CARPATHIAN J. MATH.
Volume **37** (2021), No. 3,
Pages 477 - 487

Online version at `https://www.carpathian.cunbm.utcluj.ro/`
Print Edition: ISSN 1584 - 2851; Online Edition: ISSN 1843 - 4401
DOI: https://doi.org/10.37193/CJM.2021.03.10

Strong convergence theorems for variational inequalities and fixed point problems in Banach spaces

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ABSTRACT. In this paper, using a sunny generalized non-expansive retraction which is different from the metric projection and generalized metric projection in Banach spaces, we present a retractive iterative algorithm of Krasnosel'skii-type, whose sequence approximates a common solution of a mono-variational inequality of a finite family of *η-strongly-pseudo-monotone-type maps* and fixed points of a countable family of *generalized non-expansive-type maps*. Furthermore, some new results relevant to the study are also presented. Finally, the theorem proved complements, improves and extends some important related recent results in the literature.

1. INTRODUCTION

In this paper, we study the *mono*-variational inequality problem of Jouymandil and Moradloul [26] and J-fixed points for a countable family of generalized non-expansive-type maps in L_p spaces, $2 \leq p < \infty$.

The first problem involving variational inequality was developed to solve equilibrium problems, precisely, the Signorini problem, posed in the year 1959, by Signorini [46], and was solved in the year 1963, by Fichera [22]. Hartman and Stampacchia studied the existence of solutions of variational inequality; see, for example, [24,35].

Variational inequality has been found to have numerous applications in many areas of mathematics, such as in partial differential equations, optimal control, optimization, mathematical programming and some other nonlinear problems; see, for example, [3,5,6,8,15,16,23,53].

Numerous iterative methods have been proposed for solving variational inequality in the setting of real Hilbert spaces and more general Banach spaces; see, for example [9,32].

Let $\mathcal{Q}^*$ be the dual space of a real normed linear space, $\mathcal{Q}$. A map $A : \mathcal{Q} \to 2^{\mathcal{Q}}$ with domain $D(A)$ in $\mathcal{Q}$ is called *accretive*, if for all $u, v \in D(A)$, there exists $j(u-v) \in J(u-v)$ such that

$$\langle \eta_u - \eta_v, j(u-v) \rangle \geq 0, \ \eta_u \in Au, \ \eta_v \in Av, \tag{1.1}$$

where $J : \mathcal{Q} \to 2^{\mathcal{Q}^*}$ is the normalized duality map.

The map A is called *m-accretive*, if A is accretive and $R(I + sA) = \mathcal{Q}$, for all $s > 0$, where $R(I + sA)$ denotes the range of $(I + sA)$ and I is the identity map on $\mathcal{Q}$. In real Hilbert spaces, accretive maps are called *monotone.*

Accretive maps were introduced independently in the year 1967 by Browder [6] and Kato [31]. Interest in this class of maps stems mainly from their firm connection with the existence theory for nonlinear equations of evolution.

Received: 15.06.2020. In revised form: 20.03.2021. Accepted: 27.03.2021
2010 *Mathematics Subject Classification.* 47H09, 47H10, 47J25, 47J05, 47J20.
Key words and phrases. *J-non-expansive maps, J-pseudo-monotone maps, J-variational inequalities, J-fixed points.*
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A fundamental problem in the study of accretive maps is the following:

$$\text{find} \quad u \in D(A) \quad \text{such that} \quad 0 \in Au. \tag{1.2}$$

To study the inclusion (1.2) when A is accretive, Browder [6] defined a self map T on $\mathcal{Q}$ by $T := (I - A)$ and called such a map *pseudo-contractive.* It is obvious that fixed points of T correspond to solutions of the inclusion (1.2), assuming existence of a solution. For earlier and more recent results on the approximation of fixed points of pseudo-contractive maps, the reader may consult any of the following; [5,13,33,38,41,44].

A map $A : \mathcal{Q} \to 2^{\mathcal{Q}^*}$ is called *monotone,* if for each $u, v \in \mathcal{Q}$, the following inequality holds:

$$\big\langle \eta_u - \eta_v, u - v \big\rangle \geq 0, \;\; \eta_u \in Au, \;\; \eta_v \in Av. \tag{1.3}$$

We recall in [43] that the sub-differential of a proper and convex function f, denoted by ∂f, is a monotone map, and for each $u \in \mathcal{Q}$, $0 \in \partial f(u)$ if and only if u is a minimizer of f. In particular, setting $\partial f \equiv A$, where A is monotone, we have the fact that, $0 \in Au$. This also reduces to $Au = 0$, where A is single-valued.

