = F_{f^{nk}} = \{x^*\}$, for all $n \in \mathbb{N}^*$ and $(f^{nk}(x), x^*) \in R_n$, for all $n \in \mathbb{N}^*$ and $x \in X$. Since f^k has a unique fixed point, x^*, it follows that (see Chu-Diaz [36]), $F_f = \{x^*\}$. If we take in the relations, $(f^{nk}(x), x^*) \in R_n$, for all $x \in X$ and instead of x, the following, $f(x), \ldots, f^{k-1}(x)$, we have that $(f^n(x), x^*) \in R_n$, for all $n \in \mathbb{N}$ and all $x \in X$. If we suppose that there exists $y^* \in X$ and $n_0 \in \mathbb{N}^*$ such that $f^{n_0}(y^*) = y^*$, then the sequence $(f^n(y^*))_{n\in\mathbb{N}}$ has the constant subsequence $(y^*)_{in_0\in\mathbb{N}}$. It follows that $y^* = x^*$. So, $F_f = F_{f^n} = \{x^*\}$, for all $n \in \mathbb{N}^*$. □

For other set-theoretical results in the fixed point theory see: [27], [28], [29], [48], [46], [56], [65], [79], [154], [102], [103], [104], [108], [117], [155], [6], [99], [156], [146], [77], ...

For set-theoretical aspects of fixed point theory of multivalued operators see [125] and the references therein.

The above considerations give rise to the following open problems.

Problem I. By P. Urysohn, an L-space $(X, \to)$ is an L^*-space if the following implication holds:

$(x_n)_{n\in\mathbb{N}} \subset X$, $x_n \not\to x \Rightarrow$ there exists a subsequence $(x_{n_i})_{i\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that for any subsequence $(z_n)_{n\in\mathbb{N}}$ of $(x_{n_i})_{i\in\mathbb{N}}$ we have that $z_n \not\to x$.

We have seen that each R-convergence on X is an L-convergence on X. The problem is which type of R-convergence on X is an L^*-convergence ?

References: [43], [21] and the references therein.

Problem II. Conversions between R-spaces and metric spaces, and between classes of operators on R-spaces to classes of operators on metric spaces.

Commentaries:

Let (X,R) be an R-space and $f:X\to X$ be an operator. Then by definition:

• f is continuous iff,

$$(x_n)_{n\in\mathbb{N}}\subset X,\ x\in X,\ (x_n,x)\in R_n,\ \text{for all } n\in\mathbb{N}\ \Rightarrow\ (f(x_n),f(x))\in R_n,\ \text{for all } n\in\mathbb{N};$$

• f is R-Kannan iff,

$$x,y\in X,\ (x,f(x))\in R_n,\ (y,f(y))\in R_n,\ \text{for all } n\in\mathbb{N}\ \Rightarrow\ (f(x),f(y))\in R_n,\ \text{for all } n\in\mathbb{N}.$$

On the other hand, in [66], J. Jachymski constructs on an R-space a metric. Moreover, the following result is given.

Jachymski Theorem. *Let X be a nonempty set, $f:X\to X$ be an operator and $l\in]0,1[$. The following statements are equivalent:*

(i) *There exists an R-structure on X such as in Eilenberg's Theorem.*

(ii) *There exists a non-Archimedean bounded and complete metric d on X such that f is an l-contraction.*

Let (X,d) be a bounded metric space, $f:X\to X$ be an operator and $l\in]0,1[$. For $n\in\mathbb{N}$, let $R_n:=\{(x,y)\mid d(x,y)\le l^n\delta(X)\}$. Then (X,R) is an R-space. If f is an R-contraction, then f satisfies the following metric condition:

$$x,y\in X,\ d(x,y)\le l^n\delta(X)\ \Rightarrow\ d(f(x),f(y))\le l^{n+1}\delta(X),\ \text{for all } n\in\mathbb{N}.$$

The problem is to study the conversions between fixed point theorems in R-spaces to fixed point theorems in generalized metric spaces.

Problem III. Structured sets with an R-space structure.

Let $(X,+)$ be an abelian group. Let $R=(R_n)_{n\in\mathbb{N}}$ be a complete R-space structure on X. In addition we suppose that:

(C_4) $z\in X$, $(x,y)\in R_n\Rightarrow(x+z,y+z)\in R_n$, for all $n\in\mathbb{N}$.

We call $(X,+,R)$, with $(X,+)$ and (X,R) as above, an R-group.

We have the following result (see [116]).

Theorem 2.3. *Let $(X,+,R)$ be an R-group, with a complete R-structure satisfying (C_4) and let $f:X\to X$ be an R-contraction. Then we have that:*

(i) *$1_X-f:X\to X$ is a bijective operator;*

(ii) *for each $y\in X$, the operator $g_y:X\to X$ defined by $x\mapsto f(x)+y$, has a unique fixed point x^*_y;*

(iii) *$(g^n_y(x),x^*_y)\in R_n$, for all $n\in\mathbb{N}$.*

Proof. From (C_4) we remark that g_y is an R-contraction, for each $y\in X$. From Theorem 2.1 we have (ii) and (iii). To finish the proof we observe that $(ii)\Rightarrow(i)$. □

The problem is to study the properties of operators on structured sets with R-space structure.

3. Equivalent statements with the set-theoretical one

In the paper [119] we have presented the following deep generalization of a well known converse to the contraction principle due to C. Bessaga (1959, [121], [140]).

Theorem 3.4. *Let X be a nonempty set and $f:X\to X$ be an operator. The following statements are equivalent:*

(1) *$F_f = F_{f^n} \neq \emptyset$, for all $n \in \mathbb{N}^*$.*
(2) *For each $l \in]0,1[$ there exists a complete metric d on X such that:*
 (a) *$f : (X,d) \to (X,d)$ is orbitally continuous;*
 (b) *$d(f^2(x), f(x)) \leq ld(x, f(x))$, for all $x \in X$, i.e., f is an l-graphic contraction;*
(3) *There exists a complete metric d on X such that $f : (X,d) \to (X,d)$ is a weakly Picard operator.*

Theorem 1 in [119] contains three other equivalent statements.

Revisiting this result and others that are similar, in what follows we give the following result.

Theorem 3.5 (Theorem of equivalent statements)**.** *Let X be a nonempty set and $f : X \to X$ be an operator. The following statements are equivalent:*

(1) *$F_f = F_{f^n} \neq \emptyset$, for all $n \in \mathbb{N}^*$.*
(2) *There exists an L-space structure on X with respect to which f is weakly Picard operator.*
(3) *For each $l \in]0,1[$ there exists a complete metric d on X such that:*
 (a) *f is an l-graphic contraction with respect to the metric d;*
 (b) *f is orbitally continuous with respect to the metric d.*
(4) *$F_f \neq \emptyset$ and there exists a metric d on X with respect to which f is asymptotically regular.*
(5) *There exists a metric d on X and a partition of X, $X = \bigcup_{\lambda \in \Lambda} X_\lambda$, such that:*
 (a) *$f(X_\lambda) \subset X_\lambda$, for all $\lambda \in \Lambda$;*
 (b) *$f\big|_{X_\lambda} : X_\lambda \to X_\lambda$ is a Picard operator with respect to the metric d, for all $\lambda \in \Lambda$.*

Proof. For $(1) \Leftrightarrow (3)$ and $(3) \Leftrightarrow (5)$ see [119].

$(2) \Rightarrow (1)$. Since f is a weakly Picard operator in $(X, \to)$, it follows that,
$$f^n(x) \to f^\infty(x) \text{ as } n \to \infty.$$
Let us suppose that there exists $n_0 \in \mathbb{N}^*$ and $y \in X$ such that $f^{n_0}(y) = y$. The sequence $(f^n(y))_{n\in\mathbb{N}}$ converges to the fixed point of f, $f^\infty(y)$. But the constant sequence (y) is a subsequence of $(f^n(y))_{n\in\mathbb{N}}$. So, $y = f(y)$.

