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Measures of noncompactness and infinite systems of integral equations of Urysohn type in $L^\infty(\mathfrak{G})$

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ABSTRACT. In this article, applying the concept of measure of noncompactness, some fixed point theorems in the Fréchet space $L^\infty(\mathfrak{G})$ (where $\mathfrak{G} \subseteq \mathbb{R}^\omega$) have been proved. We handle our obtained consequences to inquiry the existence of solutions for infinite systems of Urysohn type integral equations. Our results extend some famous related results in the literature. Finally, to indicate the effectiveness of our results we present a genuine example.

1. INTRODUCTION AND PRELIMINARIES

Measure of noncompactness (MNC) approaches ([8], [17]) have an substantial role in nonlinear functional analysis and fixed point theory. Heretofore, applying MNC approaches many articles have been extracted on the existence and behavior of solutions for nonlinear differential and integral equations. Some of these papers are [2, 3, 6, 7, 11, 14].

In this paper, we extract some fixed point theorems in Fréchet spaces with the assistance of MNC approaches and the Tychonoff fixed point theorem (TFPT), which are extensions of the results presented in [18, 19, 20, 21].

The conformation of this paper is as follows. In part 1, some preliminaries and concepts are summoned. Part 2 is allocated to stating some fixed point theorems of Darbo-type in the space $L^\infty(\mathfrak{G})$. Finally, in part 3, we apply our results to contemplate the existence of solutions for the following infinite system of nonlinear integral equations:

$$\sigma_n(\iota) = \rho_n\Big(\iota, \sigma_1(\iota)\dots, \sigma_n(\iota), \dots, \int_{\mathfrak{G}} \eta_n(\iota, \kappa, (\sigma_j(\kappa))_{j=1}^{\infty})d\kappa\Big) \tag{1.1}$$

where $\mathfrak{G} \subseteq \mathbb{R}^\omega$ in which $\mathbb{R}^\omega$ denotes the countable cartesian product of $\mathbb{R}$ with itself. Note that some classes of infinite system of nonlinear integral equations have been investigated in [10, 12, 15].

All over this paper, $\mathfrak{B}$ is assumed to be an infinite dimensional Banach space or a Fréchet space. As well as, $\overline{B}(x,r)$ marks the closed ball centered at x with radius r. The symbol $\overline{B}_r$ stands for the ball $\overline{B}(0,r)$. If $\mathcal{Q}$ be a subset of $\mathfrak{B}$, then the closure and closed convex hull of $\mathcal{Q}$, are announced by $\overline{\mathcal{Q}}$ and $Conv\mathcal{Q}$, respectively. Furthermore, the family of all nonempty bounded subsets and the collection of all relatively compact subsets of $\mathfrak{B}$ are indicated by $\mathfrak{M}_{\mathfrak{B}}$ and $\mathfrak{N}_{\mathfrak{B}}$, respectively.

A vector space $\mathcal{Q}$ over the field $\mathbb{R}$ which is endowed with a topology such that the maps $(\iota,\kappa) \to \iota+\kappa$ and $(\upsilon,\iota) \to \upsilon\iota$ are continuous from $\mathcal{Q}\times\mathcal{Q}$ and $\mathbb{R}\times\mathcal{Q}$ to $\mathcal{Q}$ is called a topological vector space (TVS). A TVS is called locally convex if the origin has a neighborhood basis (i.e. a local base) consisting of convex sets [16]. Fréchet spaces are locally convex and complete with respect to a translation invariant metric.

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Example 1.1. [4] If $\mathfrak{B}_i$ is a Banach space ($i \in \mathbb{N}$) and $d : \prod_{i\in\mathbb{N}} \mathfrak{B}_i \times \prod_{i\in\mathbb{N}} \mathfrak{B}_i \longrightarrow \mathbb{R}$ be characterized by $d(\iota, \kappa) = \sup\big\{\frac{1}{2^i} \min\{1, d(\iota_i, \kappa_i)\} : i \in \mathbb{N}\big\}$, $\iota = (\iota_1, \iota_2, \ldots)$ and $\kappa = (\kappa_1, \kappa_2, \ldots)$, then $\prod\limits_{i\in\mathbb{N}} \mathfrak{B}_i$ is a Fréchet space.

Definition 1.1. [5] Let $\mathcal{M}$ be a collection of subsets of a Fréchet space $\mathfrak{B}$. If $\mathcal{Q} \in \mathcal{M}$ implies that $Conv(\mathcal{Q}), \overline{\mathcal{Q}} \in \mathcal{M}$, then we say that $\mathcal{M}$ is an admissible set.

Definition 1.2. [5] Let $\mathcal{M}$ be an admissible subset of a Fréchet space $\mathfrak{B}$ and $\mathfrak{M} : \mathcal{M} \longrightarrow \mathbb{R}_+$. We say that $\mathfrak{M}$ is a measure of noncompactness on $\mathfrak{B}$ if

1° The family $ker\{\mathfrak{M}\} = \{\mathcal{Q} \in \mathcal{M} : \mathfrak{M}(\mathcal{Q}) = 0\}$ is nonempty and $ker\{\mathfrak{M}\} \subseteq \mathfrak{N}_{\mathfrak{B}}$.
2° $\mathcal{Q} \subset \Lambda \Longrightarrow \mathfrak{M}(\mathcal{Q}) \leq \mathfrak{M}(\Lambda)$.
3° $\mathfrak{M}(\overline{\mathcal{Q}}) = \mathfrak{M}(\mathcal{Q})$.
4° $\mathfrak{M}(Conv\mathcal{Q}) = \mathfrak{M}(\mathcal{Q})$.
5° $\mathfrak{M}(\eta X + (1-\eta)\Lambda) \leq \eta\mathfrak{M}(X) + (1-\eta)\mathfrak{M}(\Lambda)$ for all $\eta \in [0, 1]$.
6° If $(\mathcal{Q}_k)$ be a sequence of closed sets from $\mathcal{M}$ such that $\mathcal{Q}_{k+1} \subset \mathcal{Q}_k$ for all $k = 1, 2, \cdots$ and if $\lim\limits_{k\longrightarrow\infty} \mathfrak{M}(\mathcal{Q}_k) = 0$, then $\mathcal{Q}_\infty = \cap_{k=1}^{\infty} \mathcal{Q}_k \neq \emptyset$.