Approximating zeros of such monotone maps is equivalent to finding a minimizer of f.

We observe that, the fixed point technique introduced by Browder [6], in the year 1967, for approximating zeros of accretive maps in a real Hilbert, for obvious reasons, is not applicable in this case, where A is a monotone map from a real Banach space to its dual space.

Hence, there is the need to introduce and develop new techniques for approximating zeros of such monotone maps.

To approximate zeros of monotone maps, Chidume and Idu [12], in the year 2016, introduced a map, T, from a real normed space to its dual space, by $T := (J - A)$, where $A : \mathcal{Q} \to 2^{\mathcal{Q}^*}$ is monotone. They called such a map J-pseudo-contractive.

Interest in J-pseudo-contractive maps stems mainly from their firm connection with the important class of nonlinear monotone maps (see also, [40]).

Let $T : \mathcal{Q} \to \mathcal{Q}^*$ be a map. An element $u \in \mathcal{Q}$ is called a J-fixed point of T, if

$$Tu = Ju, \tag{1.4}$$

where J is single-valued in this case. Examples of J-fixed points include:

Example 1.1. [12] It is known (see e.g., Alber [3], pp.36) that in l_p spaces, $1 < p < \infty$,

$$Ju = ||u||_{l_p}^{2-p} v \in l_q, \; v = \{|u_1|^{p-2}u_1, \; |u_2|^{p-2}u_2, \; |u_3|^{p-2}u_3, \cdots\}, \; u = \{u_1, u_2, \dots\} \in l_p.$$

For $1 < q < p$, and any $\lambda \in \mathbb{R}$, set $\gamma_p := (1 + \frac{1}{2^p})^{\frac{2-p}{p}}$ and define $T : l_p \to l_q$ by

$$T(u_1, u_2, u_3, \dots) = (\gamma_p u_1, \frac{\gamma_p}{2^{p-1}} u_2, 0, 0, 0, \dots).$$

Let $u_\lambda = (\lambda, \frac{\lambda}{2}, 0, 0, \dots)$. Then, $Tu_\lambda = \gamma_p(\lambda, \frac{\lambda}{2^p}, 0, 0, \dots)$ and $Ju_\lambda = \gamma_p(\lambda, \frac{\lambda}{2^p}, 0, 0, \dots)$. Hence, $u_\lambda \in F_J(T)$.

Example 1.2. Let A, $T : L_p^{\mathbb{R}}([0,1]) \to L_q^{\mathbb{R}}([0,1])$, $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, be defined by

$$Au(t) := (1 + t)Ju(t), \; \forall \; t \in [0,1] \quad \text{and} \quad T := J - A.$$

Clearly, A is a monotone map. Hence, $u(t) = 0$ is the fixed point of T, for all $t \in [0,1]$.

We observe that, in a real Hilbert space and a strictly convex space, the notion of J-fixed point coincides with the classical definition of fixed point. However, if the space is not strictly convex, J may fail to be one-to-one. Hence, the inverse of J will not exist.

Therefore, approximating zeros of such monotone maps, is equivalent to approximating J-fixed points of J-pseudo-contractive maps, assuming existence of such zeros, which is also equivalent to finding minimizers of some convex functions.

In the year 2016, Chidume and Idu [12] studied this notion of fixed point and called it *J-fixed point*. This notion of fixed point has turned out to be very useful and applicable. For example, Chidume and Idu [12] studied the concept of J-fixed points and proved a strong convergence of a sequence, that approximates a J-fixed point of a J-pseudo-contractive map. As an application of their theorem, they proved a strong convergence of a sequence, that approximates a zero of an m-accretive map. They also applied their theorem on J-fixed points, to approximate solutions of Hammerstein integral equations. Nnakwe [36] in 2020, also applied this concept of J-fixed point and proved a strong converge theorem for approximating a common solution of variational inequality and two convex minimization problems. For more recent works on J-fixed points; see, for example, [14,37,48,55].

Motivated by the ongoing research in this direction, it is our purpose in this paper to continue with the study of J-fixed points and some of it's applications. Here, we study a *retractive iterative algorithm of Krasnosel'skii-type* and prove a *strong convergence* of the sequence generated by the algorithm for approximating a common element in the set of solutions of *mono-variational inequality* of a finite family of *η-strongly J-pseudo-monotone maps* and fixed points of a countable family of *generalized J-non-expansive maps* in L_p spaces, $2 \leq p < \infty$. Furthermore, we give some new definitions and lemmas relevant to the study. Finally, examples of η-strongly J-pseudo-monotone maps which are neither strongly J-monotone nor J-monotone and generalized J-non-expansive maps are constructed.