$(1) \Rightarrow (2)$. Follows from $(1) \Rightarrow (3) \Rightarrow (2)$.

$(4) \Rightarrow (1)$. For some $n_0 \in \mathbb{N}^*$, let $y \in F_{f^{n_0}}$. Since f is asymptotically regular, the sequence, $(d(f^{kn_0}(y), f^{kn_0+1}(y))_{k\in\mathbb{N}} = (d(y, f(y)))_{k\in\mathbb{N}} \to 0$ as $k \to \infty$. So, $y = f(y)$.

$(1) \Rightarrow (4)$. Follows from $(1) \Rightarrow (3) \Rightarrow (4)$. □

For other statements equivalent with $F_f = F_{f^n} \neq \emptyset$, $n \in \mathbb{N}^*$, see J. Jachymski [65]. See also Rus-Petruşel-Şerban [141], V.G. Angelov [8].

Here are some references for the terms which appear in the *Equivalent Statements Theorem*:

• Weakly Picard operators: [122], [121], [140], [135], [131], [21], [143], [134], [22], [29], ...

• graphic contractions: [105], [106], [140], [143], [92], ...

• asymptotic regular operators: [137], [17], [21], [23] and the references therein.

Problem IV. To give equivalent statements with the set-theoretical statement:
$$F_f = F_{f^n} \neq \emptyset, \text{ for all } n \in \mathbb{N}^*.$$

Problem V. To give equivalent statements with the set-theoretical statement:
$$F_f = F_{f^n} = \{x^*\}.$$

Problem VI. To give equivalent statements with the set-theoretical statement:
$$\bigcap_{n\in\mathbb{N}} f^n(X) = \{x^*\}.$$

Problem VII. To give equivalent statements with the set-theoretical statement:

$$\bigcap_{n\in\mathbb{N}} f^n(X) = F_f.$$

Commentaries:

Let X be a nonempty set and $f : X \to X$ be an operator. If $\bigcap_{n\in\mathbb{N}} f^n(X) = \{x^*\}$, then $F_f = F_{f^n} = \{x^*\}$.

We have the following results.

Theorem 3.6. *Let X be a nonempty set and $f : X \to X$ be an operator. The following statements are equivalent:*

(1) $F_f = F_{f^n} = \{x^*\}$.
(2) *For each $l \in]0,1[$ there exists a complete metric on X such that $f : (X,d) \to (X,d)$ is an l-contraction.*
(3) *There exist $x^* \in X$ and a metric d on X such that:*
  (a) $F_f = \{x^*\}$;
  (b) $f : (X,d) \to (X,d)$ *is asymptotically regular.*
(4) *There exists an L-space structure on X such that $f : (X,\to) \to (X,\to)$ is a Picard operator.*

Proof. $(1) \Leftrightarrow (2)$. This is Bessaga's Theorem.

$(3) \Rightarrow (1)$. From (b) we have that $F_f = F_{f^n}$, for all $n \in \mathbb{N}^*$.

$(1) \Rightarrow (3)$. Follows from $(1) \Rightarrow (2) \Rightarrow (3)$.

$(4) \Rightarrow (1)$. Follows from the definition of a Picard operator.

$(1) \Rightarrow (4)$. Follows from $(1) \Rightarrow (2)$ and the fact that $(X, \xrightarrow{d})$ is an L-space. □

It is well-known the following result.

Janos Theorem. *Let (X,d) be a compact metric space and $f : X \to X$ be such that:*

(i) *f is continuous;*
(ii) $\bigcap_{n\in\mathbb{N}} f^n(X) = \{x^*\}$.

Then for each $l \in]0,1[$ there exists a metric ρ on X such that:

(a) *d and ρ are topologically equivalent;*
(b) *$f : (X,\rho) \to (X,\rho)$ is an l-contraction.*

For the following set-theoretical notions: Bessaga operators (i.e., $F_f = F_{f^n} = \{x^*\}$, for all $n \in \mathbb{N}$), Rus operators (i.e., $F_f = F_{f^n} \neq \emptyset$, for all $n \in \mathbb{N}^*$) and Janos operators (i.e., $\bigcap_{n\in\mathbb{N}} f^n(X) = \{x^*\}$) see: [109], [121], [140], [131], [143], [85], [138], [83], ...

4. Invariant operator with respect to an operator: invariant partition of a set with respect to an operator

One of the basic problem in the weakly Picard operator theory is the following.

Problem VIII. Let $(X,\to)$ be an L-space (R-space, topological space, metric space, Banach space, Hilbert space, ordered L-space, ...) and $f : X \to X$ be an operator. The problem is to give conditions on X and f which imply that f is a weakly Picard operator.

In which terms shall we give these conditions ?

It is natural to start with set-theoretical terms. With the Theorem of Equivalent Statements (Theorem 3.5) in mind, one of them is the following.

Definition 4.4. Let X be a nonempty set and $f : X \to X$ be an operator. By definition, a partition of $X = \bigcup_{\lambda \in \Lambda} X_\lambda$ is an invariant partition with respect to the operator f if $f(X_\lambda) \subset X_\lambda$, for all $\lambda \in \Lambda$. If $F_f \neq \emptyset$, $\Lambda = F_f$ and $F_f \cap X_{x^*} = \{x^*\}$, for all $x^* \in F_f$, then the corresponding invariant partition is called fixed point invariant partition of X.

In order to generate invariant partitions, the following terms are useful.

Definition 4.5. [[38], p. 299, [133], [33], [96]] Let X and Y be two sets and $f : X \to X$ be an operator. By definition, an operator $\Phi : X \to Y$ is invariant with respect to f (or is invariant operator for f) iff $\Phi \circ f = \Phi$.

It is important to remark that each surjective invariant operator $\Phi : X \to Y$ of an operator $f : X \to X$ generates an invariant partition of X with respect to the operator f, as follows. For $y \in Y$, let $X_y := \Phi^{-1}(y)$. Then $X = \bigcup_{y \in Y} X_y$ is an invariant partition of X with respect to the operator f.

For a better understanding of the above notions it is useful to see the following examples.

Example 4.1. [Interpolation set of an operator and invariant partition]

Let X, Y be nonempty sets, $\mathbb{M}(X,Y) := \{f : X \to Y \mid f \text{ is an operator } \}$ and $T : \mathbb{M}(X,Y) \to \mathbb{M}(X,Y)$ be an operator. By definition, a subset $I \subset X$ is an interpolation set for T iff

$$T(f)(x) = f(x), \text{ for all } f \in \mathbb{M}(X,Y) \text{ and all } x \in I.$$

Let $\Phi : \mathbb{M}(X,Y) \to \mathbb{M}(I,Y)$ be defined by $f \mapsto f\big|_I$, where $f\big|_I$ is the restriction of f to I. It is clear that $\Phi \circ T = \Phi$, i.e. Φ is an invariant operator of T. So, Φ generates the following invariant partition of the set $Z := \mathbb{M}(X,Y)$,

$$Z = \bigcup_{\lambda \in \mathbb{M}(I,Y)} Z_\lambda, \text{ where } Z_\lambda := \{f \in \mathbb{M}(X,Y) \mid f\big|_I = \lambda\}.$$

Example 4.2. Let $(X, \to)$ be an L-space and $f : X \to X$ be a weakly Picard operator. Then, the operator $\Phi := f^\infty : X \to F_f$ is an invariant operator of f.