Theorem 1.1. (TFPT [1]) *Let $\mathfrak{B}$ be a Hausdorff locally convex TVS, $\mathcal{G}$ be a convex subset of $\mathfrak{B}$ and $H : \mathcal{G} \longrightarrow \mathfrak{B}$ be a continuous mapping such that*

$$H(\mathcal{G}) \subseteq A \subseteq \mathcal{G}$$

where A is compact. Then, H possesses at least one fixed point.

2. Some fixed point theorems in a Fréchet space

In this section, some Darbo-type fixed point theorems [9] in a Fréchet space have been investigated.

Theorem 2.2. *Let $\mathcal{G}$ be a nonempty, closed and convex subset of a Fréchet space $\mathfrak{B}$, $\mathfrak{M}$ be a measure of noncompactness on $\mathfrak{B}$ and $H : \mathcal{G} \longrightarrow \mathcal{G}$ be a continuous mapping such that*

$$\mathfrak{M}(H(\mathcal{Q})) \leq \zeta\big(\mathfrak{M}(\mathcal{Q})\big), \tag{2.2}$$

where $\mathcal{Q} \in \mathcal{M}$ and $\zeta : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a nondecreasing function such that $\lim\limits_{n\longrightarrow\infty} \zeta^n(t) = 0$ for all $t > 0$. Then H admits at least a fixed point in $\mathcal{G}$.

Note that $\zeta(t) = kt$ for some $k \in (0, 1)$ and $\zeta(t) = \ln(1 + t/2)$ are some examples of function ζ.

Proof. Construct $(\mathcal{P}_n)$ such that $\mathcal{P}_0 = \mathcal{G}$ and $\mathcal{P}_n = ConvH(\mathcal{P}_{n-1})$ for all $n \geq 1$, inductively. According to the suppositions of the mapping $\mathfrak{M}$, we have

$$\mathfrak{M}(\mathcal{P}_n) = \mathfrak{M}(ConvH(\mathcal{P}_{n-1})) = \mathfrak{M}(H(\mathcal{P}_{n-1})) \leq \zeta(\mathfrak{M}(\mathcal{P}_{n-1})).$$

Therefore,

$$\mathfrak{M}(\mathcal{P}_n) \leq \zeta^n(\mathfrak{M}(\mathcal{P}_0)).$$

Abandoning $n \longrightarrow \infty$ in the above relation, we understand that $\lim_{n\longrightarrow\infty} \mathfrak{M}(\mathcal{P}_n) = 0$. If $\mathcal{P}_\infty = \cap_{n=1}^{\infty} \mathcal{P}_n$, then Definition 1.2 warrantees that $\mathcal{P}_\infty$ is nonempty. Obviously, $\mathcal{P}_\infty$ is a convex compact subset of $L^\infty(\mathfrak{G})$. Now, TFPT insinuates that H possesses a fixed point. □

Corollary 2.1. *Let $\mathcal{G}$ be a nonempty, closed and convex subset of a Fréchet space $\mathfrak{B}$, $\mathfrak{M}$ be a measure of noncompactness on $\mathfrak{B}$ and $H : \mathcal{G} \longrightarrow \mathcal{G}$ be a continuous mapping such that*

$$\mathfrak{M}(H(\mathcal{Q})) \leq k\mathfrak{M}(\mathcal{Q}), \tag{2.3}$$

where $\mathcal{Q} \in \mathcal{M}$ and $k \in [0, 1)$. Then H admits at least a fixed point in $\mathcal{G}$.

Theorem 2.3. *Suppose that $\mathfrak{M}_i$ be a measure of noncompactness on Banach space $\mathfrak{B}_i$ for all $i \in \mathbb{N}$. Let*

$$\mathcal{M} = \{\mathcal{G} \subseteq \prod_{i=1}^{\infty} \mathfrak{B}_i : \sup_{i\geq 1}\{\mathfrak{M}_i(\varpi_i(\mathcal{G}))\} < \infty\},$$

where $\varpi_i(\mathcal{G})$ announces the natural projection of $\prod_{i=1}^{\infty} \mathfrak{B}_i$ into $\mathfrak{B}_i$ and $\widetilde{\mathfrak{M}} : \mathcal{M} \longrightarrow \mathbb{R}_+$ be defined by

$$\widetilde{\mathfrak{M}}(\mathcal{G}) = \sup_{i\geq 1}\{\mathfrak{M}_i(\varpi_i(\mathcal{G}))\}, \tag{2.4}$$

then $\widetilde{\mathfrak{M}}$ is a measure of noncompactness on $\mathcal{Q} = \prod_{i=1}^{\infty} \mathfrak{B}_i$.

Proof. The proof of (2°) is obvious and the properties (3°)-(5°) are immediate consequences of

$$\begin{aligned}
&\varpi_i(\alpha\mathcal{U} + (1-\alpha)\mathcal{V}) = \alpha\varpi_i(\mathcal{U}) + (1-\alpha)\varpi_i(\mathcal{V}),\\
&\varpi_i(Conv\mathcal{G}) = Conv\varpi_i(\mathcal{G}),\\
&\varpi_i(\mathcal{G}) \subseteq \varpi_i(\overline{\mathcal{G}}) \subseteq \overline{\varpi_i(\mathcal{G})}.
\end{aligned}$$

If $\widetilde{\mathfrak{M}}(\mathcal{G}) = 0$ for some $\mathcal{G} \in \mathcal{M}$, then $\mathfrak{M}_i(\varpi_i(\mathcal{G})) = 0$ for each $1 \leq i \leq n$. Hence, according to (1°) of Definition 1.2 for the measure of noncompactness $\mathfrak{M}_i$ we deduce that $\varpi_i(\mathcal{G})$ is relatively compact for all $i \in \mathbb{N}$. Now, exploiting the Tychonoff's theorem[13], we see that $\mathcal{G}$ is relatively compact. Eventually, it sufficient to show (6°). Let $(\mathcal{G}_n)$ be a sequence of closed sets from $\mathcal{M}$ such that $\mathcal{G}_{n+1} \subseteq \mathcal{G}_n$ for all $n \in \mathbb{N}$ and $\lim_{n\longrightarrow\infty} \widetilde{\mathfrak{M}}(\mathcal{G}_n) = 0$ for all $n \in \mathbb{N}$. So, we conclude that $\lim_{n\longrightarrow\infty} \mathfrak{M}_i(\varpi_i(\mathcal{G}_n)) = 0$ (or, $\lim_{n\longrightarrow\infty} \mathfrak{M}_i(\varpi_i(\overline{\mathcal{G}_n})) = 0$) and $\mathcal{Q}_i^\infty = \bigcap_{n=1}^{\infty} \varpi_i(\mathcal{G}_n) \neq \emptyset$ for all $i \in \mathbb{N}$. Therefore, $\prod_{i=1}^{\infty} \mathcal{Q}_i^\infty = \mathcal{G}_\infty \neq \emptyset$. This ends the proof. □

Remark 2.1. The Proof of Theorems 2.2 and 2.3 are parallel to the Proof of Theorems 3.2 and 3.1 of reference [5], respectively.