2. PRELIMINARIES

Let $\mathcal{Q}^*$ be the dual space of a smooth real Banach space $\mathcal{Q}$. Consider a map $\phi : \mathcal{Q} \times \mathcal{Q} \to \mathbb{R}$ defined by

$$\phi(u,v) = ||u||^2 - 2\langle u, Jv\rangle + ||v||^2, \text{ for all } u, v \in \mathcal{Q}.$$

This map which was introduced by Alber [3] will play a central role in the sequel. The following is a property of ϕ :

$$(P_1) \qquad \phi(u,w) = \phi(u,v) + \phi(v,w) + 2\langle u - v, Jv - Jw\rangle, \text{ for all, } u, v, w \in \mathcal{Q}.$$

The following lemmas and definitions will be needed in the sequel.

Definition 2.1. Let $V : \mathcal{Q} \times \mathcal{Q}^* \to \mathbb{R}$ be defined by $V(u, u^*) = ||u||^2 - 2\langle u, u^*\rangle + ||u^*||$, for all $u \in \mathcal{Q}$, $u^* \in \mathcal{Q}^*$. It is easy to see that $V(u, u^*) = \phi(u, J^{-1}u^*), \ \forall\, u \in \mathcal{Q}, \ u^* \in \mathcal{Q}^*$.

Definition 2.2. [26] Let M be a nonempty, closed and convex subset of $\mathcal{Q}$. The *mono-variational inequality* is to find an element $u \in M$, such that $\langle \psi(u), Jz - Ju\rangle \geq 0, \ \forall\, z \in M$, where $\psi : M \to \mathcal{Q}$. The set of solutions of *mono*-variational inequality will be denoted by $V_J(\psi, M)$.

Definition 2.3. Let $\mathcal{Q}$ and Y be real normed spaces, and $\psi : \mathcal{Q} \to Y$ be a map.

(a) ψ is called *L-Lipschitz,* if there exists $L > 0$ such that $||\psi(u) - \psi(v)|| \leq L||u - v||, \ \forall\, u, v \in \mathcal{Q}$.

It is known that in L_p spaces, $p \geq 2$, J_p is Lipschitz.

Definition 2.4. [14] A map $T : M \to \mathcal{Q}^*$ is called *generalized J-non-expansive*, if $F_J(T) \neq \emptyset$ and $\phi((J^{-1}oT)x, p) \leq \phi(x, p)$, for all $x \in M$, for all $p \in F_J(T)$.

NST-Condition. Let $\{S_n\}$ and Γ be two families of generalized J-non-expansive maps from M into $\mathcal{Q}^*$ such that $\cap_{n=1}^{\infty} F(S_n) = F(\Gamma) \neq \emptyset$, where $F(S_n)$ is the set of fixed points of S_n and $F(\Gamma)$ is the set of fixed points of Γ. A sequence $\{S_n\}$ from M to $\mathcal{Q}^*$ is said to satisfy the *NST-condition with* Γ, if for each bounded sequence $\{x_n\} \subset \mathcal{Q}$, $\lim||Jx_n - S_n x_n|| = 0 \implies \lim||Jx_n - Sx_n|| = 0, \forall\, S \in \Gamma$.

Lemma 2.1. *[34] Let M be a nonempty closed and convex subset of a smooth, strictly convex and reflexive Banach space $\mathcal{Q}$. Then, the following are equivalent.*
(i) M is a sunny generalized non-expansive retract of $\mathcal{Q}$
(ii) M is a generalized non-expansive retract of $\mathcal{Q}$
(iii) JM is closed and convex.

Lemma 2.2. *[25] Let M be a nonempty closed and convex subset of a smooth and strictly convex Banach space $\mathcal{Q}$ such that there exists a sunny generalized non-expansive retraction R from $\mathcal{Q}$ onto M. Then, the following hold:*
(i) $z = Rx$ iff $\langle y - z, Jz - Jx\rangle \geq 0$, for all $y \in M$ and,
(ii) $\phi(x, Rx) + \phi(Rx, z) \leq \phi(x, z)$, for all $z \in M$.

