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An inertial iterative scheme for solving variational inclusion with application to Nash-Cournot equilibrium and image restoration problems

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ABSTRACT. Variational inclusion is an important general problem consisting of many useful problems like variational inequality, minimization problem and nonlinear monotone equations. In this article, a new scheme for solving variational inclusion problem is proposed and the scheme uses inertial and relaxation techniques. Moreover, the scheme is self adaptive, that is, the stepsize does not depend on the factorial constants of the underlying operator, instead it can be computed using a simple updating rule. Weak convergence analysis of the iterates generated by the new scheme is presented under mild conditions. In addition, schemes for solving variational inequality problem and split feasibility problem are derived from the proposed scheme and applied in solving Nash-Cournot equilibrium problem and image restoration. Experiments to illustrate the implementation and potential applicability of the proposed schemes in comparison with some existing schemes in the literature are presented.

1. INTRODUCTION

Variational inclusion (VI) problem is a problem of determining a point $u^* \in \mathbb{H}$, such that

$$0 \in (Au^* + Bu^*), \tag{1.1}$$

where $A : \mathbb{H} \longrightarrow 2^{\mathbb{H}}$ is a multi-valued operator and $B : \mathbb{H} \longrightarrow \mathbb{H}$ is a single-valued operator. The problem (1.1) is an important general problem that can be interpreted and modeled as several problems in different areas of research, such as optimization theory, optimal control, transportation problem and so on (Refs. [8, 12, 13, 17, 18, 20, 23, 25, 29, 30, 31, 34, 36, 43, 59, 55, 57]). Applications of variational inclusion problem in solving different real-world problems ranging from compressed sensing, image processing, radiation therapy treatment planning, have led to increasing interest in devising iterative methods for solving problem (1.1).

The popular method for solving (1.1) is the forward-backward splitting method proposed in [37] and [50]. The scheme generates a sequence u_n for $n \geq 1$ as follows:

$$u_{n+1} = (I + \lambda A)^{-1}(I - \lambda B)u_n, \tag{1.2}$$

where $(I + \lambda A)^{-1}$ is the resolvent operator associated with the operator A and λ is a positive parameter. It has been shown that, the convergence of (1.2) requires some restrictive assumptions on the underlying operators. For instance, A is required to be cocoercive or B to be inverse strongly monotone operator. In an effort to weaken some of these strong assumptions in (1.2), Tseng in [58] introduced a modified forward-backward method for

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solving (1.1). Weak convergence of the scheme in [58] is established under the condition that, A is maximal monotone and B is a monotone operator.

Recently, the authors in [41] successfully introduced another modified forward backward method called forward-reflected-backward splitting method which also does not requires the cocoercivity assumption on the operator B. Although, the convergence of the modified forward-backward schemes in [58] and [41] are formulated without the strong assumptions, it can be observed that the stepsizes considered in these works are either a fixed stepsize chosen in $(0, \frac{1}{L})$ (L is the Lipschitz constant of B) or the stepsize can be computed using a line search procedure with finite stopping criterion. It is known that, line search procedures involve extra functions evaluations or computations of resolvent of the considered operator, thereby reducing the computational performance of a given scheme. A modification of the forward-reflected-backward splitting method is proposed in [61] by considering variable stepsizes which are updated over each iteration by some simple computations given as: choose $\lambda_0 > 0$ and $\nu \in (0, \frac{1}{2})$ such that

$$\lambda_{n+1} = \min\left\{\frac{\nu\|u_{n+1} - u_n\|}{\|Bu_{n+1} - Bu_n\|}, \lambda_n\right\} \qquad n \geq 1. \tag{1.3}$$

The stepsize (1.3) is updated without the prior knowledge of the Lipschitz constant of the underlying operator. Several variants of the forward-backward-forward method have been studied and a number of schemes have been presented to solve (1.1) (Refs. [1, 6, 19, 23, 24, 27, 32, 33, 49, 52, 53, 55, 58]).

Inertial extrapolation techniques are introduced as a process of accelerating the convergence rate of an iterative scheme. These methods trace back to the pioneering work of Polyak [51] who introduced the heavy ball method to speed up the gradient algorithm's convergence behavior and allow the identification of various critical points. The inertial idea was later used and developed by Nesterov [45] and Alvarez and Attouch (Refs. [4, 5]) in the sense of solving smooth convex minimization problems and monotone inclusions/non-smooth convex minimization problems respectively. A considerable amount of literature has been contributed to inertial algorithms over the last decade [2, 3, 22, 56, 60].

Fixed points iterative methods play a vital role in designing many iterative methods (Ref. [21, 10]). For example, Krasnoselskii–Mann (KM) iterative method introduced in [42, 35] for solving fixed point of quasi-nonexpansive mappings T given by the following scheme

$$u_{n+1} = (1 - t_n)u_n + t_n T u_n, \qquad n \geq 1 \tag{1.4}$$

is applied to derive iterative methods for solving convex feasibility and monotone inclusion (Refs. [35, 10, 39]). Under the condition that the sequence $t_n \subset [0, 1]$ and $\sum_{n=1}^{\infty} t_n(1 - t_n) = \infty$, the sequence generated by (1.4) is shown to converge weakly to a fixed point of the operator T. A more general setting of KM-type scheme that considers incorporation of inertial extrapolation step is studied in [39]. The inertial scheme is as follows:

$$\begin{cases} \bar{u}_n = u_n + \varrho_n(u_n - u_{n-1}) \\ u_{n+1} = (1 - t_n)\bar{u}_n + t_n T\bar{u}_n, \qquad n \geq 1. \end{cases} \tag{1.5}$$

The convergence of the sequence generated by the scheme (1.5) is established in [39] under the assumptions that $0 < \inf t_n \leq \sup t_n < 1$ and $\varrho_n \in (0, \varrho)$, where $\varrho \in [0, 1)$ and $\sum_{n=0}^{\infty} \varrho_n\|u_n - u_{n-1}\|^2 < \infty$. In addition, iterative schemes for solving convex feasibility problem and monotone inclusion are derived from (1.5) in [39]. The scheme (1.5) is similarly studied in [10] with the strict assumptions on the parameters given as

$0 < t \leq t_n \leq \frac{\delta-\varrho[\varrho(1+\varrho)+\varrho\delta+\sigma]}{\delta[1+\varrho(1+\varrho)+\varrho\delta+\sigma])}$ where $\delta > \frac{\varrho^2(1+\varrho)+\varrho\sigma}{1-\varrho^2}$, $\sigma, t > 0$ with $\{\varrho_n\} \subset [0,1)$ is a nonincreasing sequence. Moreover, a Douglas Rachford splitting method for solving monotone inclusion problem is derived from (1.5). The observations above informed the following question.

Can we apply the KM-type scheme that considers the incorporation of inertial extrapolation step to construct a new scheme for solving variational inclusion problem (1.1)? The purpose of this paper is to give a positive answer to this question. The main contribution of this paper is to design a new scheme for solving the variational inclusion problem (1.1) as well as to derive a modified version of the proposed scheme for solving variational inequality problem and split convex feasibility problem. The proposed scheme utilizes the structure of (1.4) and the effect of inertial extrapolation step. Moreover, the proposed scheme uses self adaptive stepsize that does not involve any line search technique or does not require prior knowledge of the Lipschitz constant as in [61]. Additionally, we illustrate the potential applicability and performance of the proposed method in comparison to the existing methods in the literature by applying the proposed schemes to solve the Nash-Cournot equilibrium problem and the problem of image recovery.