Corollary 2.2. *Let B_i $(i \in \mathbb{N})$ be a nonempty, closed, convex and bounded subset of Banach space $\mathfrak{B}_i$ and $\mathfrak{M}_i$ be an arbitrary (MNC) on $\mathfrak{B}_i$. Let $H_i : \prod_{i=1}^{\infty} B_i \longrightarrow B_i$ $(i \in \mathbb{N})$ be a continuous operator such that*

$$\mathfrak{M}_i\Big(H_i(\prod_{i=1}^{\infty} \mathcal{Q}_i)\Big) \leq (\sup_{i\geq 1} \zeta_i)(\mathfrak{M}_i(\mathcal{Q}_i)), \tag{2.5}$$

where $\mathcal{Q}_i \subseteq B_i$ $(i \in \mathbb{N})$ and $\zeta_i : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a bounded mapping which satisfies the conditions of Theorem 2.2 for all $i \in \mathbb{N}$. Then there exists $(\iota_j^*)_{j=1}^{\infty} \in \prod_{j=1}^{\infty} B_j$ such that

$$H_i\Big((\iota_j^*)_{j=1}^{\infty}\Big) = \iota_i^*, \tag{2.6}$$

for all $i \in \mathbb{N}$.

Proof. Let $\widetilde{H} : \prod_{i=1}^{\infty} B_i \longrightarrow \prod_{i=1}^{\infty} B_i$ be characterized by

$$\widetilde{H}((\iota_j)_{j=1}^{\infty}) = (H_1((\iota_j)_{j=1}^{\infty}), H_2((\iota_j)_{j=1}^{\infty}), \ldots, H_i((\iota_j)_{j=1}^{\infty}), \ldots) \tag{2.7}$$

for all $(\iota_j)_{j=1}^{\infty} \in \prod_{i=1}^{\infty} B_i$. We will investigate that all conditions of Theorem 2.2 hold. Let $\mathcal{Q} \subseteq \prod_{i=1}^{\infty} B_i$. Let $\varpi_i(\mathcal{Q})$ signifies the natural projection of $\prod_{i=1}^{\infty} \mathcal{Q}_i$ into $\mathcal{Q}_i$. For each $n \in \mathbb{N}$, we have

$$\begin{aligned}\widetilde{\mathfrak{M}}(\widetilde{H}(\mathcal{Q})) &\leq \widetilde{\mathfrak{M}}\Big(\widetilde{H}(\prod_{k=1}^{\infty} \varpi_k(\mathcal{Q}))\Big),\\ &= \sup_{i\geq 1}\Big\{\mathfrak{M}_i\Big(H_i(\prod_{k=1}^{\infty} \varpi_k(\mathcal{Q}))\Big)\Big\}\\ &\leq (\sup_{i\geq 1}\zeta_i)(\widetilde{\mathfrak{M}}(\mathcal{Q})).\end{aligned} \tag{2.8}$$

Taking $\widetilde{\zeta} = \sup_{i\geq 1}\zeta_i$, all conditions of Theorem 2.2 are satisfied. Therefore, $\widetilde{H}$ possesses a fixed point and (2.6) holds. □

3. Application

In this section, we study the existence of a solution for integral equation system (1.1) in the space $L^{\infty}(\mathfrak{G})$, where $\mathfrak{G} \subseteq \mathbb{R}^{\omega}$ to indicate the applicability of presented results.

Let $L^{\infty}(\mathfrak{G})$ be the space of all real valued Lebesgue measurable functions on an open subset $\mathfrak{G}$ of $\mathbb{R}^{\omega}$ which are essentially bounded on $\mathbb{R}^{\omega}$, endowed with the norm

$$\|\rho\|_{\infty} = \inf\{M > 0 : |\rho| \leq M \ a.e. \ on \ \mathfrak{G}\}.$$

Let $\mathcal{Q}$ be a bounded subset of the space $L^{\infty}(\mathfrak{G})$. Let $\mathcal{Z}$ be a positive real number and $\bar{B}_{\mathcal{Z}}$ be the closed ball with center 0 and radius $\mathcal{Z}$. For all $\rho \in \mathcal{Q}$ and for all $\varepsilon > 0$, let:

$$\begin{aligned}\omega^{\mathcal{Z}}(\rho,\varepsilon) &= \sup\{\|\upsilon_h\rho - \rho\|_{L^{\infty}(\bar{B}_{\mathcal{Z}})} : \|h\| < \varepsilon\},\\ \omega^{\mathcal{Z}}(\mathcal{Q},\varepsilon) &= \sup\{\omega^{\mathcal{Z}}(\rho,\varepsilon) : \rho \in \mathcal{Q}\},\\ \omega_0^{\mathcal{Z}}(\mathcal{Q}) &= \lim_{\varepsilon\longrightarrow 0}\omega^{\mathcal{Z}}(\mathcal{Q},\varepsilon),\\ \omega_0(\mathcal{Q}) &= \lim_{\mathcal{Z}\longrightarrow\infty}\omega_0^{\mathcal{Z}}(\mathcal{Q}),\end{aligned}$$

and

$$\begin{aligned}d_{\mathcal{Z}}(\mathcal{Q}) &= \sup\{ess \sup_{\|\iota\|>\mathcal{Z}} |\rho(\iota) - \varrho(\iota)| : \rho, \varrho \in \mathcal{Q}\},\\ d(\mathcal{Q}) &= \lim_{\mathcal{Z}\longrightarrow\infty} d_{\mathcal{Z}}(\mathcal{Q})\\ \mathfrak{M}(\mathcal{Q}) &= \omega_0(\mathcal{Q}) + d(\mathcal{Q}).\end{aligned}$$

The function $\mathfrak{M}$ is a measure of noncompactness in the space $L^{\infty}(\mathfrak{G})$ [3].

