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Dedicated to Prof. Ioan A. Rus on the occasion of his 85^{th} anniversary

Application of a fixed point theorem on infinite cartesian product to an infinite system of differential equations

MARCEL-ADRIAN ŞERBAN

ABSTRACT. In the paper *Operators on infinite dimensional cartesian product*, (Analele Univ. Vest Timişoara, Mat. Inform., **48** (2010), 253–263), by I. A. Rus and M. A. Şerban, the authors give a generalization of the Fibre contraction theorem on infinite dimensional cartesian product. In this paper we give an application of this abstract result to an infinite system of differential equations.

1. INTRODUCTION

For a triangular operator $A : X \times Y \to X \times Y$, $A = (B, C)$, where $B : X \to X$ and $C : X \times Y \to Y$ we can formulate the following problem:

Problem 1.1 (Fibre Picard operator problem). *Let $\left(X, \overset{1}{\to}\right)$ and $\left(Y, \overset{2}{\to}\right)$ be two L-spaces.*

Let $B : X \to X$ be a WPO and $C : X \times Y \to Y$ be such that $C(x, \cdot) : Y \to Y$ is a WPO for every $x \in X$.

In which conditions the operator $A = (B, C)$ is a WPO ?

Results concerning this problem in different settings were obtained by M. W. Hirsch, C. C. Pugh [5], J. Sotomayor [16], I. A. Rus [6], [7], [8], S. András [1], C. Bacoţiu [2], I. A. Rus and M. A. Şerban [13], [15], M. A. Şerban [17], [18], [19], [20].

In the paper I. A. Rus and M. A. Şerban [14], the authors give a generalization of the Fibre contraction theorem for triangular operators defined on infinite dimensional cartesian product spaces. The aim of this paper is to present an application of this abstract result to an infinite system of differential equations.

2. PRELIMINARIES

In this paper we shall use the terminologies and notations from [10] and [11]. For the convenience of the reader we shall recall some of them.

Let $(X, \to)$ be an L-space and $f : X \to X$ an operator. We denote by $f^0 := 1_X$, $f^1 := f$, $f^{n+1} := f \circ f^n$, $n \in \mathbb{N}$ the iterate operators of the operator A. Also:

$$P(X) := \{Y \subseteq X \mid Y \neq \emptyset\}$$

$$F_f := \{x \in X \mid f(x) = x\}$$

By $(X, \to)$ we will denote an L-space. Actually, an L-space is any set endowed with a structure implying a notion of convergence for sequences (for examples of L-spaces see Fréchet [4], Blumenthal [3] and I. A. Rus [10]).

Let $(X, \to)$ be an L-space.

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Definition 2.1. $f : X \to X$ is called a Picard operator (briefly PO) if:
(i) $F_f = \{x^*\}$;
(ii) $f^n(x) \to x^*$ as $n \to \infty$, for all $x \in X$.

Definition 2.2. $f : X \to X$ is said to be a weakly Picard operator (briefly WPO) if the sequence $(f^n(x))_{n\in N}$ converges for all $x \in X$ and the limit (which may depend on x) is a fixed point of f.

If $f : X \to X$ is a WPO, then we may define the operator $f^\infty : X \to X$ by

$$f^\infty(x) := \lim_{n\to\infty} f^n(x).$$

Obviously $f^\infty(X) = F_f$. Moreover, if f is a PO and we denote by x^* its unique fixed point, then $f^\infty(x) = x^*$, for each $x \in X$.

For a triangular operator $A : X \times Y \to X \times Y$, $A = (B, C)$, where $B : X \to X$ and $C : X \times Y \to Y$ we have the following result:

Theorem 2.1 (Fibre contraction principle)**.** ([5], [7], [8], [16]) *We suppose that:*
(i) (Y, d_Y) *is a complete metric space;*
(ii) B *is a WPO;*
(iii) $C(x,\cdot) : Y \to Y$ *is* $\alpha-$ *contraction for every* $x \in X$*;*
(iv) $C : X \times Y \to Y$ *is continuous.*

Then
(a) A *is a WPO;*
(b) If B *is a PO then* A *is a PO.*

Theorem 2.1 was extended in [14] to the case of triangular operators defined on infinite dimensional cartesian product spaces in the following way.