In what follows, we present our propose scheme which is motivated by [39, 10, 61].

$$(1.6) \qquad \begin{cases} \bar{u}_n = u_n + \varrho_n(u_n - u_{n-1}), \\ z_n = (1-s_n)\bar{u}_n + s_n(I+\lambda_n B)^{-1}(I-\lambda_n A)\bar{u}_n, \\ u_{n+1} = (1-t_n)\bar{u}_n + t_n(I+\lambda_n B)^{-1}(I-\lambda_n A)z_n, \qquad n \geq 1. \end{cases}$$

where $\{s_n\}, \{t_n\}$ are sequences in $(0,1)$ and $\varrho_n(u_n - u_{n-1})$ denotes the inertial extrapolation term. The step size λ_n is defined to be self-adaptively updated according to a new simple step size rule that does not depend on the Lipschitz constant of the underlying operator and does not involve any line search. The proposed scheme (1.6) can be a generalization of many important schemes, for instance, observe that, (1.6) with $\varrho_n = 0$ and $s_n = 0$ reduces to the KM-type scheme. Moreover, scheme (1.6) with $s_n = 0$ reduces to the particular setting derived from KM-type scheme proposed in [39] and [10] to solve monotone inclusion problems. Additionally, (1.6) with $s_n = 0$, and $t_n = 1$ is a form of inertial forward-backward scheme for (1.1) proposed for example, in [38] with an identity operator as the linear self-adjoint and positive definite operator.

This work is outlined as follows: In the next Section, preliminaries consisting of lemmas, definitions and some characterizations which are essential for the convergence analysis of the proposed scheme are recalled. The main scheme and its convergence analysis are presented in Section 3. In Section 4, the computational illustrations to show the implementation and performance of the proposed scheme in comparison with some existing schemes in the literature are presented as an application in solving Nash-Cournot equilibrium problem and image deblurring problem.

2. Preliminaries

In this section, we recall some characterizations, basic facts and lemmas that will aid in showing the convergence analysis of our proposed method. In the sequel, $\mathbb{H}$ is a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$, norm $\|\cdot\|$ and $\mathbb{E}$ is a nonempty closed and convex subset of $\mathbb{H}$. We use $u_i \longrightarrow u^*$ (resp. $u_i \rightharpoonup u^*$) to denote that, $\{u_i\}$ converges strongly (resp. weakly) to u^*. The following holds in a Hilbert space:

$$(2.7) \qquad \|\bar{u} \pm u\|^2 = \|\bar{u}\|^2 + \|u\|^2 \pm 2\langle \bar{u}, u \rangle,$$

and

$$\|\varrho\bar{u}+(1-\varrho)u\|^2=\varrho\|\bar{u}\|^2+(1-\varrho)\|u\|^2-\varrho(1-\varrho)\|\bar{u}-u\|^2, \tag{2.8}$$

for every $\bar{u}, u \in \mathbb{H}$.

Definition 2.1. Let a mapping $B : \mathbb{H} \longrightarrow \mathbb{H}$ be defined on a real Hilbert space $\mathbb{H}$. For all $u, v \in \mathbb{E}$, B is said to be:

(1) Monotone if:

$$\langle Bu - Bv, u - v\rangle \geq 0.$$

(2) Firmly nonexpansive if:

$$\|Bu - Bv\|^2 \leq \langle Bu - Bv, u - v\rangle,$$

or equivalently,

$$\|Bu - Bv\|^2 \leq \|u - v\|^2 - \|(I - B)u - (I - B)v\|^2.$$

(3) L-Lipschitz continuous on $\mathbb{H}$ if there exists a constant $L > 0$ such that:

$$\|Bu - Bv\| \leq L\,\|u - v\|\,.$$

If $L = 1$, then B is said to be nonexpansive.

Definition 2.2 ([9])**.** A multi-valued mapping $A : \mathbb{H} \longrightarrow 2^{\mathbb{H}}$ is said to be monotone, if for every $u, v \in \mathbb{H},\ x \in Au$ and $y \in Av \Rightarrow \langle x - y, u - v\rangle \geq 0$. Furthermore, A is said to be maximal monotone if it is monotone and if for every $(u, x) \in \mathbb{H}$, $\langle x - y, u - v\rangle \geq 0$ for every $(v, y) \in \text{Graph}(A) \Rightarrow x \in Au$.

Definition 2.3. Let $A : \mathbb{H} \longrightarrow 2^{\mathbb{H}}$ be a multi-valued maximal monotone mapping. Then, the resolvent mapping $J^A_\lambda : \mathbb{H} \longrightarrow \mathbb{H}$ associated with A is defined by:

$$J^A_\lambda(u) = (I + \lambda A)^{-1}(u),$$

for some $\lambda > 0$, where I stands for the identity operator on $\mathbb{H}$.

It is worth mentioning that if $A : \mathbb{H} \longrightarrow 2^{\mathbb{H}}$ is a set-valued maximal monotone mapping and $\lambda > 0$, then $\text{Dom}(J^A_\lambda) = \mathbb{H}$, and J^A_λ is a single-valued and firmly nonexpansive mapping (see. [54] for more details).

Lemma 2.1 ([11])**.** *Let $B : \mathbb{H} \longrightarrow \mathbb{H}$ be a Lipschitz continuous and monotone mapping and $A : \mathbb{H} \longrightarrow 2^{\mathbb{H}}$ be a maximal monotone mapping, then the mapping $A + B$ is a maximal monotone mapping.*

Lemma 2.2. [21] *Let $\mathbb{H}$ be a real Hilbert space then for any $\tau > 0$ and $\gamma \in \mathbb{R}$, we have*

$$\|u \pm \gamma v\|^2 \geq (1 - \gamma\tau)\|u\|^2 + \gamma(\gamma - \frac{1}{\tau})\|v\|^2,$$

for all $u, v \in \mathbb{H}$.

Lemma 2.3 ([47])**.** *Suppose $\{\zeta_n\}$, $\{\phi_n\}$ and $\{\varrho_n\}$ are sequences in $[0, \infty)$ such that, for all $n \geq 1$,*

$$\zeta_{n+1} \leq \zeta_n + \varrho_n(\zeta_n - \zeta_{n-1}) + \phi_n, \quad \sum \phi_n < \infty,$$

and there exists $\varrho \in \mathbb{R}$ with $0 \leq \varrho_n \leq \varrho \leq 1$ for all $n \geq 1$. Then, the following are satisfied:

(i) $\sum[\zeta_n - \zeta_{n-1}]_+ < \infty$, where $[a]_+ = \max\{a, 0\}$
(ii) there exists $\zeta^* \in [0, \infty)$ with $\lim \zeta_n = \zeta^*$.

Lemma 2.4. *Let $\mathbb{H}$ be a nonempty closed and convex subset of a Hilbert space $\mathbb{H}$ and $T : \mathbb{E} \to \mathbb{H}$ be a nonexpansive mapping. Let $\{u_n\}$ be a subsequence in $\mathbb{E}$ such that $u_n \rightharpoonup u$ for $u \in \mathbb{H}$ and $\{u_n - Tu_n\} \to 0$ as $n \to \infty$. Then $u \in Fix(T)$, where $Fix(T)$ denotes the set of fixed point of T.*

Lemma 2.5 ([48]). *Let $\mathbb{E}$ be a nonempty subset of a real Hilbert space $\mathbb{H}$ and a sequence $\{\bar{u}_i\}$ in $\mathbb{H}$ such that:*

(a) for every $\bar{u} \in \mathbb{E}, \lim_{i\to\infty} \|\bar{u}_i - u^*\|$ exists;
(b) every sequentially weak cluster point of $\{\bar{u}_i\}$ is in $\mathbb{E}$.

Then, $\{\bar{u}_i\}$ weakly converges in $\mathbb{E}$.

3. New algorithm for solving VI problem

This section presents the detail description of our proposed algorithm and the convergence analysis of the iterates generated by the algorithm which involves a maximally monotone operator A and a continuous Lipschitz and monotone operator B. We assume the following for the analysis of the proposed method.

Assumption 3.1.