Definition 3.3. A function $\rho : \mathfrak{G} \times \mathbb{R}^{\omega} \longrightarrow \mathbb{R}$ is said to have the Carathéodory property if

(a) The function $\iota \longrightarrow \rho(\iota,\sigma)$ is measurable on $\mathfrak{G}$ for all $\sigma \in \mathbb{R}^{\omega}$.

(b) The function $\sigma \longrightarrow \rho(\iota,\sigma)$ is continuous on $\mathbb{R}^{\omega}$ for almost all $\iota \in \mathbb{R}^{\omega}$.

Let:

$(\mathcal{P}_1)$ $\rho_n : \mathfrak{G} \times \mathbb{R}^{\omega} \times \mathbb{R} \longrightarrow \mathbb{R}$ ($n \in \mathbb{N}$ and $\mathfrak{G}$ is an unbounded subset of $\mathbb{R}^{\omega}$) satisfies the Carathéodory conditions with $\rho_n(.,0,\ldots) \in L^{\infty}(\mathfrak{G})$. Moreover, for a bounded nondecreasing, concave and upper semicontinuous mapping χ with $\chi(t) < t$,

$$|\rho_n(\iota,\sigma_1,\ldots,\sigma_n,\ldots,\varrho) - \rho_n(\iota,\varsigma_1,\ldots,\varsigma_n,\ldots,q)| \quad\leq\quad \chi(\sup_{i\geq 1}|\sigma_i - \varsigma_i|) + |\varrho - q|,$$

for a.e. $\iota \in \mathfrak{G}$.

$(\mathcal{P}_2)$ $\eta_n : \mathfrak{G} \times \mathfrak{G} \times \mathbb{R}^\omega \longrightarrow \mathbb{R}$ $(n \in \mathbb{N})$ satisfies the Carathéodory conditions, $\eta_n \in L^\infty_{loc}(\mathfrak{G} \times \mathfrak{G} \times \mathbb{R}^\omega)$ and for a positive constant D

$$ess \sup_{\iota\in\mathfrak{G}} \Big\{\Big| \int_{\mathfrak{G}} \eta_n(\iota, \kappa, (\sigma_j(\kappa))_{j=1}^\infty) d\kappa\Big|\Big\} \leq D. \tag{3.9}$$

Moreover,

$$\lim_{\mathcal{Z}\longrightarrow\infty} ess \sup_{\|\iota\|>\mathcal{Z}} \Big| \int_{\mathfrak{G}} \bigg(\eta_n(\iota, \kappa, (\sigma_j(\kappa))_{j=1}^\infty) - \eta_n(\iota, \kappa, (\varsigma_j(\kappa))_{j=1}^\infty)\bigg) d\kappa\Big| = 0 \tag{3.10}$$

uniformly with respect to $\sigma_j, \varsigma_j \in L^\infty(\mathfrak{G})$ and for all $r > 0$ with $\max\{\|\sigma_j\|_\infty, \|\varsigma_j\|_\infty\} \leq r$;

$(\mathcal{P}_3)$

$$\lim_{\mathcal{Z}\longrightarrow\infty} ess \sup_{\iota\in\mathfrak{G}} \int_{\mathfrak{G}\setminus\bar{B}_{\mathcal{Z}}} |\eta_n(\iota, \kappa, (\sigma_j(\kappa))_{j=1}^\infty)| d\kappa = 0.$$

$(\mathcal{P}_4)$ For each $n \in \mathbb{N}$

$$\chi(\sup_{i\geq1} \lambda_i) + E_n + D \leq \lambda_{n_0}$$

for some n_0 such that $\lambda_n \leq \lambda_{n+1}$ where $E_n := ess \sup_{\iota\in\mathfrak{G}} |\rho_n(\iota, 0, 0, \ldots)|$.

Note that we say that a function $f : \mathcal{G} \longrightarrow \mathbb{R}$ belongs to $L^\infty_{loc}(\mathbb{R}^\omega)$ if $f\chi K \in L^\infty(\mathbb{R}^\omega)$ for every compact set K contained in $\mathcal{G}$, where $\mathcal{G}$ is an open subset of $\mathbb{R}^\omega$.

Theorem 3.4. *Having suppositions $(\mathcal{P}_1) - (\mathcal{P}_4)$, the infinite system (1.1) admits at least one solution $(\sigma_i = \sigma_i(\iota))_{i=1}^\infty \in (L^\infty(\mathfrak{G}))^\omega$.*

Proof. First, consider arbitrary $n \in \mathbb{N}$. Let $H_n : (L^\infty(\mathfrak{G}))^\omega \longrightarrow L^\infty(\mathfrak{G})$ be defined by

$$H_n((\sigma_j)_{j=1}^\infty)(\iota) = \rho_n\Big(\iota, \sigma_1(\iota), \ldots, \sigma_n(\iota), \ldots, \int_{\mathfrak{G}} \eta_n\big(\iota, \kappa, (\sigma_j(\kappa))_{j=1}^\infty\big) d\kappa\Big). \tag{3.11}$$

According to the Carathéodory conditions, we conclude that $H_n((\sigma_j)_{j=1}^\infty)$ is measurable for any $(\sigma_j)_{j=1}^\infty \in (L^\infty(\mathfrak{G}))^\omega$.

Now, we prove that $H_n((\sigma_j)_{j=1}^\infty) \in L^\infty(\mathfrak{G})$. Handling suppositions $(\mathcal{P}_1)$ -$(\mathcal{P}_4)$ we have

$$\begin{aligned}
&|H_n((\sigma_j)_{j=1}^\infty)(\iota)| \\
&\leq |\rho_n\big(\iota, \sigma_1(\iota), \ldots, \sigma_n(\iota), \ldots, \int_{\mathfrak{G}} \eta_n(\iota, \kappa, (\sigma_j(\kappa))_{j=1}^\infty) d\kappa\big) - \rho_n(\iota, 0, \ldots)| + |\rho_n(\iota, 0, \ldots)| \\
&\leq \chi(\sup_{i\geq1} |\sigma_i(\iota)|) + |\int_{\mathfrak{G}} \eta_n(\iota, \kappa, (\sigma_j(\kappa))_{j=1}^\infty) d\kappa| + |\rho_n(\iota, 0, \ldots)| \\
&\leq \chi(\sup_{i\geq1} \|\sigma_i\|_\infty) + D + E_n.
\end{aligned}$$

for a.e. $\iota \in \mathfrak{G}$. Therefore, we conclude that

$$\|H_n((\sigma_j)_{j=1}^\infty)\|_\infty \leq \chi(\sup_{i\geq1} \|\sigma_i\|_\infty) + D + E_n. \tag{3.12}$$

Thus, $H_n((\sigma_j)_{j=1}^\infty) \in L^\infty(\mathfrak{G})$ for any $(\sigma_j(\iota))_{i=1}^\infty \in (L^\infty(\mathfrak{G}))^\omega$.