Lemma 2.3. *[29] Let $\mathcal{Q}$ be a uniformly convex and uniformly smooth real Banach space and $\{u_n\}$, $\{v_n\}$ be sequences in $\mathcal{Q}$ such that either $\{u_n\}$ or $\{v_n\}$ is bounded. If $\lim\phi(u_n, v_n) = 0$, then, $\lim||u_n - v_n|| = 0$.*

Lemma 2.4. *[11] Let $\mathcal{Q} = L_p$, $p \geq 2$. Then, this inequality holds:*

$$||u - v||^2 \geq \phi(u, v) - p||u||^2, \text{ for all } u, v \in \mathcal{Q}.$$

Lemma 2.5. *[10] Let $\mathcal{Q} = L_p$, $1 < p < 2$. Then, this inequality holds:*

$$||u + v||^2 \geq 2||u||^2 + 2\langle v, Ju\rangle + c_p||v||^2, \text{ for all } u, v \in \mathcal{Q}, \text{ and some constant } c_p > 0.$$

Lemma 2.6. *[14] Let $\mathcal{Q}^*$ be the dual space of a uniformly convex and uniformly smooth real Banach space $\mathcal{Q}$. Let M be a closed subset of $\mathcal{Q}$ such that JM is closed and convex. Let T be a generalized J-non-expansive map from M to $\mathcal{Q}^*$ with $F_J(T) \neq \emptyset$. Then, $F_J(T)$ is closed and $JF_J(T)$ is closed and convex.*

3. MAIN RESULT

Let M be a nonempty, closed and convex subset of $\mathcal{Q} = L_p$ spaces, $1 < p < \infty$, with dual space $\mathcal{Q}^*$. In this section, we present some new definitions, prove some new lemmas which are used to prove the main result of the section.

Definition 3.5. Let ψ be a map from M to $\mathcal{Q}$.

(a) ψ is called *η-strongly J-monotone*, if there exists $\eta > 0$ such that $\langle\psi(u) - \psi(v), Ju - Jv\rangle \geq \eta||u - v||^2$, $\forall\, u, v \in M$,

(b) ψ is called *J-monotone*, if $\langle\psi(u) - \psi(v), Ju - Jv\rangle \geq 0$, $\forall\, u, v \in M$,

(c) ψ is called *η-strongly J-pseudo-monotone*, if there exists $\eta > 0$, such that $\langle\psi(u), Jv - Ju\rangle \geq 0 \implies \langle\psi(v), Jv - Ju\rangle \geq \eta||u - v||^2$, $\forall\, u, v \in M$,

(d) ψ is *J-pseudo-monotone*, if $\langle\psi(u), Jv - Ju\rangle \geq 0 \implies \langle\psi(v), Jv - Ju\rangle \geq 0$, $\forall\, u, v \in M$.

In a real Hilbert space H, J is the identity map on H and (a)-(d) of Definition 3.5 coincides with the usual definition of (a)-(d) in the literature.

We observe that $(a) \implies (b) \implies (d)$ and $(a) \implies (c) \implies (d)$. The converse is not true.

Lemma 3.7. *Let $\mathcal{Q}$ be a uniformly convex and uniformly smooth real Banach space with dual space $\mathcal{Q}^*$. Let M be a closed subset of $\mathcal{Q}$ such that JM is closed and convex. Let ψ be a J-pseudo-monotone and Lipschitz map from M to $\mathcal{Q}$ with $V_J(\psi, M) \neq \emptyset$. Then, $V_J(\psi, M)$ is closed and $JV_J(\psi, M)$ is closed and convex.*

Proof. First, we prove that $V_J(\psi, M)$ and $JV_J(\psi, M)$ are closed. Obviously, $V_J(\psi, M)$ is closed. Let $\{v_n^*\} \subset JV_J(\psi, M)$ such that $v_n^* \to v^*$, for some $v^* \in \mathcal{Q}^*$. Since JM is closed, we have that $v^* \in JM$. Hence, there exist $v \in M$ and $\{v_n\} \subset V_J(\psi, M)$ such that $v^* = Jv$ and $v_n^* = Jv_n$, $\forall\, n \in \mathbb{N}$. Utilizing the definition of ψ and the fact that J^{-1} is uniformly continuous on bounded subset of $\mathcal{Q}^*$, we have

$$0 \leq \lim_{n\to\infty} \langle \psi J^{-1} v_n^*, Jy - v_n^* \rangle = \langle \psi J^{-1} v^*, Jy - v^* \rangle.$$

Thus, we have that $v^* = Jv \in JV_J(\psi, M)$. Hence, $JV_J(A, M)$ is closed.