A_1 *The feasible set of problem* (1.1) *is a nonempty closed and convex subset of* $\mathbb{H}$.
A_2 *The solution set Γ of* (1.1) *is nonempty.*
A_3 $A : \mathbb{H} \longrightarrow 2^{\mathbb{H}}$ *is maximally monotone, and* $B : \mathbb{H} \longrightarrow \mathbb{H}$ *is L-Lipschitz continuous and monotone on* $\mathbb{H}$.
A_4 *Suppose the sequences s_n and t_n in $(0,1)$ where $0 < s \le s_n$ and $0 < t \le t_n \le \frac{\eta-\varrho(2p\varrho(1+\varrho)+\varrho\eta(1-t)+2p\sigma)}{\eta[1+2p\varrho(1+\varrho)+\varrho\eta(1-t)+2p\sigma]}$, with $\{\varrho_n\} \subset [0,\varrho]$ for $\varrho, \eta, p, \sigma > 0$. Define the sequence*

$$\zeta_n = \frac{(1-t_n)(1-\varrho_n\tau_n)}{2pt_n}$$

and

$$\vartheta_n = \varrho_n(1+\varrho_n) + \frac{\varrho_n(1-t_n)(1-\varrho_n\tau_n)}{2pt_n},$$

where $p = \left(1+\frac{1}{s}\right)$ and $\tau_n = \frac{1}{\varrho_n+\eta t_n}$.

Algorithm 1: An Inertial Algorithm for Solving VI Problem

Initialization: Choose $u_{-1}, u_0 \in \mathbb{H}, \varrho_n \subset (0,\varrho), \varrho \in [0,1)$ $\nu \in (0,1)$ and $\lambda_0 > 0$.

Iterative Steps: For current iterates u_{n-1} and $u_n \in \mathbb{H}$.

Step 1. Set $\bar{u}_n$ as:

$$\bar{u}_n = u_n + \varrho_n(u_n - u_{n-1}), \tag{3.9}$$

Step 2. Compute

$$z_n = (1-s_n)\bar{u}_n + s_n(I+\lambda_n A)^{-1}(I-\lambda_n B)\bar{u}_n.$$

Step 3. Compute

$$u_{n+1} = (1-t_n)\bar{u}_n + t_n(I+\lambda_n A)^{-1}(I-\lambda_n B)z_n.$$

Update

$$\lambda_{n+1} := \begin{cases} \min\left\{\frac{\nu\|\bar{u}_n - z_n\|}{\|B\bar{u}_n - Bz_n\|}, \lambda_0\right\} & \text{if } B\bar{u}_n \neq Bz_n \\ \lambda_n & \text{otherwise.} \end{cases} \tag{3.10}$$

If $\bar{u}_n = z_n = u_{n+1}$ then stop and $\bar{u}_n$ is a solution of the VI problem, otherwise set $n := n+1$ and go back to step 1.

Remark 3.1. It can be seen that the sequence $\{\lambda_n\}$ is monotonically decreasing. Moreover, since A is a Lipschitz continuous with Lipschitz's constant L, for $B\bar{u}_n \neq Bz_n$, we have:

$$\frac{\nu\|\bar{u}_n - z_n\|}{\|B\bar{u}_n - Bz_n\|} \geq \frac{\nu}{L} \tag{3.11}$$

It is obvious that when $B\bar{u}_n = Bz_n$ the inequality (3.11) is satisfied. Hence, it follows that $\lambda_n \geq \min\{\frac{\nu}{L}, \lambda_0\}$. Therefore, the update (3.10) is well defined.

Next, we give some important lemmas (with their proofs) that we use for the convergence analysis of the sequence generated by Algorithm 1.

Lemma 3.6. *Let $\{\bar{u}_n\}$ be a sequence generated by Algorithm 1 and Assumption 3.1 be satisfied. For any $\breve{u} \in \Gamma$, we have*

$$\begin{aligned}\|u_{n+1} - \breve{u}\|^2 &\leq (1+\varrho_n)\|u_n - \breve{u}\|^2 - \varrho_n\|u_{n-1} - \breve{u}\|^2 + \varrho_n(1+\varrho_n)\|u_n - u_{n-1}\|^2 \\ &\quad - s_n^2 t_n(1-t_n)\|\bar{u} - T_n z_n\|^2. \end{aligned} \tag{3.12}$$

Proof. For each $n \geq 1$. Let the resolvent $T_{\lambda_n} = (I + \lambda_n A)^{-1}(I - \lambda_n B) = J^A_{\lambda_n}(I - \lambda_n B)$, then for all $\breve{u} \in \mathbb{H}$ together with the fact that T_{λ_n} is nonexpansive, we have

$$\begin{aligned}\|z_n - \breve{u}\| &= \|(1-s_n)\bar{u}_n + s_n T_{\lambda_n}\bar{u}_n - \breve{u}\| \\ &\leq (1-s_n)\|\bar{u}_n - \breve{u}\| + s_n\|T_{\lambda_n}\bar{u}_n - \breve{u}\| \\ &\leq (1-s_n)\|\bar{u}_n - \breve{u}\| + s_n\|\bar{u}_n - \breve{u}\| \\ &\leq \|\bar{u}_n - \breve{u}\|. \end{aligned} \tag{3.13}$$

On the other hand, from (3.9) and (2.8), we have

$$\begin{aligned}\|\bar{u}_n - \breve{u}\|^2 &= \|(1+\varrho_n)(u_n - \breve{u}) - \varrho_n(u_{n-1} - \breve{u})\|^2, \\ &= (1+\varrho_n)\|u_n - \breve{u}\|^2 - \varrho_n\|u_{n-1} - \breve{u}\|^2 + \varrho_n(1+\varrho_n)\|u_n - u_{n-1}\|^2. \end{aligned} \tag{3.14}$$

Similarly, from the definition of u_{n+1} and Equation (2.8), we have

$$\begin{aligned}\|u_{n+1} - \breve{u}\|^2 &= \|(1-t_n)\bar{u}_n + t_n T_{\lambda_n} z_n - \breve{u}\|^2, \\ &= (1-t_n)\|\bar{u}_n - \breve{u}\|^2 + t_n\|T_{\lambda_n} z_n - \breve{u}\|^2 - t_n(1-t_n)\|\bar{u}_n - T_{\lambda_n} z_n\|^2 \\ &\leq (1-t_n)\|\bar{u}_n - \breve{u}\|^2 + t_n\|z_n - \breve{u}\|^2 - t_n(1-t_n)\|\bar{u}_n - T_{\lambda_n} z_n\|^2 \\ &\leq (1-t_n)\|\bar{u}_n - \breve{u}\|^2 + t_n\|\bar{u}_n - \breve{u}\|^2 - t_n(1-t_n)\|\bar{u}_n - T_{\lambda_n} z_n\|^2 \\ &= \|\bar{u}_n - \breve{u}\|^2 - t_n(1-t_n)\|\bar{u}_n - T_{\lambda_n} z_n\|^2 \\ &\leq \|\bar{u}_n - \breve{u}\|^2 - s_n^2 t_n(1-t_n)\|\bar{u}_n - T_{\lambda_n} z_n\|^2. \end{aligned} \tag{3.15}$$

The fourth and last inequality follows by substituting the estimate (3.13) and by the assumption on the sequence s_n (Assumption (A_4)) respectively. Hence, the results follows by substituting (3.14) in (3.15). □

Lemma 3.7. *Let the solution set of the VI problem $\Gamma \neq \emptyset$ and the sequences $\{\vartheta_n\}$, $\{\zeta_n\}$ as defined in Assumption 3.1 (A_4) and $\{s_n\}, \{t_n\}$ satisfy the conditions in Assumption (A_4), then the sequence $\{u_n\}$ generated by Algorithm 1 satisfies*

$$Q_{n+1} - (1+\varrho_n)Q_n - \varrho_n Q_{n-1} \leq \zeta_n\|u_{n+1} - u_n\|^2 + \vartheta_n\|u_n - u_{n-1}\|^2, \tag{3.16}$$

where $Q_n = \|u_n - \breve{u}\|^2$ for all $\breve{u} \in \Gamma$.