Propounding relation (3.12) and exploiting $(\mathcal{P}_4)$, the function H_n maps $\prod_{i=1}^{\infty} \bar{B}_{\lambda_i}$ into $\bar{B}_{\lambda_{n_0}}$.

Now, we prove that H_n is a continuous operator. Let $0 < \varepsilon$ for which $0 < \varepsilon < \frac{1}{2^n}$ and take arbitrary $((\sigma_j)_{j=1}^\infty)$ and $((\varsigma_j)_{j=1}^\infty) \in (L^\infty(\mathfrak{G}))^\omega$ such that

$$d(((\sigma_j)_{j=1}^\infty), ((\varsigma_j)_{j=1}^\infty)) = \sup \Big\{\frac{1}{2^i} \min\{1, \|\sigma_i - \varsigma_i\|_\infty\} : i \in \mathbb{N}\Big\} < \varepsilon.$$

Then,

$$\begin{aligned}\Big|H_n((\sigma_j)_{j=1}^{\infty})(\iota)-H_n((\varsigma_j)_{j=1}^{\infty})(\iota)\Big| &\le \Big|\rho_n\big(\iota,\sigma_1(\iota),\ldots,\sigma_n(\iota),\ldots,\int_{\mathfrak{G}}\eta_n(\iota,\kappa,(\sigma_j(\kappa))_{j=1}^{\infty})d\kappa\big)\\ &\quad -\rho_n\big(\iota,\varsigma_1(\iota),\ldots,\varsigma_n(\iota),\ldots,\int_{\mathfrak{G}}\eta_n(\iota,\kappa,(\varsigma_j(\kappa))_{j=1}^{\infty})d\kappa\big)\Big|\\ &\le \chi(\sup_{i\ge 1}|\sigma_i(\iota)-\varsigma_i(\iota)|)+|\int_{\mathfrak{G}}\eta_n(\iota,\kappa,(\sigma_j(\kappa))_{j=1}^{\infty}d\kappa-\int_{\mathfrak{G}}\eta_n(\iota,\kappa,(\varsigma_j(\kappa))_{j=1}^{\infty})d\kappa|.\end{aligned}$$

Exerting supposition $(\mathcal{P}_1)$ we can choose $\mathcal{Z}_1>0$ such that

$$\operatorname*{ess\,sup}_{\|\iota\|>\mathcal{Z}}\Big|\int_{\mathfrak{G}}[\eta_n(\iota,\kappa,(\sigma_j(\kappa))_{j=1}^{\infty})-\eta_n(\iota,\kappa,(\varsigma_j(\kappa))_{j=1}^{\infty})]d\kappa\Big|<\varepsilon. \tag{3.13}$$

Exerting suppositions $(\mathcal{P}_2)$ we can choose $\mathcal{Z}_2>0$ such that

$$\operatorname*{ess\,sup}_{\iota\in\mathfrak{G}}\int_{\mathfrak{G}\setminus\bar{B}_{\mathcal{Z}}}|\eta_n(\iota,\kappa,(\sigma_j(\kappa))_{j=1}^{\infty})|d\kappa<\varepsilon. \tag{3.14}$$

Now, let $\mathcal{Z}=\max\{\mathcal{Z}_1,\mathcal{Z}_2\}$.

From (3.13), we conclude that

$$\operatorname*{ess\,sup}_{\|\iota\|>\mathcal{Z}}\Big|H_n((\sigma_j)_{j=1}^{\infty})(\iota)-H_n((\varsigma_j)_{j=1}^{\infty})(\iota)\Big|\le\chi(\varepsilon)+\varepsilon. \tag{3.15}$$

For almost all $\iota\in\bar{B}_{\mathcal{Z}}\cap\mathfrak{G}$, we have

$$\begin{aligned}\Big|H_n((\sigma_j)_{j=1}^{\infty})(\iota)-H_n((\varsigma_j)_{j=1}^{\infty})(\iota)\Big| &\le \chi(\sup_{i\ge 1}|\sigma_i(\iota)-\varsigma_i(\iota)|)\\ &+|\int_{\bar{B}_{\mathcal{Z}}}[\eta_n(\iota,\kappa,(\sigma_j(\kappa))_{j=1}^{\infty})-\eta_n(\iota,\kappa,(\varsigma_j(\kappa))_{j=1}^{\infty})]d\kappa|\\ &+|\int_{\mathfrak{G}\setminus\bar{B}_{\mathcal{Z}}}[\eta_n(\iota,\kappa,(\sigma_j(\kappa))_{j=1}^{\infty})-\eta_n(\iota,\kappa,(\varsigma_j(\kappa))_{j=1}^{\infty})]d\kappa|\\ &\le \chi(\varepsilon)+|\int_{\bar{B}_{\mathcal{Z}}}[\eta_n(\iota,\kappa,(\sigma_j(\kappa))_{j=1}^{\infty})-\eta_n(\iota,\kappa,(\varsigma_j(\kappa))_{j=1}^{\infty})]d\kappa|\\ &+2\operatorname*{ess\,sup}_{\iota\in\mathfrak{G}}\int_{\mathfrak{G}\setminus\bar{B}_{\mathcal{Z}}}|\eta_n(\iota,\kappa,(\sigma_j(\kappa))_{j=1}^{\infty})|d\kappa\end{aligned}$$

$$\le\chi(\varepsilon)+\vartheta_n(\varepsilon)+2\operatorname*{ess\,sup}_{\iota\in\mathfrak{G}}\int_{\mathfrak{G}\setminus\bar{B}_{\mathcal{Z}}}|\eta_n(\iota,\kappa,(\sigma_j(\kappa))_{j=1}^{\infty})|d\kappa \tag{3.16}$$

where

$$\begin{aligned}\vartheta_n(\varepsilon)=\inf\{\mathcal{G}\ge 0:|\eta_n(\iota,\kappa,(\sigma_j(\kappa))_{j=1}^{\infty})-\eta_n(\iota,\kappa,(\varsigma_j(\kappa))_{j=1}^{\infty})|\le\mathcal{G}\\ for\ \ a.e.\ \iota,\kappa\in\bar{B}_{\mathcal{Z}}\subset\mathfrak{G},\ \text{and}\ (\varsigma_j(\kappa)_{j=1}^{\infty})\in\bar{B}((\sigma_j(\kappa)_{j=1}^{\infty}),\varepsilon)\}.\end{aligned}$$

Exerting the Carathéodory reservations for η_n on $\bar{B}_{\mathcal{Z}}\times\bar{B}_{\mathcal{Z}}\times\bar{B}((\sigma_j(\kappa)_{j=1}^{\infty}),\varepsilon)$, one has $\vartheta_n(\varepsilon)\to 0$ as $\varepsilon\to 0$. Therefore, from (3.14), (3.15) and (3.16) we conclude that H_n is continuous on $(L^{\infty}(\mathfrak{G}))^{\omega}$.