Let u^*, $v^* \in JV_J(\psi, M)$. Then, $u^* = Ju$, $v^* = Jv \in JM$, for some $u, v \in M$. For $t, k \in (0,1)$, let $z^* = ku^* + (1-k)v^* \in JM$, and for any $y \in M$, we set $x_t^* = tJy + (1-t)z^*$. We compute as follows:

$$\begin{aligned} 0 = \langle \psi J^{-1} x_t^*, x_t^* - x_t^* \rangle &= \langle \psi J^{-1} x_t^*, Jy - x_t^* \rangle - \langle \psi J^{-1} x_t^*, Jy - x_t^* \rangle \\ &\leq (1-t)\langle \psi J^{-1} x_t^*, Jy - z^* \rangle \leq \langle \psi J^{-1}(z^* + t(Jy - z^*)), Jy - z^* \rangle. \end{aligned} \tag{3.5}$$

Taking $\limsup_{t\downarrow 0}$ on (3.5); $\langle \psi J^{-1} z^*, Jy - z^* \rangle \geq 0$, $\forall\, y \in M$. Thus, $z^* \in JV_J(\psi, M)$. ☐

Remark 3.1. From Lemmas 2.6 and 3.7, we have that $JF_J(T)$ and $JV_J(\psi, M)$ are closed and convex. Since J is one-to-one, we have that $J(F_J(T) \cap V_J(\psi, M)) = JF_J(T) \cap JV_J(\psi, M)$. By Lemma 2.1, we have $F_J(T) \cap V_J(\psi, M)$ is a sunny generalized J-nonexpansive retract of $\mathcal{Q}$.

Lemma 3.8. *Let $\mathcal{Q} = L_p$ spaces, $1 < p < 2$ or $p \geq 2$, and $\mathcal{Q}^*$ be the dual space of $\mathcal{Q}$. Then, for each $u, v \in \mathcal{Q}$, $u^* \in \mathcal{Q}^*$, the following inequalities hold:*

$$V_p(u, u^*) + 2\langle v, Ju - u^* \rangle \leq V_p(u + v, u^*), \quad 1 < p < 2, \tag{3.6}$$

$$V_p(u, u^*) + 2\langle v, Ju - u^* \rangle \leq V_p(u + v, u^*) + (p-1)||v||^2, \quad p \geq 2. \tag{3.7}$$

Proof. For $1 < p < 2$; from Definition 2.1, property of ϕ, (P_1) and Lemma 2.5, we have

$$\begin{aligned} V_p(u, u^*) + 2\langle v, Ju - u^* \rangle &= \phi(u, u+v) + \phi(u+v, J^{-1}u^*) - 2\langle v, J(u+v) - u^* \rangle \\ &\quad + 2\langle v, Ju - u^* \rangle \\ &= V_p(u+v, u^*) + \phi(u, u+v) + 2\langle v, Ju - J(u+v) \rangle \\ &= V_p(u+v, u^*) + ||u||^2 + 2\langle v, Ju \rangle - ||u+v||^2 \qquad (3.8) \\ &\leq V_p(u+v, u^*) - c_p||v||^2, \quad c_p > 0. \end{aligned}$$

Also, for $p \geq 2$, from inequality (3.8), Lemma 2.4 and interchanging $u, v \in \mathcal{Q}$, we have

$$\begin{aligned} V_p(u, u^*) + 2\langle v, Ju - u^* \rangle &= V_p(u+v, u^*) - ||u+v||^2 + ||u||^2 + 2\langle v, Ju \rangle \\ &\leq V_p(u+v, u^*) - \phi(v, -u) + p||v||^2 + ||u||^2 + 2\langle v, Ju \rangle \\ &= V_p(u+v, u^*) + (p-1)||v||^2. \end{aligned}$$

☐

We shall make the following assumptions.

C_1. The map ψ_i is η_i-strongly-J-pseudo-monotone and L-Lipschitz on M, with $L > 0$, $\eta_i > 0,\ i = 1, 2, \cdots, N$.

C_2. $||\psi_i(v)|| \leq ||\psi_i(u) - \psi_i(v)||$, for all $v \in M$, $u \in V_J(\psi_i, M),\ i = 1, 2, \cdots, N$.

Convergence theorem in L_p spaces, $p \geq 2$.