Proof. Observe that from the definition of z_n, we have

$$\begin{aligned}\|\bar{u}_n - T_{\lambda_n}\bar{u}_n\|^2 &= \frac{1}{s_n^2}\|z_n - \bar{u}_n\|^2,\\ &\geq \frac{1}{s_n^2}\|T_{\lambda_n}z_n - T_{\lambda_n}\bar{u}_n\|^2,\end{aligned}$$

notice that, we have $T_{\lambda_n}z_n = \frac{1}{t_n}(u_{n+1} - \bar{u}_n) + \bar{u}_n$. Therefore, we have

$$\begin{aligned}\|\bar{u}_n - T_{\lambda_n}\bar{u}_n\|^2 &= \frac{1}{s_n^2}\|\frac{1}{t_n}(u_{n+1} - \bar{u}_n) + \bar{u}_n - T_{\lambda_n}\bar{u}_n\|^2,\\ &= \frac{1}{s_n^2 t_n^2}\|u_{n+1} - \bar{u}_n + t_n(\bar{u}_n - T_{\lambda_n}\bar{u}_n)\|^2.\end{aligned}$$

Now, using Lemma 2.2 with $\tau = \frac{1}{2t_n}$ and $\gamma = t_n$, we get

$$\|\bar{u}_n - T_{\lambda_n}\bar{u}_n\|^2 \geq \frac{1}{s_n^2 t_n^2}\left\{\frac{1}{2}\|u_{n+1} - \bar{u}_n\|^2 - t_n^2\|\bar{u}_n - T_{\lambda_n}\bar{u}_n\|^2\right\}. \tag{3.17}$$

This implies that

$$\left(1 + \frac{1}{s_n}\right)\|\bar{u}_n - T_{\lambda_n}\bar{u}_n\|^2 \geq \frac{1}{2s_n^2 t_n^2}\|u_{n+1} - \bar{u}_n\|^2,$$

It follows from the last inequality and assumption (A_4), that

$$\begin{aligned}\left(1 + \frac{1}{s}\right)\|\bar{u}_n - T_{\lambda_n}\bar{u}_n\|^2 &\geq \frac{1}{2s_n^2 t_n^2}\|u_{n+1} - \bar{u}_n\|^2,\\ &= \frac{1}{2s_n^2 t_n^2}\|u_{n+1} - u_n - \varrho_n(u_n - u_{n-1})\|^2.\end{aligned}$$

Similarly, using Lemma 2.2 as in (3.17) with $\gamma = \varrho_n$, we get

$$\begin{aligned}\left(1 + \frac{1}{s}\right)\|\bar{u}_n - T_{\lambda_n}\bar{u}_n\|^2 &\geq \frac{(1 - \varrho_n\tau_n)}{2s_n^2 t_n^2}\|u_{n+1} - u_n\|^2 + \frac{\varrho_n}{2s_n^2 t_n^2}\left(\varrho_n - \frac{1}{\tau_n}\right)\|u_n - u_{n-1}\|^2\\ &= \frac{(1 - \varrho_n\tau_n)}{2s_n^2 t_n^2}\|u_{n+1} - u_n\|^2 - \frac{\varrho_n(1 - \varrho_n\tau_n)}{2s_n^2 t_n^2\tau_n}\|u_n - u_{n-1}\|^2. \qquad (3.18)\end{aligned}$$

Multiplying both sides of (3.18) by $-s_n^2 t_n(1 - t_n)$, we get

$$\begin{aligned}-ps_n^2 t_n(1 - t_n)\|\bar{u}_n - T_{\lambda_n}\bar{u}_n\|^2 \leq &-\frac{(1 - t_n)(1 - \varrho_n\tau_n)}{2t_n}\|u_{n+1} - u_n\|^2\\ &+ \frac{\varrho_n(1 - t_n)(1 - \varrho_n\tau_n)}{2t_n\tau_n}\|u_n - u_{n-1}\|^2. \qquad (3.19)\end{aligned}$$

Substituting the estimate (3.19) in (3.12), we have

$$\begin{aligned}\|u_{n+1} - \breve{u}\|^2 \leq\ &(1 + \varrho_n)\|u_n - \breve{u}\|^2 - \varrho_n\|u_{n-1} - \breve{u}\|^2 + \varrho_n(1 + \varrho_n)\|u_n - u_{n-1}\|^2\\ &- \frac{(1 - t_n)(1 - \varrho_n\tau_n)}{2pt_n}\|u_{n+1} - u_n\|^2\\ &+ \frac{\varrho_n(1 - t_n)(1 - \varrho_n\tau_n)}{2pt_n\tau_n}\|u_n - u_{n-1}\|^2.\end{aligned}$$

Thus,

$$\begin{aligned}Q_{n+1} \leq (1+\varrho_n)Q_n - \varrho_n Q_{n-1} + \varrho_n(1+\varrho_n)\|u_n - u_{n-1}\|^2 \\ - \frac{(1-t_n)(1-\varrho_n\tau_n)}{2pt_n}\|u_{n+1}-u_n\|^2 \\ + \frac{\varrho_n(1-t_n)(1-\varrho_n\tau_n)}{2pt_n\tau_n}\|u_n - u_{n-1}\|^2,\end{aligned}$$

which can be simply expressed as

$$Q_{n+1} - (1+\varrho_n)Q_n + \varrho_n Q_{n-1} \leq \zeta_n\|u_{n+1}-u_n\|^2 + \vartheta_n\|u_n - u_{n-1}\|^2,$$

with the sequences ζ_n and ϑ_n as defined in assumption (A_4). Hence the proof. □

Lemma 3.8. *Let $\{\vartheta_n\}$ and $\{\zeta_n\}$ be sequences as defined in Assumption 3.1 (A_4), and the sequences $\{s_n\}$ and $\{t_n\}$ satisfy assumption (A_4), then for all $n \geq 1$, we have*

$$\zeta_n + \vartheta_{n+1} \leq -\sigma_n. \tag{3.20}$$

Proof. Observe that, since $\varrho_n\tau_n < 1$ and $t_n \in (0,1)$,

$$\varrho_n(1+\varrho_n) + \frac{\varrho_n(1-t_n)(1-\varrho_n\tau_n)}{2pt_n\tau_n} > 0.$$

Taking into consideration the choice $\tau_n = \frac{1}{\varrho_n + \eta t_n}$ in Assumption (3.1)(A_4), we have $\eta = \frac{1-\varrho_n\tau_n}{\tau_n t_n}$. Observe again that

$$\begin{aligned}\vartheta_n &= \varrho_n(1+\varrho_n) + \frac{\varrho_n(1-t_n)\eta}{2p}, \\ &\leq \varrho(1+\varrho) + \frac{\varrho\eta(1-t)}{2p}. \qquad (3.21)\end{aligned}$$

Therefore,

$$\begin{aligned}\zeta_n + \vartheta_{n+1} &\leq -\sigma_n, \\ &\Leftrightarrow \frac{(1-t_n)(1-\varrho_n\tau_n)}{2pt_n} + \vartheta_{n+1} + \sigma \leq 0 \\ &\Leftrightarrow (1-t_n)(1-\varrho_n\tau_n) + 2pt_n(\vartheta_{n+1}+\sigma) \leq 0 \\ &\Leftrightarrow -(1-t_n)\eta\tau_n t_n + 2pt_n(\vartheta_{n+1}+\sigma) \leq 0 \\ &\Leftrightarrow \frac{-(1-t_n)\eta}{\varrho_n + \eta t_n} + 2p(\vartheta_{n+1}+\sigma) \leq 0 \\ &\Leftrightarrow -(1-t_n)\eta + 2p(\vartheta_{n+1}+\sigma)(\varrho_n+\eta t_n) \leq 0 \\ &\Leftrightarrow 2p(\vartheta_{n+1}+\sigma)(\varrho_n+\eta t_n) + \eta t_n \leq \eta. \qquad (3.22)\end{aligned}$$