Now, we demonstrate that H_n has all the reservations of Corollary 2.2. Let Δ_n be a nonempty subset of $\bar{B}_{\lambda_n}$ for all $n\in\mathbb{N}$ with $\Delta_n\subset\Delta_{n+1}$. Assume that $\mathcal{Z}>0$ and $\varepsilon>0$ be given. For almost all $\iota \in \mathfrak{G}$, all h with $\|h\| \leq \varepsilon$ and all $\sigma_n \in \Delta_n$ we have

$$\begin{aligned}
&|H_n((\sigma_j)_{j=1}^{\infty})(\iota) - H_n((\sigma_j)_{j=1}^{\infty})(\iota+h)| \\
&\leq \Big|\rho_n\big(\iota, \sigma_1(\iota), \ldots, \sigma_n(\iota), \ldots \textstyle\int_{\mathfrak{G}} \eta_n(\iota, \kappa, (\sigma_j(\kappa))_{j=1}^{\infty})d\kappa\big) \\
&-\rho_n\big(\iota+h, \sigma_1(\iota), \ldots, \sigma_n(\iota), \ldots \textstyle\int_{\mathfrak{G}} \eta_n(\iota, \kappa, (\sigma_j(\kappa))_{j=1}^{\infty})d\kappa\big)\Big| \\
&+\Big|\rho_n\big(\iota+h, \sigma_1(\iota), \ldots, \sigma_n(\iota), \ldots \textstyle\int_{\mathfrak{G}} \eta_n(\iota, \kappa, (\sigma_j(\kappa))_{j=1}^{\infty})d\kappa\big) \\
&-\rho_n\big(\iota+h, \sigma_1(\iota+h), \ldots, \sigma_n(\iota+h), \ldots, \textstyle\int_{\mathfrak{G}} \eta_n(\iota+h, \kappa, (\sigma_j(\kappa))_{j=1}^{\infty})d\kappa\big)\Big| \\
&\leq H^{\mathcal{Z}}_{\{\lambda_n\}}(\rho_n, \varepsilon) + \chi(\sup_{i\geq 1} |\sigma_i(\iota) - \sigma_i(\iota+h)|) \\
&+\Big| \textstyle\int_{\bar{B}_{\mathcal{Z}}} \eta_n(\iota, \kappa, (\sigma_j(\kappa))_{j=1}^{\infty}) - \eta_n(\iota+h, \kappa, (\sigma_j(\kappa))_{j=1}^{\infty})d\kappa\Big| \\
&+\Big| \textstyle\int_{\mathfrak{G}\setminus\bar{B}_{\mathcal{Z}}} \eta_n(\iota, \kappa, (\sigma_j(\kappa))_{j=1}^{\infty}) - \eta_n(\iota+h, \kappa, (\sigma_j(\kappa))_{j=1}^{\infty})d\kappa\Big| \\
&\leq H^{\mathcal{Z}}_{\{\lambda_n\}}(\rho_n, \varepsilon) + \chi(\sup_{i\geq 1} \omega^{\mathcal{Z}}(\sigma_i, \varepsilon)) \\
&+\omega^{\mathcal{Z}}_{\{\lambda_n\}}(\eta_n, \varepsilon) + 2ess\sup_{\iota\in\mathfrak{G}} \textstyle\int_{\mathfrak{G}\setminus\bar{B}_{\mathcal{Z}}} |\eta_n(\iota, \kappa, (\sigma_j(\kappa))_{j=1}^{\infty})|d\kappa,
\end{aligned}$$

where

$$\begin{aligned}
\omega^{\mathcal{Z}}_{(\lambda_n)}(\rho_n, \varepsilon) = \inf\{&\mathcal{G} \geq 0 : |\rho_n(\iota, \sigma_1, \ldots, \sigma_n, \ldots, v) - \rho(\iota+h, \sigma_1, \ldots, \sigma_n, \ldots, v)| \leq \mathcal{G} \\
&for\ a.e.\ \iota \in \bar{B}_{\mathcal{Z}}, \|h\| \leq \varepsilon, |\sigma_i| \leq \lambda_i, |v| < D\}
\end{aligned}$$

and

$$\begin{aligned}
\omega^{\mathcal{Z}}_{(\lambda_n)}(\eta_n, \varepsilon) = \inf\{&C \geq 0 : |\eta_n(\iota, \kappa, (\sigma_j(\kappa))_{j=1}^{\infty}) - \eta_n(\iota+h, \kappa, (\sigma_j(\kappa))_{j=1}^{\infty})| \leq C \\
&for\ a.e.\ \iota, \kappa \in \bar{B}_{\mathcal{Z}} \cap \mathfrak{G}, \|h\| \leq \varepsilon, |\sigma_j| \leq \lambda_j\}.
\end{aligned}$$

Since σ_n was an arbitrary element of Δ_n for all $n \in \mathbb{N}$ in the above inequality, we subsume that

$$\begin{aligned}
\omega^{\mathcal{Z}}\Big(H_n(\prod_{i=1}^{\infty}\Delta_i), \varepsilon\Big) &\leq \omega^{\mathcal{Z}}_{(\lambda_n)}(\rho_n, \varepsilon) + \chi(\sup_{i\geq 1} \omega^{\mathcal{Z}}(\sigma_i, \varepsilon)) \\
&\quad + \omega^{\mathcal{Z}}_{(\lambda_n)}(\eta_n, \varepsilon) + 2ess\sup_{\iota\in\mathfrak{G}} \int_{\mathfrak{G}\setminus\bar{B}_{\mathcal{Z}}} |\eta_n(\iota, \kappa, (\sigma_j(\kappa))_{j=1}^{\infty})|d\kappa.
\end{aligned}$$