Let (X_i, d_i), $i \in \mathbb{N}^*$, be metric spaces and $(X_0, \to)$ an L-space and we denote by $X = \prod_{i\in\mathbb{N}} X_i$, the cartesian product of X_i, $i \in \mathbb{N}$. We organize $X = \prod_{i\in\mathbb{N}} X_i$ as an L-space by termwise convergence, i.e.

$$x^k \xrightarrow{t} x,\ as\ k \to +\infty \Leftrightarrow \begin{cases} x_0^k \to x_0, as\ k \to +\infty, \\ x_n^k \xrightarrow{d_n} x\ as\ k \to +\infty,\ \forall n \in \mathbb{N}^*. \end{cases}$$

We consider the operators $A_k : X_0 \times X_1 \times \ldots \times X_k \to X_k$, $k \in \mathbb{N}$, and

$$\begin{gathered} A : X \to X \\ A(x_0, x_1, \ldots, x_n, \ldots) = (A_0(x_0), A_1(x_0, x_1), \ldots, A_n(x_0, x_1, \ldots, x_n), \ldots) \end{gathered} \tag{2.1}$$

Theorem 2.2. [14] *Suppose that:*
(i) (X_k, d_k) *is a complete metric space for every* $k \in \mathbb{N}^*$*;*
(ii) A_0 *is WPO;*
(iii) $A_k(x_0, x_1, \ldots, x_{k-1}, \cdot) : X_k \to X_k$ *is* α_k-*contraction, for every* $k \in \mathbb{N}^*$*;*
(iv) A *is continuous.*

Then A *is WPO. Moreover, if* A_0 *is PO then* A *is PO.*

3. Cauchy problem for a system of infinite differential equations

We consider the Cauchy problem:

$$\begin{cases} x_i'(t) = & f_i(t, x_1(t), \ldots, x_m(t)), & t \in [a;b],\ i = \overline{1,m} \\ x_{m+k}'(t) = & f_{m+k}(t, x_1(t), \ldots, x_{m+k}(t)), & t \in [a;b],\ k \in \mathbb{N}^* \\ x(a) = & x^0 \end{cases} \tag{3.2}$$

where $x = (x_1, x_2, \ldots)$ and $x^0 = \left(x_1^0, x_2^0, \ldots\right) \in \mathbb{R}^\infty$.

The Cauchy problem (3.2) is equivalent with the following infinite system of integral equations:

$$\text{(3.3)}\quad \begin{cases} \begin{pmatrix} x_1(t) \\ \vdots \\ x_m(t) \end{pmatrix} = \begin{pmatrix} x_1^0 \\ \vdots \\ x_m^0 \end{pmatrix} + \begin{pmatrix} \int_a^t f_1(s, x_1(s), \ldots, x_m(s))\, ds \\ \vdots \\ \int_a^t f_m(s, x_1(s), \ldots, x_m(s))\, ds \end{pmatrix}, \\ x_{m+k}(t) = x_{m+k}^0 + \int_a^t f_{m+k}(s, x_1(s), \ldots, x_{m+k}(s))\, ds,\ k \in \mathbb{N}^*,\ t \in [a; b]. \end{cases}$$

We consider the Banach spaces $X_0 = \left(C([a, b], \mathbb{R}^m), \|\cdot\|_{\tau_0}\right)$, $X_k = \left(C[a, b], \|\cdot\|_{\tau_k}\right)$, $k \in \mathbb{N}^*$, where $\|\cdot\|_{\tau_k}$ are the corresponding Bielecki norms:

$$\|y\|_{\tau_0} = \max_{i=\overline{1,m}} \max_{t\in[a,b]} \left\{|y_i(t)| \cdot e^{-\tau_0(t-a)}\right\},\ y \in C([a, b], \mathbb{R}^m),$$

$$\|z\|_{\tau_k} = \max_{t\in[a,b]} \left\{|z(t)| \cdot e^{-\tau_k(t-a)}\right\},\ z \in C[a, b],\ k \in \mathbb{N}^*,$$

and the operators $A_0 : X_0 \to X_0$,

$$\text{(3.4)}\qquad A_0(x_1, \ldots, x_m)(t) = \begin{pmatrix} x_1^0 \\ \vdots \\ x_m^0 \end{pmatrix} + \begin{pmatrix} \int_a^t f_1(s, x_1(s), \ldots, x_m(s))\, ds \\ \vdots \\ \int_a^t f_m(s, x_1(s), \ldots, x_m(s))\, ds \end{pmatrix},$$

and $A_k : X_0 \times X_1 \times \ldots \times X_k \to X_k$, $k \in \mathbb{N}^*$,

$$\text{(3.5)}\qquad A_k(x_1, \ldots, x_{m+k})(t) = x_{m+k}^0 + \int_a^t f_{m+k}(s, x_1(s), \ldots, x_{m+k}(s))\, ds.$$

It is clear that for operator $A : X \to X$, where $X = \prod_{i\in\mathbb{N}} X_i$ and $A = (A_0, A_1, \ldots, A_k, \ldots)$, the infinite system of integral equations (3.3) is equivalent with fixed point equation in X:

$$x = A(x).$$

We have the following result:

Theorem 3.3. *We consider the Cauchy problem (3.2) and we suppose that:*

(i) *f_i are continuous functions, $i \in \mathbb{N}^*$;*

(ii) *there exists $L_0 > 0$ such that*

$$|f_i(t, u_1, \ldots, u_m) - f_i(t, v_1, \ldots, v_m)| \le L_0 \sum_{j=1}^m |u_j - v_j|,$$

for all $t \in [a; b]$, $u_1, \ldots, u_m, v_1, \ldots, v_m \in \mathbb{R}^m$, $i = \overline{1, m}$;