Theorem 3.1. *Let M be a nonempty closed convex subset of $\mathcal{Q} = L_p$, $p \geq 2$, such that JM is closed and convex. Let $\psi_i : M \to \mathcal{Q}$, $i = 1, 2, \cdots, N$ be a finite family of maps satisfying conditions C_1 and C_2. Let $S_n : M \to \mathcal{Q}^*$, $n = 1, 2, \ldots$ be a countable family of generalized J-non-expansive maps and Γ be a family of closed and generalized J-non-expansive maps from M to $\mathcal{Q}^*$ such that $F =: \big[\cap_{n=1}^{\infty} F_J(S_n)\big] \cap \big[\cap_{i=1}^{N} V_J(\psi_i, M)\big] = F_J(\Gamma) \cap \big[\cap_{i=1}^{N} V_J(\psi_i, M)\big] \neq \emptyset$. Let $\{v_n\}$ be a sequence generated iteratively by*

$$\begin{cases} v_1 \in M, \\ y_n = R_M(v_n - \tau\psi_n(v_n)), \\ v_{n+1} = \beta v_n + (1-\beta)J^{-1}oS_n(y_n),\ \forall\ n \geq 1, \end{cases} \tag{3.9}$$

where $\psi_n =: \psi_n(mod\, N)$, $\beta \in (0, 1)$ and $\tau \in \big(0, \frac{2\eta}{(2+p)L^2}\big)$, where $\eta := \min\{\eta_i,\ i = 1, 2, \cdots, N\}$. Assume $\{S_n\}$ satisfies NST-condition with Γ, then, $\{v_n\}$ converges strongly to $R_{F_J(\Gamma)\cap V_J(\psi,M)}v_1$; $R_{F_J(\Gamma)\cap V_J(\psi,M)}$ is the sunny generalized J-non-expansive retraction of $\mathcal{Q}$ onto $F_J(\Gamma) \cap V_J(\psi, M)$.

Proof. Let $u \in F$. Set $z_n = v_n - \tau\psi_n(v_n)$. By Lemma 3.8, we have

$$\begin{aligned} \phi(y_n, u) &\leq V(v_n - \tau\psi_n(v_n), Ju) \\ &\leq V(v_n, Ju) - 2\tau\langle\psi_n(v_n), J(v_n - \tau\psi(v_n)) - Ju\rangle + \tau^2(p-1)||\psi_n(v_n)||^2 \\ &\leq \phi(v_n, u) - 2\tau\langle\psi_n(v_n), J(v_n - \tau\psi_n(v_n)) - Jv_n\rangle - 2\tau\langle\psi_n(v_n), Jv_n - Ju\rangle \\ &\quad + \tau^2 p||\psi_n(v_n)||^2. \end{aligned} \tag{3.10}$$

since $u \in V_J(\psi_n, M)$, then, $\langle\psi(u), Jx - Ju\rangle \geq 0,\ \forall\ x \in M$. By η-pseudo-J-monotonicity of ψ_n on M, we have that $\langle\psi_n(x), Jx - Ju\rangle \geq \eta||x - u||^2$. In particular, for $x = v_n$, we have that $\langle\psi_n(v_n), Jv_n - Ju\rangle \geq \eta||v_n - u||^2$. From inequality (3.10), conditions C_2, C_1 and the fact that J is Lipschitz on $\mathcal{Q}$, we have

$$\begin{aligned} \phi(y_n, u) &\leq \phi(v_n, u) - 2\tau\langle\psi_n(v_n), J(v_n - \tau\psi_n(v_n)) - Jv_n\rangle - 2\tau\eta||v_n - u||^2 \\ &\quad + \tau^2 p||\psi_n(v_n) - \psi_n(u)||^2 \\ &\leq \phi(v_n, u) + 2\tau^2L^2||v_n - u||^2 - 2\tau\eta||v_n - u||^2 + \tau^2L^2p||v_n - u||^2 \\ &= \phi(v_n, u) - \tau\big(2\eta - \tau L^2(2+p)\big)||v_n - u||^2. \end{aligned} \tag{3.11}$$

$$\begin{aligned} \phi(v_{n+1}, u) &= \phi(\beta v_n + (1-\beta)J^{-1}oS_n(y_n), u) \\ &\leq \beta\phi(v_n, u) + (1-\beta)\phi(J^{-1}oS_ny_n, u) - \beta(1-\beta)g(||v_n - J^{-1}oS_n(y_n)||) \\ &\leq \beta\phi(v_n, u) + (1-\beta)\phi(y_n, u) - \beta(1-\beta)g(||v_n - J^{-1}oS_n(y_n)||) \quad (3.12) \\ &\leq \beta\phi(v_n, u) + (1-\beta)\big[\phi(v_n, u) - \tau\big(2\eta - \tau L^2(2+p)\big)||v_n - u||^2\big] \\ &= \phi(v_n, u) - \tau(1-\beta)\big(2\eta - \tau L^2(2+p)\big)||v_n - u||^2 \leq \phi(v_n, u). \quad (3.13) \end{aligned}$$

Hence, $\lim\phi(v_n, u)$ exists. Furthermore, $\{v_n\}$, $\{y_n\}$ are bounded.