Example 4.3. Let B be a Banach space and $T : C([a,b], B) \to C([a,b], B)$ be defined by

$$T(f)(x) := f(a) + \int_a^x K(x, s, f(s))ds, \; x \in [a,b],$$

where $K \in C([a,b] \times [a,b] \times B, B)$. Then a is an interpolation point of the operator T and $\Phi : C([a,b], B) \to B$, defined by $\Phi(f) := f(a)$, is an invariant operator for T. For $\lambda \in B$, let $Z_\lambda := \{f \in C([a,b], B) \mid f(a) = \lambda\}$. Then $C([a,b], B) = \bigcup_{\lambda \in B} Z_\lambda$ is an invariant partition with respect to the operator T. If $K(t, s, \cdot) : B \to B$ is L-Lipschitz for $t, s \in [a,b]$, then $T\big|_{Z_\lambda} : Z_\lambda \to Z_\lambda$ is a contraction with respect to a suitable Bielecki norm on Z_λ. From these we have that the operator T is a weakly Picard operator with respect to the uniform convergence on $C([a,b], B)$.

Example 4.4. Let B be a Banach space, $h > 0$ and $T : C([a-h, b], B) \to C([a-h, b], B)$, be defined by

$$T(f)(x) := \begin{cases} f(x), & x \in [a-h, a] \\ f(a) + \displaystyle\int_a^x K(x, s, f(s), f(s-h))ds, & x \in [a, b] \end{cases}$$

where $K \in C([a,b] \times [a,b] \times B \times B, B)$. Then $I := [a-h,a]$ is an interpolation set for T and $\Phi : C([a-h,b],\mathbb{R}) \to C([a-h,a],B)$, $f \mapsto f\big|_{[a-h,b]}$ is an invariant operator of T. For $\lambda \in C([a-h,b],B)$, let $Z_\lambda := \{f \in C([a-h,b],B) \mid f\big|_{[a-h,a]} = \lambda\}$. Then $C([a-h,b],B) = \bigcup\limits_{\lambda \in C([a-h,a],B)} Z_\lambda$ is an invariant partition with respect to the operator T. If $K(t,s,\cdot,\cdot) : B \times B \to B$ is L-Lipschitz for all $t,s \in [a,b]$ then $T\big|_{Z_\lambda} : Z_\lambda \to Z_\lambda$ is a contraction with respect to a suitable Bielecki norm on Z_λ. These imply that the operator T is a weakly Picard operator with respect to the uniform convergence on $C([a-h,b],B)$.

Example 4.5. Let $T : C[0,1] \to C[0,1]$ be a linear positive operator such that:

$$F_T = \{f \in C[0,1] \mid f(x) = c_1 x + c_2,\ c_1, c_2 \in \mathbb{R}\}.$$

Then it is well known that (see Rus [127] and the references therein) the set $I = \{0,1\}$ is an interpolation set for T. Then $\Phi : C[0,1] \to \mathbb{R}^2$, defined by $\Phi(f) := (f(0), f(1))$, is an invariant operator of T. For $\lambda \in \mathbb{R}^2$, let $Z_\lambda := \{f \in C[0,1] \mid f(0) = \lambda_1,\ f(1) = \lambda_2\}$. Then $C[a,b] = \bigcup\limits_{\lambda \in \mathbb{R}^2} Z_\lambda$ is an invariant partition with respect to T. If, for example, $T = B_n$, is a Bernstein operator, $n \in \mathbb{N}^*$ (see Rus [123]),

$$B_n(f)(x) := \sum_{k=0}^{n} f\left(\tfrac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k},\ x \in [0,1],$$

then we are in the above conditions, and $B_n : Z_\lambda \to Z_\lambda$, is a contraction with respect to the max-norm on $C[0,1]$. So, the Bernstein operators are weakly Picard operators with respect to the uniform convergence on $C[0,1]$.

Example 4.6. [Petruşel-Rus-Şerban [95]] Let $f : \mathbb{R}^m \to \mathbb{R}^m$ be a linear positive stochastic operator. Then the functional $\Phi : \mathbb{R}^m \to \mathbb{R}$, defined by $\Phi(x_1, \ldots, x_m) := \sum\limits_{k=1}^{m} x_k$ is an invariant functional of f. For $\lambda \in \mathbb{R}$, let $X_\lambda := \{x \in \mathbb{R}^m \mid \sum\limits_{k=1}^{m} x_k = \lambda\}$. Then $\mathbb{R}^m = \bigcup\limits_{\lambda \in \mathbb{R}} X_\lambda$ is an invariant partition with respect to f and $f\big|_{X_\lambda} : X_\lambda \to X_\lambda$ is a contraction with respect to the $\|\cdot\|_1$ on $\mathbb{R}^m$, for all $\lambda \in \mathbb{R}$. It follows that f is a weakly Picard operator.

Example 4.7. [András-Rus [5]] Let $(\mathbb{B}, \|\cdot\|)$ be a Banach space. For $x \in \mathbb{R}$, $\tilde{x} := (x, \ldots, x, \ldots)$ is the constant sequence defined by x. We denote by $s(\mathbb{B})$ the set of all sequences with elements in $\mathbb{B}$. We consider the L-space, $(s(\mathbb{B}), \xrightarrow{t})$, where $\xrightarrow{t}$ is the termwise convergence on $s(\mathbb{B})$. Also, we consider on $s(\mathbb{B})$ the family of pseudometrics, $\mathscr{D} := \{d_k \mid k \in \mathbb{N}$, where $d_k(u,v) := \max\limits_{0 \le n \le k} \|u_n - v_n\|\}$. Then the gauge space $(s(\mathbb{B}), \mathscr{D})$ is separated and complete. Moreover for $(u^n)_{n \in \mathbb{N}}$, $u \in s(\mathbb{B})$, we have that

$$u^n \xrightarrow{\mathscr{D}} u \ as\ n \to \infty \ \Rightarrow\ u^n \xrightarrow{t} u \ as\ n \to \infty.$$

Let us consider the Cesáro operator, $C : s(\mathbb{B}) \to s(\mathbb{B})$. This operator is defined by

$$(u_0, u_1, \ldots, u_n, \ldots) \mapsto (u_0, \tfrac{1}{2}(u_0 + u_1), \ldots, \tfrac{1}{n+1}(u_0 + u_1 + \ldots + u_n), \ldots).$$

We remark that the fixed point set of C is $F_C = \{\tilde{x} \mid x \in \mathbb{B}\}$.

For $x \in \mathbb{B}$ we consider $X_x := \{u \in s(\mathbb{B}) \mid u_0 = x\}$. Then $s(\mathbb{B}) = \bigcup\limits_{x \in \mathbb{B}} X_x$ is an invariant partition with respect to C. We remark that X_x is a closed subset of $s(\mathbb{B})$ and $C\big|_{X_x} : X_x \to X_x$ is a contraction with respect to the family of pseudometrics $\mathscr{D}$.

So, the operator $C : (s(\mathbb{B}), \xrightarrow{t}) \to (s(\mathbb{B}), \xrightarrow{t})$ is a weakly Picard operator.

A particular case of the Problem VIII is the following.

Problem IX (Rus, [131]). Let $(X, \to)$ be an L-space, $f : X \to X$ be an operator with $F_f \neq \emptyset$. The problem is in which conditions there exists an invariant subset Y of f such that:

$$f\big|_Y : Y \to Y \text{ is } WPO \;\Rightarrow\; f : X \to X \text{ is } WPO\,?$$

Regarding the technique of invariant partitions in studying the Probelms VIII and IX see: Rus ([123], [133], [122]), Agratini-Rus [3], J. Jachymski [67], András-Rus [5], Gonska-Kacsó-Piţul [52], Gonska-Piţul [53], Cătinaş-Otrocol-Rus [33], Petruşel-Rus-Şerban ([96], [95]), Bacoţiu [12], Cătinaş-Otrocol [32], ...