From (3.21), we get

$$\begin{aligned}&2p(\vartheta_{n+1}+\sigma)(\varrho_n+\eta t_n) + \eta t_n \\ &\leq 2p\left(\varrho(1+\varrho) + \frac{\varrho\eta(1-t)}{2p} + \sigma\right)(\varrho_n + \eta t_n) + \eta t_n \\ &\leq \eta.\end{aligned}$$

where the last inequality follows by using the upper bound of $\{t_n\}$ in assumption (A_4). Hence the result follows. □

Theorem 3.1. *Suppose that Assumption 3.1 is satisfied. Then, for all $\breve{u} \in \Gamma \neq \emptyset$, the sequence $\{u_n\}$ generated by Algorithm 1, converges weakly to $\breve{u}$.*

Proof. Let $\breve{u} \in \Gamma$ and set $R_n = Q_n - \varrho_n Q_{n-1} + \vartheta_n \|u_n - u_{n-1}\|^2$, we make the following claims

Claim 1: $\{R_n\}$ is a non-increasing sequence. So, to see to that, we observe that

$$\begin{aligned} R_{n+1} - R_n &= Q_{n+1} - \varrho_{n+1} Q_n + \vartheta_{n+1}\|u_{n+1} - u_n\|^2 - Q_n + \varrho_n Q_{n-1} \\ &\quad - \vartheta_n \|u_n - u_{n-1}\|^2, \\ &= Q_{n+1} - (1 + \varrho_{n+1}) Q_n + \varrho_n Q_{n-1} - \vartheta_{n+1}\|u_{n+1} - u_n\|^2 \\ &\quad - \vartheta_n \|u_n - u_{n-1}\|^2. \end{aligned}$$

From Lemma 3.7, we obtain

$$\begin{aligned} R_{n+1} - R_n &\leq \zeta_n \|u_{n+1} - u_n\|^2 + \vartheta_{n+1}\|u_{n+1} - u_n\|^2, \\ &= (\zeta_n + \vartheta_{n+1})\|u_{n+1} - u_n\|^2. \end{aligned}$$

Now, by Lemma 3.8, the last inequality implies that for all $n \geq 1$

$$R_{n+1} - R_n \leq -\sigma \|u_{n+1} - u_n\|^2. \tag{3.23}$$

Thus, $\{R_n\}$ is non-increasing. Now we claim also that

Claim 2: $\sum_{n=1}^{\infty} \|u_{n+1} - u_n\|^2 \leq \infty$. In fact, since $\{R_n\}$ is non-increasing and the sequence $\{\varrho_n\}$ is bounded, we have

$$-\varrho Q_{n-1} \leq Q_n - \varrho Q_{n-1} \leq R_n \leq R_1.$$

Thus, we get

$$\begin{aligned} Q_n &\leq \varrho Q_{n-1} + R_1, \\ &\leq \varrho(\varrho Q_{n-2} + R_1) + R_1, \\ &\vdots \\ &\leq \varrho^n Q_0 + R_1 \sum_{i=0}^{n-1} \varrho^i \leq \varrho^n Q_0 + \frac{R_1}{1-\varrho}. \end{aligned}$$

It can be deduced from (3.23) that

$$\begin{aligned} \sigma \sum_{i=1}^{n} \|u_{n+1} - u_n\|^2 &\leq R_1 - R_{n+1}, \\ &\leq R_1 - \varrho Q_n, \\ &\leq R_1 - \varrho \left(\varrho Q_0 + \frac{R_1}{1-\varrho} \right), \\ &\leq \varrho^{n+1} Q_0 + \frac{R_1}{1-\varrho}. \end{aligned}$$

Since $\varrho_{n+1} \to 0$ as $n \to \infty$, we obtain the claim. Next, we claim that

Claim 3: $\lim_{n\to\infty} \|u_n - \breve{u}\|$ exists and every sequential weak cluster point of sequence $\{u_n\}$ is in Γ. Indeed, it follows from (3.16), Claim 2 and Lemma 2.3 that, $\lim_{n\to\infty} \|u_n - \breve{u}\|$ exists. From Claim 2, we have

$$\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0, \tag{3.24}$$

from the definition of $\bar{u}_n$, we have

$$\begin{aligned} \|\bar{u}_n - u_{n+1}\| &\leq \|u_n - u_{n+1}\| + \varrho_n \|u_n - u_{n-1}\|, \\ &\leq \|u_n - u_{n+1}\| + \varrho \|u_n - u_{n-1}\|, \end{aligned}$$

it follows from (3.24) that $\lim_{n\to\infty} \|u_{n+1} - \bar{u}_n\| = 0$. Using the definition of u_{n+1}, we have

$$
\begin{aligned}
\|T_{\lambda_n}\bar{u}_n - \bar{u}_n\| &= \|T_{\lambda_n}\bar{u}_n - u_{n+1} + u_{n+1} - \bar{u}_n\|,\\
&\leq \|T_{\lambda_n}\bar{u}_n - u_{n+1}\| + \|u_{n+1} - \bar{u}_n\|,\\
&= \|T_{\lambda_n}\bar{u}_n - [(1-t_n)\bar{u}_n + t_nT_{\lambda_n z_n}]\,\| + \|u_{n+1} - \bar{u}_n\|,\\
&\leq \|T_{\lambda_n}\bar{u}_n - \bar{u}_n\| + t_n\|\bar{u}_n + T_{\lambda_n}z_n\| + \|u_{n+1} - \bar{u}_n\|,\\
&= \|T_{\lambda_n}\bar{u}_n - \bar{u}_n\| + t_n\|\bar{u}_n + T_{\lambda_n}\left[(1-s_n)\bar{u}_n + s_nT_{\lambda_n}\bar{u}_n\right]\| + \|u_{n+1} - \bar{u}_n\|,\\
&\leq \|T_{\lambda_n}\bar{u}_n - \bar{u}_n\| + s_nt_n\|\bar{u}_n - T_{\lambda_n}\bar{u}_n\| + \|u_{n+1} - \bar{u}_n\|,\\
&= (1 - s_nt_n)\|T_{\lambda_n}\bar{u}_n - \bar{u}_n\| + \|u_{n+1} - \bar{u}_n\|.
\end{aligned}
\tag{3.25}
$$

Thus,

$$s_nt_n\|T_{\lambda_n}\bar{u} - \bar{u}\| \leq \|u_{n+1} - \bar{u}_n\|.$$

It follows from the assumptions on the sequences s_n, t_n and (3.24) that

$$\lim_{n\to\infty} \|T_{\lambda_n}\bar{u}_n - \bar{u}_n\| = 0. \tag{3.26}$$

Let u^* be an arbitrary cluster point of $\{u_n\}$, then there exists a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ such that $u_n \rightharpoonup u^* \in \mathbb{E}$. Using (3.26), it follows from Lemma 2.4 that $u^* \in Fix(T_{\lambda_n})$ and we conclude that $u^* \in \mathit{\Gamma}$. Hence, it follows from Lemma 2.5 that $\{u_n\}$ converges weakly. □

4. Numerical experiments with applications

Here, we apply the proposed method to two main applications; we derived methods for solving variational inequality and split convex feasibility problem from the proposed method.

4.1. **Application to variational inequality problem.** A *Variational inequality problem* is a problem of finding a point $\breve{u} \in \mathbb{E}$ such that

$$\langle B\breve{u}, u - \breve{u}\rangle \geq 0, \qquad \forall\ u \in \mathbb{E}, \tag{4.27}$$

where $\mathbb{E}$ is a nonempty closed convex subset of a real Hilbert space $\mathbb{H}$ and $B : \mathbb{H} \longrightarrow \mathbb{H}$ is an operator. Problem (4.27) is an important problem in optimization theory with several applications in different areas of study such as economics, equilibrium, transportation, control system and so on (Ref. [7, 8, 17, 20, 29, 30, 31, 34, 36, 46, 59]).