Having Carathéodory provisions for ρ_n and η_n on $\bar{B}_{\mathcal{Z}} \times \prod_{i=1}^{\omega} \bar{B}_{\lambda_i} \times [-D, D]$ and $\bar{B}_{\mathcal{Z}} \times \bar{B}_{\mathcal{Z}} \times \prod_{i=1}^{\infty} \bar{B}_{\lambda_i}$ respectively, one has $\omega^{\mathcal{Z}}_{(\lambda_n)}(\rho_n, \varepsilon) \to 0$ and $\omega^{\mathcal{Z}}_{(\lambda_n)}(\eta_n, \varepsilon) \to 0$ as $\varepsilon \to 0$. Thus, we procure that

$$\omega^{\mathcal{Z}}_0\Big(H_n(\prod_{i=1}^{\infty}\Delta_i)\Big) \leq \chi(\sup_{i\geq 1} \omega^{\mathcal{Z}}_0(\Delta_i)) + 2ess\sup_{\iota\in\mathfrak{G}} \int_{\mathfrak{G}\setminus\bar{B}_{\mathcal{Z}}} |\eta_n(\iota, \kappa, (\sigma_j(\kappa))_{j=1}^{\infty})|d\kappa.$$

If $\mathcal{Z} \to \infty$, handling supposition ($\mathcal{P}_3$) we attain that

$$\omega_0\Big(H_n(\prod_{i=1}^{\infty}\Delta_i)\Big) \leq \chi(\sup_{i\geq 1} \omega_0(\Delta_i)). \tag{3.17}$$

On the other hand, for all $\sigma_i, \varsigma_i \in \Delta_i$, we have

$$\begin{aligned}
ess\sup_{\|\iota\|>\mathcal{Z}}\Big|H_n((\sigma_j)_{j=1}^{\infty})(\iota) - H_n((\varsigma_j)_{j=1}^{\infty})(\iota)\Big| &\leq ess\sup_{\|\iota\|>\mathcal{Z}}\Big(\chi(\sup_{i\geq 1} |\sigma_i(\iota) - \varsigma_i(\iota)|)\Big) \\
&+ess\sup_{\|\iota\|>\mathcal{Z}}| \textstyle\int_{\mathfrak{G}}(\eta_n(\iota, \kappa, (\sigma_j(\kappa))_{j=1}^{\infty}) - \eta_n(\iota, \kappa, (\varsigma_j(\kappa))_{j=1}^{\infty}))d\kappa|.
\end{aligned}$$

Accordingly, we attain that

$$d_{\mathcal{Z}}(H_n(\prod_{i=1}^{\infty}\Delta_i)) \leq \chi(\sup_{i\geq 1} d_{\mathcal{Z}}(\Delta_i)) + ess\sup_{\|\iota\|>\mathcal{Z}} |\int_{\mathfrak{G}}(\eta_n(\iota,\kappa,(\sigma_j(\kappa))_{j=1}^{\infty}) - \eta_n(\iota,\kappa,(\varsigma_j(\kappa))_{j=1}^{\infty}))d\kappa|. \tag{3.18}$$

If $\mathcal{Z}\to\infty$ in (3.18), exploiting $(\mathcal{P}_3)$ we have

$$d\Big(H_n(\prod_{i=1}^{\infty}\Delta_i)\Big) \leq \chi\Big(\sup_{i\geq 1} d(\Delta_i)\Big). \tag{3.19}$$

Subsequently, constituting (3.17) and (3.19) we acquire that

$$d\Big(H_n(\prod_{i=1}^{\infty}\Delta_i)\Big) + H_0\Big(H_n(\prod_{i=1}^{\infty}\Delta_i)\Big) \leq \chi(\sup_{i\geq 1} H_0(\Delta_i)) + \chi\Big(\sup_{i\geq 1} d(\Delta_i)\Big). \tag{3.20}$$

Since χ is a concave function, from (3.20) we acquire that

$$\begin{aligned}\frac{1}{2}\Big(\omega_0\Big(H_n(\prod_{i=1}^{\infty}\Delta_i)\Big) + diam\Big(H_n(\prod_{i=1}^{\infty}\Delta_i)\Big)\Big) &\leq \frac{1}{2}\Big[\chi(\sup_{i\geq 1}\omega_0(\Delta_i)) + \chi(\sup_{i\geq 1} diam(\Delta_i))\Big]\\ &\leq \chi\Big(\frac{1}{2}(\sup_{i\geq 1}\omega_0(\Delta_i)) + \frac{1}{2}\limsup_{i} diam(\Delta_i)\Big),\end{aligned}$$

and we get

$$\frac{1}{2}\mathfrak{M}_n\Big(H_n(\prod_{i=1}^{\infty}\Delta_i)\Big) \leq \chi\Big(\frac{1}{2}\mathfrak{M}_n(\Delta_i)\Big).$$

Taking $\mathfrak{M}'_n = \frac{1}{2}\mathfrak{M}_n$, we find that

$$\mathfrak{M}'_n\Big(H_n(\prod_{i=1}^{\infty}\Delta_i)\Big) \leq \chi\Big(\mathfrak{M}'_n(\Delta_i)\Big).$$

Now, from Corollary 2.2 and taking $\zeta_i = \chi$ for all $i\in\mathbb{N}$, for a $(\sigma_i = \sigma_i(\iota))_{i=1}^{\infty} \in (L^{\infty}(\mathfrak{G}))^{\omega}$ one has

$$\sigma_n(\iota) = \rho_n\Big(\iota, \sigma_1(\iota)\dots,\sigma_n(\iota),\dots,\int_{\mathfrak{G}}\eta_n(\iota,\kappa,(\sigma_j(\kappa))_{j=1}^{\infty})d\kappa\Big).$$

□

Example Let:

$$\begin{aligned}\sigma_n(\iota) &= \frac{||\iota||_{\infty}^2}{2+||\iota||_{\infty}^4} + \tanh|\sigma_n(\iota)|\\ &+\frac{1}{e^{||\iota||_{\infty}}}\arctan\Big(\int_{\mathfrak{G}}\frac{\sin(||\kappa||_{\infty}^3).\cos((\sigma_n(\iota))_{n=1}^{\infty}) + \cos^3(||\iota||_{\infty}).\sin((\sigma_n(\iota))_{n=1}^{\infty})}{3}d\kappa\Big),\end{aligned} \tag{3.21}$$

where $i,n\in\mathbb{N}$. Eq. (3.21) is a special case of Eq. (1.1) with

$$\rho_n(\iota,\sigma_1,\dots,\sigma_n,\dots,z) = \frac{||\iota||_{\infty}^2}{2+||\iota||_{\infty}^4} + \tanh|\sigma_n(\iota)| + \frac{1}{e^{||\iota||_{\infty}}}\arctan z,$$

$$\eta_n(\iota,\kappa,\sigma_1,\sigma_2,\dots) = \frac{\sin(||\kappa||_{\infty}^3).\cos((\sigma_n(\iota))_{n=1}^{\infty}) + \cos^3(||\iota||_{\infty}).\sin((\sigma_n(\iota))_{n=1}^{\infty})}{3},$$

and $\mathfrak{G}\subseteq\mathbb{R}^{\omega}$ is bounded.