Set $\Theta = \tau(1-\beta)\big(2\eta - \tau L^2(2+p)\big) > 0$, from inequality (3.13), we have

$$||v_n - u|| \leq \sqrt{\Theta^{-1}\big(\phi(v_n,u) - \phi(v_{n+1},u)\big)} \to 0 \text{ as } n \to \infty. \tag{3.14}$$

Furthermore, using condition C_2 and Lipschitz property of J on $\mathcal{Q}$, we have

$$\begin{aligned} \phi(y_n, v_n) &= \phi(v_n - \tau\psi_n(v_n), v_n) = V(v_n - \tau\psi_n(v_n), Jv_n) \\ &\leq V(v_n, Jv_n) - 2\tau\langle\psi_n(v_n), J(v_n - \tau\psi_n(v_n) - Jv_n)\rangle + \tau^2 p||\psi_n(v_n)||^2 \\ &\leq \phi(v_n, v_n) + (2+p)\tau^2 L^2 ||v_n - u||^2. \end{aligned} \tag{3.15}$$

From inequality (3.14), $\lim \phi(y_n, v_n) = 0$. By Lemma 2.3, $\lim ||y_n - v_n|| = 0$.

From inequalities (3.12) and (3.13), we have

$$g(||v_n - J^{-1}oS_n(y_n)||) \leq (\beta(1-\beta))^{-1}\big[\phi(v_n,u) - \phi(v_{n+1},u)\big]. \tag{3.16}$$

Thus, $\lim g(||v_n - J^{-1}oS_n(y_n)||) = 0$. By the property of g, $\lim ||v_n - J^{-1}oS_n(y_n)|| = 0$.

Now,

$$||y_n - J^{-1}oS_n(y_n)|| \leq ||y_n - v_n|| + ||v_n - J^{-1}oS_n(y_n)|| \to 0 \text{ (as } n \to \infty).$$

By Lipschitz property of J on $\mathcal{Q}$, we get that $||Jy_n - S_n(y_n)|| \to 0$ as $n \to \infty$. Since $\{S_n\}$ satisfies the NST-condition with Γ, then, $\lim||Jy_n - Sy_n|| = 0$, $\forall\ S \in \Gamma$. But S is closed, hence, $u \in F_J(S)$. □

4. EXAMPLES

We demonstrate the applicability of our result obtained in Theorem 3.1 to this example.

Example 4.3. Let $\mathcal{Q} = L_p([0,1])$, $p \geq 2$. Let $\alpha, \beta \in \mathbb{R}$ such that $0 < \frac{\beta}{2} < \alpha \leq \beta \leq 1$. Define $M_\alpha := \{v \in L_p([0,1]) : ||v||_{L_p} \leq \alpha\} \subseteq \overline{B_{L_p}(0,1)}$ and $\psi_\beta : M_\alpha \to \mathcal{Q}$ be defined by

$$(\psi_\beta(v))(t) := (\beta - ||v||)v(t).$$

Clearly, $VI(\psi_\beta, M_\alpha) = \{0\}$ and $u \in VI(\psi_\beta, M_\alpha)$ if and only if $u \in \psi_\beta^{-1}(0)$ satisfying conditions C_2. Also, let $u, v \in M_\alpha$, we have

$$\begin{aligned} ||\psi_\beta(u) - \psi_\beta(v)|| &= ||(\beta - ||u||)v_1 - (\beta - ||v||)v|| \\ &\leq \beta||u-v|| + ||u||||u-v|| + ||v||||u-v|| \leq (\beta + 2\alpha)||u-v||. \end{aligned}$$

Hence, ψ_β is L-Lipschitz with $L = (\beta + 2\alpha)$.

Furthermore, let $u, v \in M_\alpha$ such that $\langle\psi_\beta(u), Jv - Ju\rangle \geq 0$. Since $||u||_{L_p} \leq \alpha \leq \beta$, this implies that $\langle u, Jv - Ju\rangle \geq 0$. Applying a result of Xu [54], we have

$$\begin{aligned} \langle\psi_\beta(v), J_p v - J_p u\rangle &= (\beta - ||v||)\langle v, J_p v - J_p u\rangle \\ &\geq (\beta - ||v||)(\langle v, J_p v - J_p u\rangle - \langle u, J_p v - J_p u\rangle) \\ &\geq (\beta - \alpha)\langle v - u, J_p v - J_p u\rangle \geq \frac{p^{-1}c_p}{2}(\beta - \alpha)||v-u||^p,\ c_p > 0. \end{aligned}$$

In particular, for $p = 2$, ψ_β is γ-strongly-J-pseudo-monotone with $\gamma = \frac{p^{-1}c_p}{2}(\beta - \alpha) > 0$ satisfying condition C_1.