Suppose $f : \mathbb{H} \longrightarrow (-\infty, +\infty]$ is proper lower semi-continuous and convex function. Then, for all $u \in \mathbb{H}$, the subdifferential ∂f of f is defined as:

$$\partial f(u) = \{\bar{u} \in \mathbb{H} : f(u) \leq \langle \bar{u}, u - v\rangle + f(v) \quad \forall v \in \mathbb{H}\}.$$

For a nonempty closed and convex subset $\mathbb{E}$ of $\mathbb{H}$, the indicator function $i_{\mathbb{E}}$ of $\mathbb{E}$ is given by:

$$i_{\mathbb{E}}(u) = \begin{cases} 0 & \text{if} \quad u \in \mathbb{E} \\ \infty & \text{if} \quad u \notin \mathbb{E}. \end{cases} \tag{4.28}$$

Furthermore, the normal cone of $\mathbb{E}$ at u, $N_{\mathbb{E}}u$ is given as:

$$N_{\mathbb{E}}u = \{\bar{u} \in \mathbb{H} : \langle \bar{u}, u - v\rangle \leq 0 \quad \forall v \in \mathbb{H}\}.$$

It is known that the indicator function $i_{\mathbb{E}}$ is a proper lower semi-continuous and convex function on $\mathbb{H}$. Thus, the subdifferential $\partial i_{\mathbb{E}}$ of $i_{\mathbb{E}}$ is a maximal monotone operator and

$$\begin{aligned}\partial i_{\mathbb{E}} u &= \{\bar{u} \in \mathbb{H} : i_{\mathbb{E}} u \leq \langle \bar{u}, u - v\rangle + i_{\mathbb{E}} v \quad \forall v \in \mathbb{H}\},\\ &= \{\bar{u} \in \mathbb{H} : \langle \bar{u}, u - v\rangle \leq 0 \quad \forall v \in \mathbb{H}\},\\ &= N_{\mathbb{E}} u.\end{aligned}$$

Therefore, for all $u \in \mathbb{H}$, we can define the resolvent of $\partial i_{\mathbb{E}}$ as

$$J_{\lambda}^{\partial i_{\mathbb{E}}} = (I + \lambda \partial i_{\mathbb{E}})^{-1}, \text{ for each } \lambda > 0.$$

Hence, we can see that for $\lambda > 0$

$$\begin{aligned}v = J_{\lambda}^{\partial i_{\mathbb{E}}} u &\Leftrightarrow u \in (v + \lambda \partial i_{\mathbb{E}} v),\\ &\Leftrightarrow u - v \in \lambda \partial i_C v,\\ &\Leftrightarrow v = P_{\mathbb{E}} u.\end{aligned}$$

Based on the above derivation, it has been shown that, problem (4.27) is equivalent to problem (1.1), where A is a normal cone to $\mathbb{E}$ at a point $u \in \mathbb{H}$, and the resolvent $(I + \lambda_n A)^{-1}$ is the projection operator. In this case, we have the following result.

Theorem 4.2. *Let $\mathbb{E}$ be a nonempty closed convex subset of a real Hilbert space $\mathbb{H}$, B is a Lipschitz continuous monotone mapping on $\mathbb{H}$ and $VI(\mathbb{E}, B)$ be the solution set of the variational inequality problem* (4.27). *Suppose that u_{-1}, u_0 are arbitrary points in $\mathbb{H}_1$, and $\nu \in (0,1), \{\varrho_n\} \subset [0, \varrho]$ with positive numbers λ_0, ϱ.*

$$\begin{cases}\bar{u}_n = u_n + \varrho_n(u_n - u_{n-1}),\\ z_n = (1 - s_n)\bar{u}_n + s_n P_{\mathbb{E}}(\bar{u}_n - \lambda_n B\bar{u}_n),\\ u_{n+1} = (1 - t_n)\bar{u}_n + t_n P_{\mathbb{E}}(z_n - \lambda_n B z_n),\end{cases} \tag{4.29}$$

where $P_{\mathbb{E}}$ is the metric projection onto $\mathbb{E}$ and the step size λ_n is updated using Equation (3.10). *If $VI(\mathbb{E}, B)$ is nonempty, then the sequence $\{u_n\}$ converges weakly to an element in $VI(\mathbb{E}, B)$.*

As a numerical illustration, we study a generated by (4.29) Nash-Cournot oligopolistic equilibrium problem. Nash-Cournot equilibrium was reformulated as a monotone variational inequality problem in [26]. Suppose there are n firms, each of them supplies a homogeneous product in a non-cooperative manner. Suppose $q_k \geq 0$ denotes the kth firm's supply at cost $c_k(q_k)$ and $\psi = \sum_{k=1}^{n} q_k$ be the total supply in the market. Suppose $\varphi(\psi)$ denotes the inverse demand curve. The variational inequality problem that corresponds to this equilibrium problem is given as: find $q^* = (q_1^*, \ldots, q_n^*)$, such that for all $q \in \mathbb{R}_+^n$

$$\langle Bq^*, q - q^*\rangle \geq 0,$$

where $Bq^* = (B(q_1^*), \ldots, B(q_n^*))$, and

$$B(q_k^*) = c_k'(q_k^*) - \varphi\left(\sum_{k=1}^{n} q_k^*\right) - q_k^* \varphi'\left(\sum_{k=1}^{n} q_k^*\right).$$

As a particular case, we supposed the inverse demand function φ and the cost function c_k be defined by:

$$\varphi(\psi) = 5000^{\frac{1}{\alpha}} \psi^{-\frac{1}{\alpha}}$$

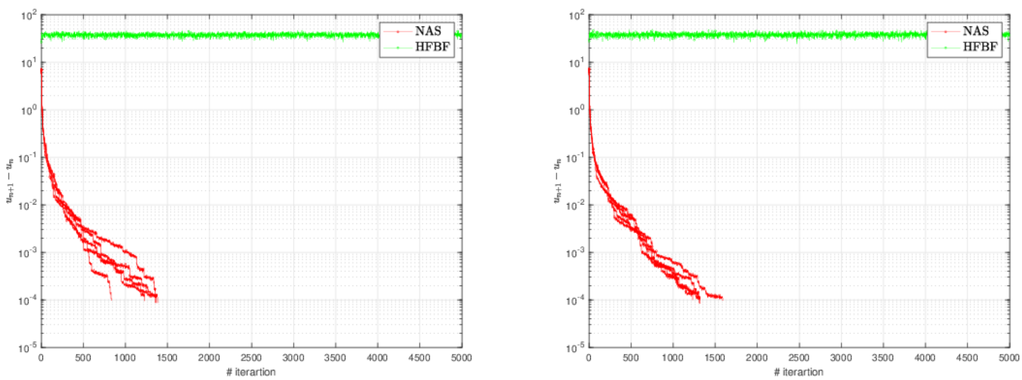
and

$$c_k(q_k) = a_k q_k + \frac{\gamma_k}{\gamma_k + 1} L_k^{\frac{1}{\gamma_k}} q^{\frac{\gamma_k + 1}{\gamma_k}}$$

respectively, where the constants α, a, γ and L are given below. In this experiments, we consider the case $n = 100$. It is important to mention that, it has been noted in [40] that many algorithms cannot be implemented for n larger than 10. We generate the data randomly for two cases, we choose each value of α, γ, a and L independently from the uniform distributions with the following parameters:

i. $\alpha_k = 1.1, \gamma_k \sim U(\frac{1}{2}, 2), a_k \sim U(1, 100)$ and $L_k \sim U(\frac{1}{2}, 5)$.
ii. $\alpha_k = 1.5, \gamma_k \sim U(\frac{1}{3}, 4), a_k$ and L_k same as case i.