Regarding the interpolation set see [127] and the references therein (Altomare-Campiti (1994), Boboc-Bucur (1976), Gavrea-Ivan (2005), Raşa (2009)).

For some applications to differential and integral equations see Rus [128] and the references therein (Bacoţiu (2008), Buică-Ilea (2007), Dincuţă (2000), Dobriţoiu (2009), E. Egri (2008), R. Gabor (2006), V. Mureşan (2003), Olaru (2010), Otrocol (2005), Şerban (2002)).

5. Fixed point structures on an abstract set

Another way to use set-theoretical terms in the fixed point theory started to be constructed in 1986 ([111], [112], [113], [118], ...). In this section we revisit this construction.

5.1. **Examples of fixed point structures.** For two sets X, Y, we denote by

$$\mathbb{M}(X, Y) := \{f : X \to Y \mid f \text{ is an operator}\}$$

and by

$$\mathbb{M}(X) := \mathbb{M}(X, X).$$

Let M be an operator defined as follows

$$M : D_M \subset P(X) \times P(X) \multimap \bigcup_{U,V \in P(X)} \mathbb{M}(U, V),$$

such that $(U, V) \mapsto M(U, V) \subset \mathbb{M}(U, V)$.

We denote by $M(U) := \mathbb{M}(U, U)$.

Definition 5.6. By a fixed point structure $(f.p.s.)$ on a nonempty set X we understand a triple $(X, S(X), M)$ with the following properties:

(i) $S(X) \subset P(X)$ and $U \in S(X) \Rightarrow (U, U) \in D_M$;

(ii) $U \in S(X)$, $f \in M(U) \Rightarrow F_f \neq \emptyset$;

(iii) M is such that
$$(Y, Y) \in D_M,\ Z \in P(Y),\ (Z, Z) \in D_M \;\Rightarrow\; M(Z) \supset \{f\big|_Z \mid f \in M(Y)\}.$$

A triple $(X, S(X), M)$ which satisfies (i) and (ii) is called a large fixed point structure $(l.f.p.s.)$.

Example 5.8. [The trivial f.p.s.] X is a set, $S(X) := \{\{x\} \mid x \in X\}$ and for $Y \in P(X)$, $M(Y) := \mathbb{M}(Y)$.

Example 5.9. [The f.p.s. of R-contractions] (X, R) is a complete R-space, $S(X) = \{X\}$ and $M(X) := \{f : X \to X \mid f \text{ is an } R-\text{contraction}\}$.

Example 5.10. [The f.p.s. of contractions] (X, d) is a complete metric space, $S(X) := P_{cl}(X) := \{Y \in P(X) \mid Y = \overline{Y}\}$ and $M(Y) := \{f : Y \to Y \mid f \text{ is a contraction}\}$.

Example 5.11. [the f.p.s of Tarski] $(X, \leq)$ is a complete lattice, $S(X) := \{X\}$ and $M(Y) := \{f : Y \to Y \mid f \text{ is increasing}\}$.

Example 5.12. [The f.p.s. of progressive operators] $(X,\leq)$ is a partially ordered set, $S(X) := \{Y \in P(X) \mid (Y,\leq)$ has a maximal element$\}$ and $M(Y) := \{f : Y \to Y \mid x \leq f(x),$ for all $x \in Y\}$.

Example 5.13. [The f.p.s. of Schauder] X is a Banach space, $S(X) := P_{cp,cv}(X)$ and $M(Y) := C(Y,Y)$.

Example 5.14. [The f.p.s. of Browder-Ghöde-Kirk] X is a uniformly convex Banach space, $S(X) := P_{b,cl,cv}(X)$ and $M(Y) := \{f : Y \to Y \mid f$ is nonexpansive$\}$.

Example 5.15. [The f.p.s. of Nemytzki-Edelstein] (X,d) is a metric space, $S(X) := P_{cp}(X)$ and $M(Y) := \{f : Y \to Y \mid f$ is a contractive operator$\}$.

It is clear that for each fixed point theorem we have at least a f.p.s.
Here are the basic problems of the f.p.s. theory.

Problem X (Maximal f.p.s.) Which are the f.p.s. $(X, S(X), M)$ with the following property: there exists $S_1(X) \supset S(X)$ such that $S(X) = \{Y \in S_1(X) \mid f \in M(Y) \ \Rightarrow\ F_f \neq \emptyset\}$?

Problem XI (f.p.s. with the common fixed point property). Which are the f.p.s. $(X, S(X), M)$ with the following property:

$$Y \in S(X),\ f,g \in M(Y),\ f \circ g = g \circ f \ \Rightarrow\ F_f \cap F_g \neq \emptyset\ ?$$

Problem XII (f.p.s. with the coincidence property). Which are the f.p.s. $(X, S(X), M)$ with the following property:

$$Y \in S(X),\ f,g \in M(Y),\ f \circ g = g \circ f \ \Rightarrow\ \{x \in Y \mid f(x) = g(x)\} \neq \emptyset\ ?$$

References: [126], [130], [111], [112], [113], [117], [118], [140], [96], [136], [149], [153], [10], [139].

5.2. **Compatible pair with a fixed point structure.** Let $(X, S(X), M)$ be a f.p.s., $\theta : Z \to \mathbb{R}_+$, with $S(X) \subset Z \subset P(X)$ and $\eta : \mathcal{P}(X) \to \mathcal{P}(X)$. By definition ([126], p. 52), the pair (θ,η) is a compatible pair with $(X, S(X), M)$ iff:

(1) η is a closure operator, $S(X) \subset \eta(Z) \subset Z$ and $\theta(\eta(Y)) = \theta(Y)$, for all $Y \in Z$;
(2) $F_\eta \cap Z_\theta \subset S(X)$, where $Z_\theta := \{A \in Z \mid \theta(A) = 0\}$.

Example 5.16. On a metric space (X,d) let $(X, S(X), M)$ be the f.p.s. of Nemytski-Edelstein. Let $Z := P_b(X)$, $\theta = \alpha_k$- the Kuratowski measure of noncompactness on X and $\eta(Y) = \overline{Y}$. Then (θ,η) is a compatible pair with $(X, S(X), M)$.

Example 5.17. On a Banach space X, let $(X, S(X), M)$ be the f.p.s. of Schauder. Let $Z := P_b(X)$, $\theta := \alpha_k$ and $\eta(Y) = \overline{co}Y$ - the closed convex hull of Y. Then (θ,η) is a compatible pair with the fixed point structure of Schauder on X.

Another problem concerning the f.p.s. theory is the following.

Problem XIII. Let $(X, S(X), M)$ be a f.p.s. on a structured set X. The problem is to look at the compatible pair with $(X, S(X), M)$.
References: Rus [126] and [130], and the references therein.

5.3. **Closure operators and invariant subsets.** Let X be a nonempty set. By definition, an operator $\eta : \mathcal{P}(X) \to \mathcal{P}(X)$ is a closure operator iff:

(i) $Y \subset \eta(Y)$, for all $Y \subset X$;
(ii) $Y, Z \subset X, Y \subset Z \Rightarrow \eta(Y) \subset \eta(Z)$;
(iii) $\eta \circ \eta = \eta$.

Here are some examples of closure operators.

In a real linear space X, a convex hull operator, $\eta : \mathcal{P}(X) \to \mathcal{P}(X)$, $Y \mapsto coY$, is a closure operator.

In a topological space X, the operator $\eta : \mathcal{P}(X) \to \mathcal{P}(X)$, $Y \mapsto \overline{Y}$ is a closure operator.