(iii) *there exist $L_k > 0$, $k \in \mathbb{N}^*$, such that:*

$$|f_{m+k}(t, u_1, \ldots, u_{m+k-1}, v_1) - f_{m+k}(t, u_1, \ldots, , u_{m+k-1}, v_2)| \le L_k |v_1 - v_2|,$$

for all $t \in [a; b]$, $u_j \in \mathbb{R}$, $j = \overline{1, m+k-1}$, $v_1, v_2 \in \mathbb{R}$.

Then the Cauchy problem (3.2) has a unique solution in X.

Proof. Condition (i) ensures the well definition of the operators A_0, $A_1, \ldots$, A_k, ..., and the continuity of the operator A. From condition (ii) we get

$$\left\|A_0(\mathbf{x}) - A_0(\mathbf{y})\right\|_{\tau_0} \leq \frac{L_0}{\tau_0}\left\|\mathbf{x}-\mathbf{y}\right\|_{\tau_0},$$

for every $\mathbf{x} = (x_1, \ldots, x_m), \mathbf{y} = (y_1, \ldots, y_m) \in X_0$. If we choose $\tau_0 > 0$ such that $\frac{L_0}{\tau_0} < 1$, (for example $\tau_0 = L_0 + 1$), then A_0 is an α_0- contraction, with $\alpha_0 = \frac{L_0}{\tau_0}$, therefore A_0 is PO.

Also, for $k \in \mathbb{N}^*$, we have

$$\left\|A_k(x_0, \ldots, x_{m+k-1}, y_1) - A_k(x_0, \ldots, x_{m+k-1}, y_2)\right\|_{\tau_k} \leq \frac{L_k}{\tau_k}\left\|y_1 - y_2\right\|_{\tau_k}$$

for every $(x_0, \ldots, x_{m+k-1}) \in X_0 \times \ldots \times X_{k-1}$ and $y_1, y_2 \in X_k$, which shows that $A_k(x_0, \ldots, x_{m+k-1}, \cdot) : X_k \to X_k$ is an α_k-contraction, for a suitable choice of $\tau_k > 0$, (for example $\tau_k = L_k + 1$), thus we apply Theorem 2.2 and we obtain that $A : X \to X$ is PO, therefore the system (3.3) has a unique solution in X, so the Cauchy problem (3.2) has a unique solution. □

Example 3.1. Let consider the Cauchy problem:

(3.6)
$$\begin{cases} x_i'(t) = & g_i(t, x_1(t), \ldots, x_m(t)), & t \in [a;b],\ i = \overline{1,m} \\ x_{m+k}'(t) = & g_{m+k}(t, x_1(t), \ldots, x_{m+k-1}(t)) + k\cos(x_{m+k}), & t \in [a;b],\ k \in \mathbb{N}^* \\ x(a) = & x^0 \end{cases}$$

$x^0 \in \mathbb{R}^\infty$. We suppose that:

(i) g_i are continuous functions, $i \in \mathbb{N}^*$;
(ii) there exists $L_0 > 0$ such that

$$\left|g_i(t, u_1, \ldots, u_m) - g_i(t, v_1, \ldots, v_m)\right| \leq L_0 \sum_{j=1}^{m} \left|u_j - v_j\right|,$$

for all $t \in [a;b]$, $u_1, \ldots, u_m, v_1, \ldots, v_m \in \mathbb{R}^m$, $i = \overline{1,m}$.

Then the Cauchy problem (3.6) has a unique solution.

Proof. We apply Theorem 3.3 for f_i, $i \in \mathbb{N}^*$,

$$f_i(t, u_1, \ldots, u_m) = g_i(t, u_1, \ldots, u_m),\ i = \overline{1,m},$$

$$f_{m+k}(t, u_1, \ldots, u_{m+k-1}, u_{m+k}) = g_{m+k}(t, u_1, \ldots, u_{m+k-1}) + k\cos(u_{m+k}),\ k \in \mathbb{N}^*.$$

It is easy to see that f_i is lipschitz with constant L_0, $i = \overline{1,m}$, and $f_{m+k}(t, u_1, \ldots, u_{m+k-1}, \cdot)$ is lipschitz with $L_k = k$, $k \in \mathbb{N}^*$, therefore all the condition from Theorem 3.3 are satisfied, so we get the conclusion. □

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DEPARTMENT OF MATHEMATICS
BABEŞ-BOLYAI UNIVERSITY OF CLUJ-NAPOCA
KOGĂLNICEANU 1, 400084 CLUJ-NAPOCA, ROMANIA
Email address: mserban@math.ubbcluj.ro, marcel.serban@ubbcluj.ro