For each case above, we generate 5 random instances (the results is shown in Figure 1a and 1b). We compare the proposed method denoted as **NAS** with a generalized Halpern forward backward algorithm proposed in [32], this we denote by **HFBF**. We use $\|q - P_{\mathbb{R}_+}(q - Bq)\| \leq 10^{-6}$ as the stopping criteria and $q_1 = (1, \ldots, 1)$ as the starting point for the comparison.



(A) case i (B) case ii

FIGURE 1. Results for 5 randam instances.

From the result in Figure 1a, we can see that the proposed method performs better than the compared method in terms of number of iterations.

4.2. **Application to convex feasibility problem.** In this subsection, we derive a scheme for solving the split convex feasibility problem (SCFP) from Algorithm 1. The SCFP is a problem of finding a point $\check{u} \in C$ such that $B\check{u} \in Q$, where C, Q are nonempty closed and convex subsets of $\mathbb{H}_1$ and $\mathbb{H}_2$, respectively, and $B : \mathbb{H}_1 \longrightarrow \mathbb{H}_2$ is a bounded linear operator. Censor and Elfving [15] introduced the problem (SCFP) in finite-dimensional Hilbert spaces by using a multi-distance method to obtain an iterative method for solving SCFP. A number of problems that arise from phase retrievals, image restoration, dynamic emission, tomographic image reconstruction, radiation therapy, treatment planning, and in medical image reconstruction can be formulated as SCFP (Ref. [13, 14, 23, 12, 16]).

Now, based on the above derivation in subsection 4.1, Algorithm 1 can be reduced to the following result.

Theorem 4.3. *Let C and Q be nonempty closed convex subsets of Hilbert spaces $\mathbb{H}_1$ and $\mathbb{H}_2$, respectively, $B : \mathbb{H}_1 \longrightarrow \mathbb{H}_2$ be a bounded linear operator with adjoint B^*, and Γ_{SFP} be the solution set of the problem (SCFP). Let u_{-1}, u_0 be arbitrary points in $\mathbb{H}_1$, and $\nu \in (0, 1), \{\varrho_n\} \subset [0, \varrho]$ with positive numbers λ_0, ϱ.*

$$\begin{cases} \bar{u}_n := u_n + \varrho_n(u_n - u_{n-1}), \\ z_n := (1 - s_n)\bar{u}_n + s_n P_C\left[\bar{u}_n - \lambda_n B^*(I - P_Q)B\bar{u}_n\right], \\ u_{n+1} := (1 - t_n)\bar{u}_n + t_n P_C\left[z_n - \lambda_n B^*(I - P_Q)Bz_n\right]. \end{cases} \tag{4.30}$$

where the step size λ_n is updated using Equation 3.10. If $\Gamma_{SFP} \neq \emptyset$, then the sequence $\{u_n\}$ converges weakly to an element of $\Gamma_{SFP} \neq \emptyset$.

In what follows, we implement Algorithm 1 and the derived scheme (4.30) in solving image deblurring problem. Furthermore, to illustrate the effectiveness of the proposed schemes, we give a comparative analysis of Algorithm 1 and the algorithms proposed in [38] and [44].

Recall that the image deblurring problem in image processing can be expressed as:

$$c = Mu + \delta, \tag{4.31}$$

where $u \in \mathbb{R}^n$ represents the original image, M is the deblurring matrix, c is the observed image, and $\delta \in \mathbb{R}^m$ is the Gaussian noise. It has been known that solving (4.31) is equivalent to solving the convex unconstrained optimization problem:

$$\min_{u \in \mathbb{R}^n} f(u) := \frac{1}{2}\|Mu - c\|_2^2 + \rho\|u\|_1^2, \tag{4.32}$$

with $\rho > 0$ as the regularization parameter. To solve (4.32), we suppose $B = \nabla S(u)$ and $A = \partial T$ where $S(u) = \frac{1}{2}\|Mu - c\|_2^2$ and $T(u) = \|u\|_1^2$, then we have $\nabla S(u) = M^t(Mu - c)$ is $\frac{1}{\|M\|^2}$-cocoercive. Therefore, for any $0 < \tau < \frac{2}{\|M\|^2}$, $(I - \tau\nabla S)$ is nonexpansive [28]. The subgradient ∂T is maximal monotone [52]. It is well known that:

$$u \text{ is a solution of (4.32)} \Leftrightarrow 0 \in (A + B)u \Leftrightarrow u = prox_{\rho T}(I - \tau\nabla S)(u)$$

where $prox_{\rho T}(u) = \arg min_{x \in \mathbb{R}^n} \left\{T(u) + \frac{1}{2\rho}\|u - x\|^2\right\}$. For more details, see [18].

To measure the quality of the recovered images, we adopted the improved signal-to-noise ratio (ISNR) [49] and structural similarity index measure (SSIM) [62]. We considered motion blur from MATLAB as the blurring function using ("fspecial('motion', $9, 40$)"). For the comparison, we considered the standard test images of Girl (768×512), Fruits (512×512), and Tulips (768×512) (see Figure 2a, 2b and 2c). For the control parameters, we took $\varrho = 0.9$, $\lambda_0 = 1$, $\nu = 0.3$, and $\rho = 0.1$, for Algorithm 1. While for the compared algorithms (Algorithm 3.1, [38] and Algorithm 1.3, [44]) the control parameters are set as reported in their respective papers. For all algorithms, we took $\frac{\|u_{n+1} - u_n\|_2}{\|u_{n+1}\|_2} < 10^{-4}$ as the stopping criterion. For reference, all codes were written using MATLAB R2019b on a personnel computer.

(A) Girl

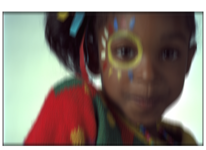
(B) Fruits

(C) Tulips

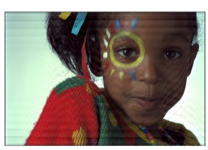
FIGURE 2. Original test images.

TABLE 1. The ISNR and SSIM values of the compared algorithms

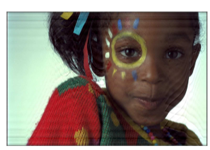
	Algorithm 1		Algorithm 1.3		Algorithm 3.1	
Images	ISNR	SSIM	ISNR	SSIM	ISNR	SSIM
Girl	5.7228	0.9277	5.5838	0.9267	5.5743	0.9267
Fruits	6.2731	0.9442	6.0838	0.9430	6.0753	0.9429
Tulips	7.3830	0.9167	7.1361	0.9145	7.1235	0.9144