In a linear topological space, the operator, $\eta : \mathcal{P}(X) \to \mathcal{P}(X)$, $Y \mapsto \overline{co}Y := \overline{coY}$, is a closure operator.

Another example is the following (see Rus-Şerban [144]).

Let X_i, $i = \overline{1,m}$ be some nonempty sets and $X := X_1 \times X_2 \times \ldots \times X_m$ be their cartesian product. Let us denote by, $\Pi_i : X \to X_i$ the canonical projection of X on X_i, $i = \overline{1,m}$. If $Y \subset X$, then the cartesian hull of Y is

$$caY := \Pi_1(Y) \times \ldots \Pi_m(Y).$$

The operator $ca : \mathcal{P}(X) \to \mathcal{P}(X)$, defined by $Y \mapsto caY$ is a closure operator.

Other examples of closure operators come from generalized notions of convexity. See for example: I. Singer [152], R. Precup [98] and I.A. Rus [132].

In the terms of closure operators we have the following general result.

General Invariant Subset Lemma (Rus [126], p. 21). *Let X be a nonempty set, $\eta : \mathcal{P}(X) \to \mathcal{P}(X)$ be a closure operator, $Y \in F_\eta$ be a set, $y \in Y$ be a point and $f : Y \to Y$ be an operator. Then there exists a subset $Y_0 \subset Y$ such that:*

(1) $y \in Y_0$;
(2) $Y_0 \in F_\eta$;
(3) $f(Y_0) \subset Y_0$;
(4) $\eta(f(Y_0) \cup \{y\}) = Y_0$.

5.4. **Fixed point theorems in terms of a compatible pair: θ-condensing operators.** Let X be a nonempty set, $Z \subset P(X)$, $Z \neq \emptyset$ and $\theta : Z \to \mathbb{R}_+$ be a functional. An operator $f : X \to X$ is strongly θ-condensing iff:

(1) $A \in Z \Rightarrow f(A) \in Z$;
(2) $A \in Z, \theta(A) \neq 0 \Rightarrow \theta(f(A)) < \theta(A)$.

If in the above definition, instead of (2) we have,

(2′) $A \in Z$, $f(A) \subset A$, $\theta(A) \neq 0 \Rightarrow \theta(f(A)) < \theta(A)$,

then the operator f is called θ-condensing.

One of the basic set-theoretical fixed point result, in terms of a fixed point structure is the following.

Theorem 5.7. *Let $(X, S(X), M)$ be a f.p.s., (θ, η) ($\theta : Z \to \mathbb{R}_+$) be a compatible pair with $(X, S(X), M)$. Let $Y \in \eta(Z)$ and $f \in M(Y)$. We suppose that:*

(i) *$A \in Z, x \in Y$ imply that $A \cup \{x\} \in Z$ and $\theta(A \cup \{x\}) = \theta(A)$;*
(ii) *f is a θ-condensing operator.*

Then we have that:

(a) *there exists $A_0 \in S(X)$ such that $f(A_0) \subset A_0$;*
(b) *$F_f \neq \emptyset$;*
(c) *if $F_f \in Z$, then $\theta(F_f) = 0$.*

Proof. $(a) + (b)$. For $x \in Y$, by the General Invariant Subset Lemma, there exists $A_0 \in F_f$ with $f(A_0) \subset A_0$ and $\eta(f(A_0) \cup \{x\}) = A_0$. We have

$$\theta(\eta(f(A_0) \cup \{x\})) = \theta(f(A_0) \cup \{x\}) = \theta(f(A_0)) = \theta(A_0)$$

Since f is θ-condensing it follows that $\theta(A_0)=0$, i.e., $A_0\in Z_\theta$. From $A_0\in F_\eta\cap Z_\theta$, $f(A_0)\subset A_0$ and $f\big|_{A_0}\in M(A_0)$, and (θ,η) is a compatible pair with $(X,S(X),M)$ we have that $F_f\neq\emptyset$.

(c). By $F_f\in Z$ and $f(F_f)=F_f$ it follows that $\theta(F_f)=0$. □

From the proof of Theorem 5.7 we have:

Theorem 5.8. *In the Theorem 5.7 instead of condition (ii) we can put the following one:*
(ii') $f:f(Y)\to f(Y)$ *is θ-condensing.*

Our Theorems 5.7 and 5.8 are set-theoretical generalizations of many known fixed point theorems of Sadowski type (B.N. Sadowski (1967), H. Amann (1973), J. Appell (2005), J. Banas and K. Goebel (1980), S. Czerwik (1980), G. Emmanuele (1981), M. Furi - A. Vignoli (1969), J.K. Hale - O. López (1973), K. Iseki (1976), ...).

For example, if X is a Banach space and $(X,S(X),M)$ is a Schauder f.p.s., $Z:=P_b(X)$, $\eta:=\overline{co}$ and $\theta:=\alpha_H$ - the Hausdorff measure of noncompactness and f is a strong α_H - condensing operator, from Theorem 5.7, we have the well known Sadowski fixed point theorem.

6. Closure operators and invariant subset in terms of iterates: (θ,l)-contractions

6.1. **Invariant subsets.** Let X be a nonempty set, $\eta:\mathcal{P}(X)\to\mathcal{P}(X)$ be a closure operator, $Y\in F_\eta$ and $f:Y\to Y$ be an operator. Let

$$Y_1:=\eta(f(Y)),\ Y_2:=\eta(f(Y_1)),\ \dots\ ,\ Y_n:=\eta(f(Y_{n-1})),\ n\in\mathbb{N}^*.$$

We remark that $Y_{n+1}\subset Y_n$, $Y_n\in F_\eta$ and $Y_n\in I(f):=\{A\subset Y\mid f(A)\subset A\}$. Let us denote $Y_\infty:=\bigcap\limits_{n\in\mathbb{N}}Y_n$. It is clear that Y_∞ is an invariant subset of f.

The following problem rises.

Problem XIV. Let $(X,S(X),M)$ be a f.p.s. on a set X and $f\in M(Y)$. The problem is in which conditions we have that: $Y_\infty\neq\emptyset$ and $Y_\infty\in S(X)$?

If f is a solution of Problem XIV, then $F_f\neq\emptyset$.

In 1986 we started to study the Problem XIV. Till now, it stills open.

To study the problem we introduced two notions: operator with intersection property and (θ,l)-contraction.

6.2. **Operators with intersection property.** Let $(O,\to,\leq)$ be an ordered L-space with the least element, 0. Let X be a nonempty set, $Z\subset P(X)$, $Z\neq\emptyset$. By definition, an operator $\theta:Z\to O$ has the intersection property if $Y_n\in Z$, $Y_{n+1}\subset Y_n$, $n\in\mathbb{N}$ and $\theta(Y_n)\to 0$ as $n\to\infty$, imply that:

$$Y_\infty:=\bigcap_{n\in\mathbb{N}}Y_n\neq\emptyset,\ Y_\infty\in Z \text{ and } \theta(Y_\infty)=0.$$

Example 6.18. Let (X,d) be a complete metric space, $Z:=P_{b,cl}(X)$, $O:=\mathbb{R}_+$. Then δ, α_k and α_H are functionals with intersection property.

Example 6.19. Let X be a locally convex space and $(p_i)_{i\in I}$ be a family of seminorms which generates the topology on X. Let $Z:=P_{b,cl}(X)$ and $O:=\mathbb{M}(I,\mathbb{R}_+)$. We define the operator, $\theta:Z\to\mathbb{M}(I,\mathbb{R}_+)$, by $\theta(A):=$ with the function $i\mapsto\alpha_K^i(A)$, where α_K^i is the Kuratowski measure of noncompactness with respect to the seminorm p_i. In a similar way we can define θ with respect to the diameter functional, δ, and to the Hausdorff measure of noncompactness, α_H. The operator θ has the intersection property.