Evidently $(\mathcal{P}_1)$ holds and $\chi(t) = \arctan(t)$ is nondecreasing, concave and upper semicontinuous such that $\chi(t) < t$ for a.e. $t > 0$. Let $\iota\in\mathbb{R}$ and $\sigma_i \geq \varsigma_i$. Thus

$$\begin{aligned}
&|\rho_n(\iota,\sigma_1,\ldots,\sigma_n,\ldots,\varrho)-\rho_n(\iota,\varsigma_1,\ldots,\varsigma_n,\ldots,q)|\\
&\leq |\tanh|\sigma_n(\iota)|-\tanh|\varsigma_n(\iota)||\\
&+\frac{1}{e^{||\iota||_\infty}}|\arctan\varrho-\arctan q|\\
&\leq \tanh|\sigma_n(\iota)-\varsigma_n(\iota)|+\tfrac{1}{e^{||\iota||_\infty}}|\varrho-q|\\
&\leq \tanh\sup_{n\geq 1}|\sigma_n(\iota)-\varsigma_n(\iota)|+|\varrho-q|\\
&=\chi(\sup_{n\geq 1}|\sigma_n(\iota)-\varsigma_n(\iota)|)+|\varrho-q|.
\end{aligned}$$

As well as,

$$E_n := ess\sup_{\iota\in\mathfrak{G}}|\rho_n(\iota,0,0,\ldots)| : n\in\mathbb{N}\} = ess\sup_{\iota\in\mathfrak{G}}\{\frac{\iota^2}{2+\iota^4}\}\leq 0.36$$

Also, ρ satisfies the Carathéodory supposition and $\rho_n(.,0,0)\in L^{\infty}(\mathbb{R}^{\omega})$. Thus, condition (ii) holds. Moreover, η_n satisfies the Carathéodory conditions and since

$$\begin{aligned}
&ess\ \sup_{\iota\in\mathfrak{G}}\Big|\int_{\mathbb{R}}\eta_n(\iota,\kappa,(\sigma_j(\kappa))_{j=1}^{\infty})d\kappa\Big|\\
&=ess\sup_{\iota\in\mathbb{R}}\Big|\frac{1}{e^{\iota}}\Big(\int_{\mathbb{R}}\frac{\sin(\kappa^3).\cos(\sigma_n(\iota))+\cos^3(\iota).\sin(\sum_{i=1}^{\infty}\frac{1}{|\sigma_i(\iota)|^2+1})}{(2+\sin(\sum_{i=1}^{\infty}\frac{1}{|\sigma_i(\iota)|^2+1})}d\kappa\Big|\\
&\leq\frac{2}{e},
\end{aligned}$$

so, we take $D=\dfrac{2}{e}$. Moreover,

$$\begin{aligned}
\lim_{\mathcal{Z}\longrightarrow\infty} ess\sup_{\|\iota\|>\mathcal{Z}}\Big|\frac{1}{e^{\iota}}\Big(\int_{\mathbb{R}}&\frac{\sin(\kappa^3).\cos(\sigma_n(\iota))+\cos^3(\iota).\sin(\sum_{i=1}^{\infty}\frac{1}{|\sigma_i(\iota)|^2+1})}{(2+\sin(\sum_{i=1}^{\infty}\frac{1}{|\sigma_i(\iota)|^2+1})}\\
&-\frac{\sin(\kappa^3).\cos(\varsigma_n)+\cos^3(\iota).\sin(\sum_{i=1}^{\infty}\frac{1}{|\varsigma_i|^2+1})}{(2+\sin(\sum_{i=1}^{\infty}\frac{1}{|\varsigma_i|^2+1})}d\kappa\Big|=0
\end{aligned}$$

uniformly with respect to $\sigma_j,\varsigma_j\in L^{\infty}(\mathbb{R})$ and for all $\lambda>0$ with $\max\{\|\sigma_j\|_\infty,\|\varsigma_j\|_\infty\}\leq\lambda$. Also,

$$\lim_{\mathcal{Z}\longrightarrow\infty} ess\sup_{\iota\in\mathbb{R}}\Big|\int_{\mathbb{R}\backslash\bar{B}_{\mathcal{Z}}}\frac{1}{e^{\iota}}\frac{\sin(\kappa^3).\cos(\sigma_n(\iota))+\cos^3(\iota).\sin(\sum_{i=1}^{\infty}\frac{1}{|\sigma_i(\iota)|^2+1})}{(2+\sin(\sum_{i=1}^{\infty}\frac{1}{|\sigma_i(\iota)|^2+1})}d\kappa\Big|=0.$$

It is easy to check that the sequence $(\lambda_n)=(3,4,5,\ldots)$ satisfies the inequality in condition $(\mathcal{P}_4)$, i.e.,

$$\chi(\sup_{n\geq 1}\lambda_n)+E_n+D=0.99505475368+1+\frac{2}{e}\leq\lambda_n,$$

for all n. Therefore, all the conditions of Theorem 3.4 hold. Hence, the integral equation (3.21) possesses at least a solution.

4. Conclusion

The existence of a solution for the integral equation system (1.1) in the space $L^{\infty}(G)$, where $G\subseteq\mathbb{R}^{\omega}$, was investigated in this study. Arab et al. [5] investigated the problem of finding solutions for an infinite system of two-variable integral equations in the space $(BC(\mathbb{R}_+\times\mathbb{R}_+))^{\omega}$. The benefit of our approach is that any function in the space $L^{\infty}(G)$ need not to be continuous.

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