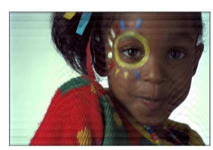
(A) degraded



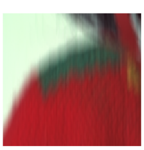
(B) Algorithm 1.0



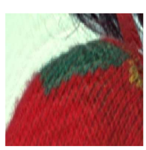
(C) Algorithm 3.1



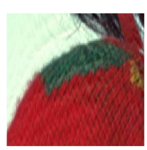
(D) Algorithm 1.3



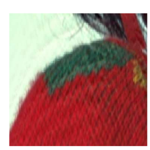
(E) degraded



(F) Algorithm 1.0

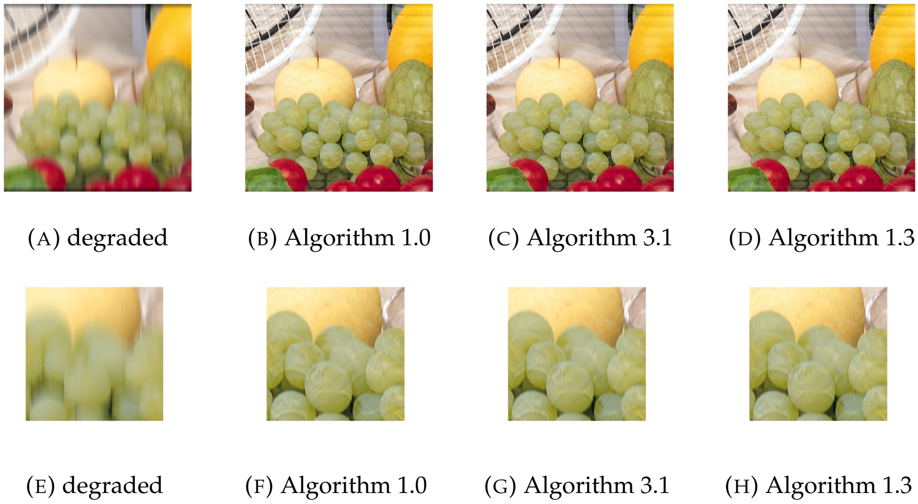


(G) Algorithm 3.1



(H) Algorithm 1.3

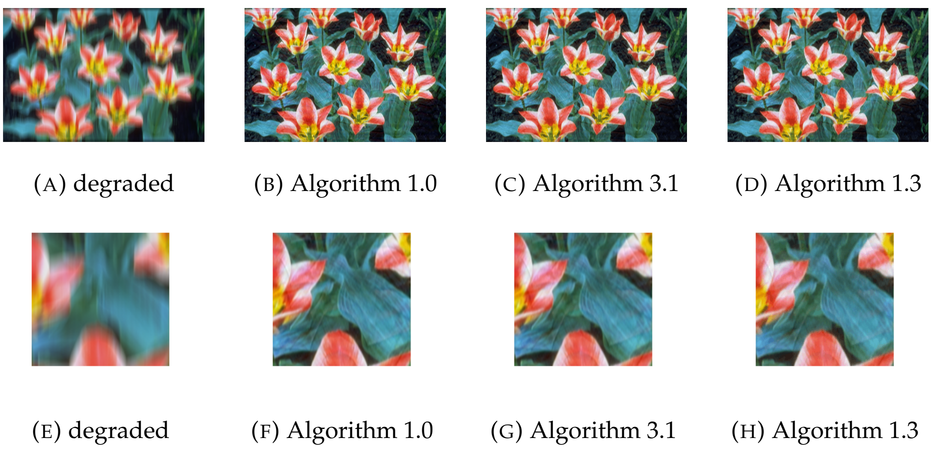
FIGURE 3. Degraded and restored Girl images by the various algorithms.



(A) degraded (B) Algorithm 1.0 (C) Algorithm 3.1 (D) Algorithm 1.3

(E) degraded (F) Algorithm 1.0 (G) Algorithm 3.1 (H) Algorithm 1.3

FIGURE 4. Degraded and restored Fruits images by the various algorithms.



(A) degraded (B) Algorithm 1.0 (C) Algorithm 3.1 (D) Algorithm 1.3

(E) degraded (F) Algorithm 1.0 (G) Algorithm 3.1 (H) Algorithm 1.3

FIGURE 5. Degraded and restored Tulips images by the various algorithms.

It can be seen from Table 1 that the recovered images (Figures 3b, 3f, 4b 4f, 5b and 5f in 3, 4 and 5 respectively) by Algorithm 1 have higher ISNR and SSIM values, which means that the quality of images recovered by Algorithm 1 is better than the compared algorithms.

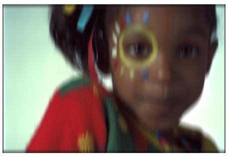
For the split convex feasibility problem (SCFP), we reformulate Problem 4.32 as a convex constrained optimization problem:

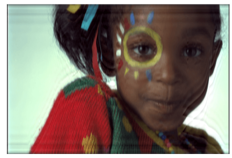
$$\min_{u\in\mathbb{R}^n} \frac{1}{2}\|Mu-c\|_2^2$$

$$\text{subject to} \|u\|_1 \leq t, \tag{4.33}$$

where $t > 0$ is a given constant, and to solve (4.33), we take $Bu = \nabla S(u)$ and consider $C := \{u \in \mathbb{R}^n : \|u\|_1 \leq t\}$ and $Q := \{c\}$.

We compare scheme 4.30 (Algorithm 2.0) with Byrne's algorithm proposed in [12] (Algorithm 1.1) for solving the SCFP. We take the same control parameters and stopping criteria as in problem (4.32) above for Algorithm 2.0 and for Algorithm 1.1 in [12] we take $\lambda = 0.5$ and $\gamma = \frac{0.7}{\|B\|^2}$.





(A) degraded (B) Algorithm 2.0 (C) Algorithm 1.1

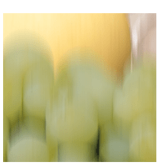


(D) degraded (E) Algorithm 2.0 (F) Algorithm 1.1

FIGURE 6. Degraded and restored Girl images by the various algorithms.

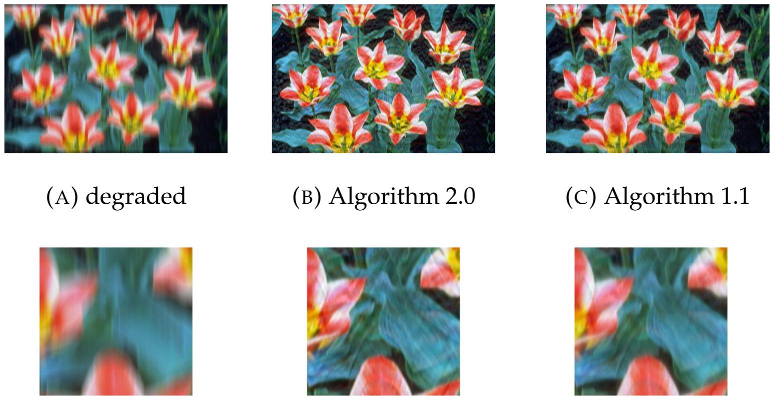


(A) degraded (B) Algorithm 2.0 (C) Algorithm 1.1



(D) degraded (E) Algorithm 2.0 (F) Algorithm 1.1

FIGURE 7. Degraded and restored Fruits images by the various algorithms.



(A) degraded (B) Algorithm 2.0 (C) Algorithm 1.1

(D) degraded (E) Algorithm 2.0 (F) Algorithm 1.1

FIGURE 8. Degraded and restored Tulips image by the various algorithms.

Figures 6, 7, 8 and Table 2 below shows that the recovered images by our algorithm (Algorithm 2) have higher ISNR and SSIM values, which means that the quality of images recovered by Algorithm 2.0 is better compared to Algorithm 1.1 of Byrne [12].

TABLE 2. The ISNR and SSIM values of the compared algorithms.

	Algorithm 2.0		**Algorithm 1.1**	
Images	**ISNR**	**SSIM**	**ISNR**	**SSIM**
Girl	4.9823	0.9150	4.0472	0.9117
Fruits	5.3341	0.9324	4.3100	0.9269
Tulips	6.1536	0.8923	4.9314	0.8853

5. CONCLUSIONS

A new accelerated method for solving variational inclusion problems is proposed in this work, and the scheme was derived by incorporating the inertial extrapolation step with a generalized KM-type method. The main advantage of this scheme is that it involves both the use of an inertial extrapolation step and relaxation technique, which makes the iterates generated by the proposed scheme to converge weakly to the solution of the zeros of the sum of a maximally monotone operator and a monotone operator. Furthermore, the proposed method does not require prior knowledge of the Lipschitz constant of the underlying operator, and the numerical experiments suggest that the iterates generated by the proposed scheme converges fast to the solution of the problem due to the combination of the inertial extrapolation step and the relaxation technique. A modified schemes derived from the proposed method were given for solving variational inequality and split feasibility problem. The application of the proposed methods in solving the Nash-Cournot equilibrium problem and image deblurring problem, and the comparison with some of the related methods in the literature suggest that the proposed methods are robust and efficient.

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