For other examples of such operators see [126], pp. 41-55.

Problem XV. To look for examples of operators with intersection property.

6.3. **Fixed point theorems in terms of an operator with intersection property:** (θ, l)-**contraction.** We are starting by introducing the notion of (θ, l)-contraction.

Let X be a nonempty set, $Z \subset P(X)$, $Z \neq \emptyset$, $\theta : Z \to \mathbb{R}_+$ be an operator and $l \in]0, 1[$.

Definition 6.7. An operator $f : X \to X$ is a (θ, l)-contraction iff:

(i) $A \in Z \Rightarrow f(A) \in Z$;
(ii) $\theta(f(A)) \leq l\theta(A)$, for all $A \in Z$ with $f(A) \subset A$.

Definition 6.8. An operator $f : X \to X$ is a strongly (θ, l)-contraction iff:

(i) $A \in Z \Rightarrow f(A) \in Z$;
(ii) $\theta(f(A)) \leq l\theta(A)$, for all $A \in Z$.

Example 6.20. Let (X, d) be a metric space, $Z := P_b(X)$ and $\theta := \delta$. Then an operator $f : X \to X$ is a (δ, l)-contraction iff: $A \in P_b(X) \Rightarrow f(A) \in P_b(X)$ and $\delta(f(A)) \leq l\delta(A)$, for all $A \in P_b(X)$ with $f(A) \subset A$.

It is important to remark that f is a strong (δ, l)-contraction iff f is an l-contraction.

Example 6.21. Let (X, d) be a metric space and $Z := P_b(X)$. If $f : X \to X$ is a compact operator, then f is an (α_K, l)-contraction, for each $l \in]0, 1[$.

Example 6.22. The radial retraction on a Banach space X to $\overline{B}(0; 1)$ is a strong α_K-nonexpansive mapping, i.e.

$$\alpha_K(\rho(A)) \leq \alpha_K(A), \text{ for all } A \in P_b(X).$$

My set-theoretical fixed point results for (strong) (θ, l)-contractions are the following (see [126], pp. 69-70).

Theorem 6.9. *Let $(X, S(X), M)$ be a f.p.s. on a set X, (θ, η) ($\theta : Z \to \mathbb{R}_+$) be a compatible pair with $(X, S(X), M)$. Let $Y \in \eta(Z)$ and $f \in M(Y)$. We suppose that:*

(i) *$\theta\big|_{\eta(Z)}$ has the intersection property;*
(ii) *f is an (θ, l)-contraction.*

Then we have that:

(a) *$I(f) \cap S(X) \neq \emptyset$;*
(b) *$F_f \neq \emptyset$;*
(c) *if $F_f \in Z$, then $\theta(F_f) = 0$.*

Theorem 6.10. *Let $(X, S(X), M)$ be a f.p.s. on a set X, (θ, η) be a compatible pair with $(X, S(X), M)$. Let $Y \in F_\eta$ and $f \in M(Y)$ be such that $f(Y) \in Z$. We suppose that:*

(i) *$\theta\big|_{\eta(Z)}$ has the intersection property;*
(ii) *$f\big|_{f(Y)} : f(Y) \to f(Y)$ is a (θ, l)-contraction.*

Then we have that:

(a) *$I(f) \cap S(X) \neq \emptyset$;*
(b) *$F_f \neq \emptyset$;*
(c) *If $F_f \in Z$, then $\theta(F_f) = 0$.*

Our Theorem 6.9 and 6.10 are set-theoretical generalizations of many known fixed point theorems of Darbo type (G. Darbo (1955), J. Appell (2005), J. Banas and K. Goebel (1980), V. Berinde (1997), O. Hadžić (1984), A. Horvat-Marc and M. Berinde (2004), J. Esenfeld - V. Lakshmikantham (1975), I.A. Rus (1983), A. Petruşel (1987), C.S. Barroso - D. O'Regan (2005), R.D. Nussbaum (1969), ...). For example, if X is a Banach space,

$S(X) := P_{cp,cv}(X)$, $\theta := \alpha_K$ and f is a strong (α_K, l)-contraction, then from Theorem 6.9 we have the well known Darbo theorem.

For other results for (θ, l)-contraction, (θ, φ)-contractions, for some type of operator in terms of measures of noncompactness, measure of weak noncompactness, measure of nonconvexity, ...see [126], [55], [74], [106], [111], [112], [113], [120], [140], [130], [101], [144], [136], [149], [99], [110], [37], [150], [11], [13], [84], ...

7. Cyclic covering of a set with respect to an operator

Another set-theoretical notion introduced in [124] is the following.

Definition 7.9. Let X be a nonempty set and $f : X \to X$ be an operator. By definition, a covering (representation in [124]) of X, $X = \bigcup_{i=1}^{m} X_i$, $m \geq 2$, is a cyclic covering of X relative to f iff:

$$f(X_1) \subset X_2, \ \dots, \ f(X_{m-1}) \subset X_m, \ f(X_m) \subset X_1.$$

In this case we call the operator f, a cyclic operator.

The following problem is studied in [124].

Problem XVI. Let $(X, S(X), M)$ be a f.p.s. on a set X, $Y \subset X$ and $f : Y \to Y$ be an operator. We suppose that $Y = \bigcup_{i=1}^{m} A_i$ is a cyclic covering of Y with respect to the operator f. The problem is in which conditions we have that:

(a) $A := \bigcap_{i=1}^{m} A_i \neq \emptyset$;

(b) $A \in S(X)$ and $f\big|_A \in M(A)$.

If f is a solution of Problem XVI, then $F_f \neq \emptyset$.

To study the Problem XVI, the following remarks are useful ([124]).

Inclusion remark. If $X = \bigcup_{i=1}^{m} X_i$ is a cyclic covering of X with respect to an operator $f : X \to X$, then $F_f \subset \bigcap_{i=1}^{m} X_i$.

Fixed point lemma for cyclic operators. *Let $(X, S(X), M)$ be a f.p.s. on an L-space $(X, \to)$. Let $A_1, \dots, A_m \in P_{cl}(X)$, $Y := \bigcup_{i=1}^{m} A_i$ and $f : Y \to Y$ be an operator. We suppose that:*

(i) *$Y = \bigcup_{i=1}^{m} A_i$ is a cyclic covering of Y relative to f;*

(ii) *there exists $x_0 \in Y$ such that $(f^n(x_0))_{n\in\mathbb{N}}$ converges;*

(iii) *if $A := \bigcap_{i=1}^{m} A_i \neq \emptyset$, then $A \in S(X)$ and $f\big|_A \in M(A)$.*

Then $F_f \neq \emptyset$.

Periodic point lemma. *Let $(X, S(X), M)$ be a f.p.s. on a set X. Let $A_1, \dots, A_m \in P(X)$, $Y := \bigcup_{i=1}^{m} A_i$ and $f \in M(Y)$. We suppose that:*

(i) *$Y = \bigcup_{i=1}^{m} A_i$ is a cyclic representation of Y with respect to the operator f;*
(ii) *at least $A_i \in S(X)$;*
(iii) *the operator M is such that, $g, h \in M \Rightarrow g \circ h \in M$.*

Then we have that $F_{f^m} \neq \emptyset$. If $F_{f^m} = \{x^*\}$, then $F_f = \{x^*\}$.

For some results for Problem XVI see, for example, Kirk et al. [75], Păcurar-Rus [86], G. Petruşel [97] and L. Pasicki [87].

8. Equivalent fixed point equations

8.1. **Admissible perturbation of an operator.** In this section we shall consider the following problem.

Problem XVII. Let $(X, \to)$ be an L-space and $f : X \to X$ be an operator. The problem is to find those conditions in which there exists an operator $g : X \to X$ such that:

(1) $F_f = F_g$;
(2) g is a weakly Picard operator.