Hence, the sequence generated by Algorithm (3.9), converges strongly to $R_{F_J(\Gamma)\cap V_J(\psi,M)}v_1$.

Let $S : M_\alpha \to \mathcal{Q}^*$ be defined by $Su = Ju,\ \forall\, u \in M_\alpha$. Let $S_n : M_\alpha \to \mathcal{Q}^*$ be defined by

$$S_n u = J(\alpha_n u + (1-\alpha_n)J^{-1}oSu),\ \forall\, n \geq 1,\ u \in M_\alpha \text{ and } \alpha_n \in (0,1).$$

Clearly, $F_J(S_n) = F_J(S),\ \forall\, n \in \mathbb{N}$. Hence, $\underset{n\geq 1}{\cap} F_J(S_n) = F_J(S)$. For any $u \in M,\ v \in F_J(S_n)$, we have

$$\begin{aligned}\phi(J^{-1}oS_n u, v) &= \phi(\alpha_n u + (1-\alpha_n)J^{-1}oSu, v)\\ &\leq \alpha_n\phi(u,v) + (1-\alpha_n)\phi(J^{-1}oSu, v)\\ &= \alpha_n\phi(u,v) + (1-\alpha_n)\phi(u,v) = \phi(u,v).\end{aligned}$$

Hence, $\{S_n\}$ is a generalized J-non-expansive map, where the map ϕ is the Alber's functional.

Let $\{u_n\}$ be a bounded sequence in M_α such that $\lim ||Ju_n - S_n u_n|| = 0$. Since $\{u_n\}$ is bounded, then, $\{J^{-1}oSu_n\}$ is bounded. Using the definition of S_n, we have

$$||u_n - J^{-1}oSu_n|| = \frac{1}{1-\beta_n}||u_n - J^{-1}oS_n u_n|| \leq 2||u_n - J^{-1}oS_n u_n||.$$

Since $\lim ||Ju_n - S_n u_n|| = 0$ and the fact that J^{-1} and J are uniformly continuous on bounded subsets of $\mathcal{Q}^*$ and $\mathcal{Q}$, respectively, we have that $\lim ||Ju_n - Su_n|| = 0$, which implies that $\{S_n\}$ satisfies NST-condition with S.

Hence, the sequence generated by Algorithm (3.9), converges strongly to $R_{F_J(\Gamma)\cap V_J(\psi,M)}v_1$.

Example 4.4. Let $M = [0,1]$ and $\psi : M \subset \mathbb{R} \to \mathbb{R}$ be given by $\psi(v) = (2-v),\ \forall\, v \in M$. Clearly, $VI(\psi, M) = \{0\}$ and ψ is 1-Lipschitz. Furthermore, suppose that $u, v \in M$ such that $\langle \psi(u), v-u\rangle \geq 0$. Since $u \in [0,1]$, this implies that $u < v$. Thus,

$$\langle \psi(v), v-u\rangle = \langle 2-v, v-u\rangle \geq |v-u| \geq |v-u|^2,\ \forall\, v \in M,$$

which implies that ψ is 1-strongly-J-pseudo-monotone. Moreover, ψ is neither strongly-J- monotone nor J-monotone. To see this, choose $u = \frac{1}{2}$ and $v = 1$.

5. DISCUSSION

Theorem 3.1 which approximate a common solution of a finite family of *mono-variational inequality problems* and a common fixed points of a *countable family* of *generalized-J-nonexpansive maps* complement and extend important recent results in the literature, in particular, the result of Khanh, [32] from a *Hilbert space* to *L_p spaces*, $2 \leq p < \infty$, respectively. Furthermore, the theorems proved are analogue of the result of Khanh, [32] in that if $\mathcal{D} = H$, a real Hilbert space, the normalized duality map is the identity on $\mathcal{D}$. Hence, the both theorems coincide. Finally, the class of *η-strongly J-pseudo-monotone maps* considered, contains the class J-monotone maps studied in [26].

Acknowledgment. The authors appreciate the support of their institutions. They are also grateful to the anonymous referees and the editor for their valuable comments and suggestions that helped to improve the quality of this paper.

Note added by Editors. To illustrate how interesting is the topic of the paper, we also included at the end of authors' list of References a selective list of some recent related published papers, see [7]- [49].

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