To study this problem, we have introduced in [129] a set-theoretical notion as follows.

Let X be a nonempty set and $G : X \times X \to X$ be an operator. We suppose that:

(A_1) $G(x, x) = x$, for all $x \in X$;
(A_2) $x, y \in X$, $G(x, y) = x$ imply $y = x$.

Here are some examples of this type of operator G.

Example 8.23. Let $(V, +, \mathbb{R})$ be a real vectorial space, $X \subset V$ be a convex subset, $\lambda \in]0, 1[$, $f : X \to X$ and $G : X \times X \to X$ be defined by:

$$G(x, y) := (1 - \lambda)x + \lambda y.$$

Example 8.24. Let (X, d) be a metric space endowed with the W-convex structure of Takahashi. The operator $W : X \times X \times [0, 1] \to X$ has the following property:

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y), \text{ for all } x, y, u \in X \text{ and all } \lambda \in [0, 1].$$

If in addition we suppose that,

$$\lambda \in]0, 1[, \; W(x, y, \lambda) = x \text{ implies } y = x,$$

then we take $G(x, y) := W(x, y, \lambda)$, $\lambda \in]0, 1[$.

Let $f : X \to X$ be an operator and $G : X \times X \to X$ be defined as above. We consider the operator $f_G : X \to X$ be defined by

$$f_G(x) := G(x, f(x)), \text{ for all } x \in X.$$

We remark that $F_{f_G} = F_f$, i.e., the fixed point equations,

$$x = f(x)$$

and

$$x = f_G(x)$$

are equivalent.

We call the operator, f_G, the admissible perturbation of f corresponding to the operator G.

For $f : (X, \to) \to (X, \to)$ we consider for the fixed point equation,

$$x = f(x)$$

the iteration algorithm,

$$x_0 \in X,\ x_{n+1} = G(x_n, f(x_n)),\ n \in \mathbb{N}.$$

By definition, this algorithm is convergent if, $x_n \to x^*(x_0) \in F_f$ as $n \to \infty$, for all $x_0 \in X$.

We remark that, $x_n = f_G^n(x_0)$. This implies that our algorithm is convergent iff f_G is WPO.

It is clear that, $f_G^\infty : X \to F_f$ is a set retraction.

To study the Problem XVII it is useful to study the following problems.

Problem XVIII. If (X, d) is a metric space, in which conditions, f_G is asymptotically regular ?

Problem XIX. If $(X, \to)$ is an L-space, in which conditions, f_G is WPO ?

For some results for Problems XVII, XVIII and XIX, see: I.A. Rus [129], [137], V. Berinde [17], [18], V. Berinde - Şt. Măruşter - I.A. Rus [20], V. Berinde - I.A. Rus [23], A. Petruşel - I.A. Rus [91], E. Toscano - C. Vetro [158], V. Berinde - A.B. Khan - M. Păcurar [19], C. Ţicală [157], ...

9. Set retraction and fixed points of nonself operators

Let us introduce some of the problems of the fixed point theory of nonself operators.

Problem XX (see [130]). Let $(X, S(X), M)$ be a f.p.s. on a set X, $Y \in S(X)$ and $f \in M(Y, X)$. In which conditions we have that, $F_f \neq \emptyset$?

For a better understanding of this problem we present the following examples.

Example 9.25. [The case of f.p.s. of R-contractions] Let (X, R) be a complete R-space, $Y \subset X$ be a closed subset and $f : Y \to X$ be an R-contraction. In which conditions we have that $F_f \neq \emptyset$?

Example 9.26. [The case of f.p.s. of contractions] Let X be a complete metric space, $Y \in P_{cl}(X)$ and $f : Y \to X$ be a contraction. In which conditions we have that, $F_f \neq \emptyset$?

Example 9.27. [The case of f.p.s. of Schauder] Let X be a Banach space, $Y \in P_{cp,cv}(X)$ and $f \in C(Y, X)$. In which conditions we have that, $F_f \neq \emptyset$?

Example 9.28. [The case of f.p.s. of Browder] Let X be a Hilbert space, $Y \in P_{b,cv}(X)$ and $f : Y \to X$ be a nonexpansive operator. In which conditions we have that, $F_f \neq \emptyset$?

Problem XXI (see [132]). Let X be a nonempty set, $Y \subset X$ be a nonempty subset of X and $f : Y \to X$ be a nonself operator. The problem is to find an operator $\rho_f : Y \to Y$ such that $F_f = F_{\rho_f}$.

To study these problems, we shall use two set-theoretical terms: retraction and retractible operator.

Let X be a nonempty set, $Y \subset X$ be a nonempty subset of X. By definition, an operator $\rho : X \to Y$ is a set retraction if, $\rho\big|_Y = 1_Y$.

If $(X, \leq)$ is an ordered set and $Y \subset X$ is a nonempty subset, then a set retraction $\rho : X \to Y$ is an ordered set retraction if ρ is an increasing operator. On an ordered set, these are some classes of set retraction. For example, let $\rho : X \to Y$ be a set retraction. Then:

- ρ is a comparable retraction if $\rho(x)$ is comparable to x, for all $x \in X$;
- ρ is progressive retraction (or u_p-retraction) if $x \leq \rho(x)$, for all $x \in X$;
- ρ is regressive retraction (or down-retraction) if $\rho(x) \leq x$, for all $x \in X$;
- ρ is decreasing retraction if ρ is a decreasing operator.

If (X, τ) is a topological space, $Y \subset X$ is a subset of X, then a set retraction $\rho : X \to Y$ is a topological retraction if ρ is continuous.

In a metric space (X, d) we have the following types of retraction: Lipschitz retraction, nonexpansive retraction, continuous retraction, ...

In the paper [28], R.F. Brown introduced the following set-theoretical notion:

Let $\rho : X \to Y$ be a set retraction and $f : Y \to X$ be an operator. By definition f is retractible with respect to the retraction ρ if, $F_f = F_{\rho\circ f}$. In this case $\rho \circ f$ is called the retract of f.

From the above considerations the following problem rises.

Problem XXII. Let $(X, S(X), M)$ be a f.p.s. on a set X. The problem is in which conditions for each $Y \in S(X)$ there exists a set retraction, $\rho : X \to Y$ such that for all $f \in M(Y, X)$, $\rho \circ f \in M(Y)$.

For this problem, we have the following result.

Theorem 9.11 (see Rus [130])**.** *Let $(X, S(X), M)$ be a f.p.s. on a set X and (θ, η) be a compatible pair with $(X, S(X), M)$. Let $Y \in \eta(Z)$, $f : Y \to X$ be an operator and $\rho : X \to Y$ be a set retraction. We suppose that:*

(i) $\theta\big|_{\eta(Z)}$ *is a functional with intersection property;*
(ii) *f is retractible with respect to ρ and $\rho \circ f \in M(Y)$;*
(iii) *ρ is (θ, l_1)-Lipschitz ($l_1 \in \mathbb{R}^*_+$);*
(iv) *f is a strong (θ, l)-contraction and $ll_1 < 1$.*

Then $F_f \neq \emptyset$ and if $F_f \in Z$, then $\theta(F_f) = 0$.

In the case of Schauder f.p.s., from Theorem 9.11 we have:

Theorem 9.12. *Let X be a Banach space, $\alpha_K : P_b(X) \to \mathbb{R}_+$ be the Kuratowski measure of noncompactness on X and $f : \overline{B}(0; R) \to X$ be a continuous operator. We suppose that:*

(1) *f is a strong (α_K, l)-conraction;*
(2) *f is retractible with respect to the radial retraction, $\rho : X \to \overline{B}(0; R)$.*

Then $F_f \neq \emptyset$ and F_f is a compact subset of X.

We remark that each of the following conditions implies the condition (ii) in the Theorem 9.12:

(a) (Leray-Schauder). $x \in \partial\overline{B}(0; R)$, $f(x) = \lambda x \Rightarrow \lambda \leq 1$;
(b) (Rothe) $f(\partial\overline{B}(0; R)) \subset \overline{B}(0; R)$;
(c) (Altman) $\|f(x) - x\|^2 \geq \|f(x)\|^2 - \|x\|^2$, for all $x \in \partial\overline{B}(0; R)$.

There are many other *boundary conditions* which appear in the fixed point theorems for nonself operators (see [55], [114], [126], [140], [132], [72], ...).

In the paper [132] we introduced a set-theoretical notion: *generalized retract of an operator*.

Let X be a nonempty set, $Y \subset X$ and $f : Y \to X$ be an operator. By definition, a self operator $\rho_f : Y \to Y$ is called a generalized retract of the nonself operator f iff:

(a) $F_f = F_{\rho_f}$;
(b) $x \in Y$, $f(x) \in Y \Rightarrow \rho_f(x) = f(x)$.

In [132] we give a generic example of generalized retract of an operator in terms of another set-theoretical notion: *interval operator*.

Let X be a nonempty set. By definition an operator $[\cdot, \cdot] : X \times X \to P(X)$ is called interval operator if it satisfies the following conditions:

(a) $[x, y] = [y, x]$, for all $x, y \in X$;

(b) $x, y \in [x, y]$, for all $x, y \in X$;
(c) $[x, x] = \{x\}$, for all $x \in X$.

Example 9.29. Let $(X, \leq)$ be a supsemilattice and $x, y \in X$. Then

$$[x, y]_{\leq} := \{z \in X \mid x \leq z \leq x \vee y\} \cup \{z \in X \mid y \leq z \leq x \vee y\},$$

defines an interval operator.

Example 9.30. Let X be a linear space and $x, y \in X$. Then

$$[x, y]_{\leq} := \{(1-\lambda)x + \lambda y \mid 0 \leq \lambda \leq 1\}$$

defines an interval operator.

Example 9.31. Let (X, d) be a metric space and $x, y \in X$. Then

$$[x, y]_d := \{z \in X \mid d(x, z) + d(z, y) = d(x, y)\}$$

defines an interval operator.

Let $(X, [\cdot, \cdot])$ be a nonempty set with an interval structure. Let $Y \subset X$ and $f : Y \to X$ be an operator. We suppose that:

(GR) $$x \in Y,\ f(x) \in X \setminus Y \ \Rightarrow]x, f(x)] \cap Y \neq \emptyset.$$

We call this condition, generalized retract condition. Supposing this condition, we define the multivalued operator,

$$R_f : Y \to P(Y),\ R_f(x) := \begin{cases} \{f(x)\} & if\ f(x) \in Y \\ ]x, f(x)] \cap Y & if\ f(x) \in X \setminus Y \end{cases}.$$

Let $\rho_f : Y \to Y$ be a selection of R_f, i.e., $\rho_f(x) \in R_f(x)$, for all $x \in Y$. Then we have that $F_f = F_{\rho_f} = F_{R_f}$. So, ρ_f is a generalized retract of f.

We have another generic example in the following way.

Let $(X, [\cdot, \cdot])$ be a nonempty set with an interval structure. Let $Y \subset X$, $f : Y \to X$ and $x_0 \in Y$. We suppose that:

(GR_{x_0}) $$x \in Y,\ f(x) \in X \setminus Y \ \Rightarrow]x_0, f(x)] \cap Y \neq \emptyset \text{ and } x \in]x_0, f(x)].$$

In the condition (GR_{x_0}), we define the multivalued operator,

$$R_{f,x_0} : Y \to P(Y),\ R_{f,x_0}(x) := \begin{cases} \{f(x)\} & if\ f(x) \in Y \\ ]x_0, f(x)] \cap Y & if\ f(x) \in X \setminus Y \end{cases}.$$

Let ρ_{f,x_0} be a selection of R_{f,x_0}. Then:

$$F_f = F_{\rho_{f,x_0}} = F_{R_{f,x_0}}.$$

These imply that ρ_{f,x_0} is a generalized retract of the nonself operator f.

It seems to me that the following set-theoretical notions are important.

Let X be a nonempty set, $Y \subset X$, $f : Y \to X$ be a nonself operator and $\rho : X \to Y$ be a set retraction. By definition, $\partial_{f,\rho}(Y) := \rho(f(Y) \setminus Y)$ is the *formal boundary of* Y with respect to f and ρ.

For other type of formal boundary see F.E. Browder [27], [26], K. Fan [47], R. Precup [99], [100], I.A. Rus [132].

We have the following results.

Lemma 9.1. *Let $f : Y \to X$ be an operator and $\rho : X \to Y$ be a set retraction. The operator f is retractible with respect to ρ iff,*

$$x \in \partial_{f,\rho}(Y),\ x = \rho(f(x)) \ \Rightarrow f(x) = x.$$

Lemma 9.2. *Let $(X, S(X), M)$ be a large f.p.s., $Y \in S(X)$, $\rho : X \to Y$ be a set retraction and $f : Y \to X$ be a nonself operator. We suppose that:*

(i) $\rho \circ f \in M(Y)$;

(ii) $x \in \partial_{f,\rho}(Y)$, $x = \rho(f(x)) \Rightarrow f(x) = x$.

Then, $F_f \neq \emptyset$.

If we take instead of $(X, S(X), M)$, in Lemma 9.2, different examples of large f.p.s. we have fixed point results in structured sets (see [132]). Here are some examples.

Theorem 9.13 (The case of Tarski f.p.s.). *Let $(X, \leq)$ be an ordered set with the least element, 0. Let $Y \in P(X)$ and $f : Y \to X$ be a nonself operator. We suppose that:*

(i) $0 \in Y$;

(ii) $(Y, \leq)$ *is a complete lattice;*

(iii) f *is increasing;*

(iv) f *satisfies* (GR_{x_0}) *condition with respect to the order interval* $[\cdot, \cdot]_{\leq}$.

Then we have that:

(a) $F_f \neq \emptyset$;

(b) $(F_f, \leq)$ *is a complete lattice.*

Theorem 9.14 (The case of f.p.s. of contractions). *Let (X, d) be a complete metric space, $Y \in P_{cl}(X)$ and $f : Y \to X$ be an operator. We suppose that:*

(i) f *is an l-contraction;*

(ii) f *satisfies the* (GR) *condition with respect to the metric interval,* $[\cdot, \cdot]_d$.

Then we have that $F_f = \{x^*\}$.

The above considerations give rise to the following conjecture.

Problem XXIII (The conjecture of the generalized retracts (Rus [132]). Let X be a Banach space, $Y \subset X$ be a subset with nonempty boundary and $f : Y \to X$ be a nonself operator. Then, each boundary condition (Leray-Schauder, Rothe, inwardness, outwardness,...) on f implies the existence of a generalized retract of f.

References: [121], [126], [140], [29], [50], [55], [106], [130], [35], ...

10. Multivalued operators

For similar problems and results in the fixed point theory of multivalued operators see:

• Set-theoretic notions and problems in the fixed point theory for multivalued operators: [125], [55], [54], [121], [140], [39], ...

• Technique of the fixed point structures for multivalued operators: [118], [126], [140], [153], ...

• Multivalued weakly Picard operators: [142], [89], [140], [90], ...

• Iterative approximation algorithms of fixed point of multivalued operators (admissible perturbations): [91], ...

• Nonself multivalued operators: [126], [55], [54], [74], [140], ...

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