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Dedicated to the memory of Academician Mitrofan M. Choban (1942-2021)

Some fixed point theorems on equivalent metric spaces

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ABSTRACT. This paper aims to analyze the existence of fixed points for mappings defined on complete metric spaces satisfying almost contractive conditions and a general contractive inequality of integral type. The existence of a fixed point is ensured by hypotheses formulated in terms of equivalent metric spaces.

1. INTRODUCTION

It can be said that 1912 (the year Brouwer's Theorem was published) and 1922 (the year Banach's Contraction Mapping Principle was published) marked the beginning of the interesting history of *Fixed Point Theory*. Numerous fixed point theorems can be found in literature, for historical aspects see [19, 20, 24], etc. There are some relevant metric conditions which appear in metric fixed point theory, for details see [1, 5, 15, 27, 28, 29]. More precisely, in a metric space (X, d) we can consider the mapping $T : X \to X$ which is

I.) a graphic l-contraction iff

$$d\left(T^2x, Tx\right) \le l \cdot d\left(x, Tx\right) \text{ for all } x \in X, \tag{1.1}$$

where $l \in [0, 1)$;

II.) a (l, L)–Berinde operator iff

$$d\left(Tx, Ty\right) \le l \cdot d\left(x, y\right) + L \cdot d\left(y, Tx\right) \text{ for all } x, y \in X, \tag{1.2}$$

where $l \in [0, 1)$ and $L \in [0, \infty)$;

III.) a Caristi operator iff there exists o function $\varphi : X \to [0, \infty)$ such that

$$d\left(x, Tx\right) \le \varphi\left(x\right) - \varphi\left(Tx\right) \text{ for all } x \in X. \tag{1.3}$$

Remark 1.1. If T is (l, L)–Berinde operator with $L = 0$, then we say that T is Banach contraction.

Remark 1.2 ([10, 9])**.** If T is (l, L)–Berinde operator with $l = \frac{b}{1-b}$ and $L = \frac{2b}{1-b}$, where $b \in \left(0, \frac{1}{2}\right)$, then we say that T is Kannan contraction.

Example 1.1. Let $1_X : [0, 1] \to [0, 1]$ be given by $1_X x = x$ for all $x \in [0, 1]$, i.e., 1_X the identity map on X. It is ovious that, 1_X does not satisfy the Banach contraction condition. In Example 2.2.3. from [6] it is prove that the identity map is a $(l, 1 - l)$–Berinde operator, where $l \in (0, 1)$.

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Example 1.2. Let $T_2 : [0, \infty) \to [0, \infty)$ be given by $T_2(t) = 2 \cdot t$ for all $t \in [0, \infty)$. If we assume that T_2 is a (l, L)–Berinde operator, then there exist $l \in (0, 1)$ and $L \in (0, \infty)$ such that

$$2|x - y| \le l \cdot |x - y| + L \cdot |y - 2x| \text{ for all } x, y \in [0, \infty). \tag{1.4}$$

Hence, for all $x, y \in [0, \infty)$ we have $(2 - l) \cdot |x - y| \le L \cdot |y - 2x|$ and this is equivalent with $(2 - l)^2 \cdot (x - y)^2 \le L^2 \cdot (y - 2x)^2$. We put $L^2 - (2 - l)^2 = K^2$, with $K < L$, and obtain the quadratic inequality

$$0 \le \left(3L^2 + K^2\right) x^2 - 2\left(L^2 + K^2\right) xy + K^2 y^2 \text{ for all } x, y \in [0, \infty) \tag{1.5}$$

The discriminant of the above quadratic equation is $\Delta = 4y^2L^2 (L - K)(L + K) > 0$ and this contradicts (1.5). So, T_2 is not a (l, L)–Berinde operator.

In [3, 4, 5, 9, 10] we can find the proof that any Banach contraction [2], any Kannan contraction any Chatterjea contraction and any Zamfirescu contraction are (l, L)–Berinde operator. We have the following fixed point theorem.

Theorem 1.1 ([5]). *Let (X, d) be a complete metric space and $T : X \to X$ be a weak contraction, i.e., a map satisfying* (1.2) *with $0 < \delta < 1$ and some $L \ge 0$. Then*

(i.) *T has at least one fixed point in X (i.e., Fix $(T) = \{x \in X : Tx = x\} \neq \emptyset$);*

(ii.) *The Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by*

$$x_{n+1} = Tx_n, \quad n = 0, 1, 2, \ldots \tag{1.6}$$

converges to some $x^* \in$ Fix (T), for any $x_0 \in X$

(iii.) *The following estimates*

$$d(x_n, x^*) \le \frac{l^n}{1 - l} d(x_0, x_1), \quad n = 0, 1, 2, \ldots \tag{1.7}$$

$$d(x_n, x^*) \le \frac{l}{1 - l} d(x_{n-1}, x_n), \quad n = 1, 2, \ldots \tag{1.8}$$

hold, where l is the constant appearing in (1.2).

In fact, the metric contractive conditions known in literature involve some of the following six displacements

$$d(Tx, Ty), d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx) \tag{1.9}$$

such that the Picard iteration $\{x_n\}_{n=0}^{\infty}$ given by (1.6) is a Cauchy sequence. So, in the setting of a complete metric space, this implies that $\{x_n\}_{n=0}^{\infty}$ is convergent. For example, the Banach contraction condition implies that for any $x, y \in X$, the element Ty must be an element of the set $\{z \in X : d(z, Tx) < l \cdot d(x, y)\}$ and $Tx \in \{z \in X : d(z, Ty) < l \cdot d(x, y)\}$, where $l \in (0, 1)$.

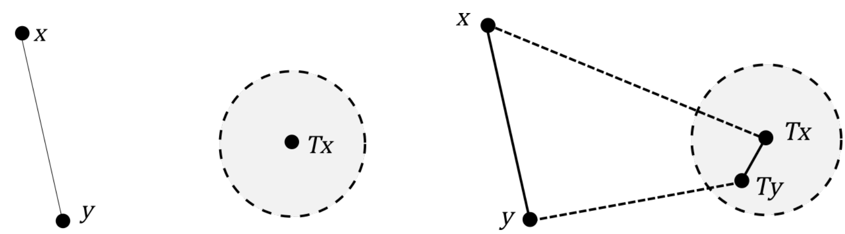

FIGURE 1. Geometric interpretatin of Banach contraction condition

In this way, the lengths of sides and diagonals of quadrilateral $x - Tx - Ty - y$ can be used to obtain some inequalities, which applied iteratively, degenerate the quadrilateral $x - Tx - Ty - y$ into a single point. This point is the fixed point of T.

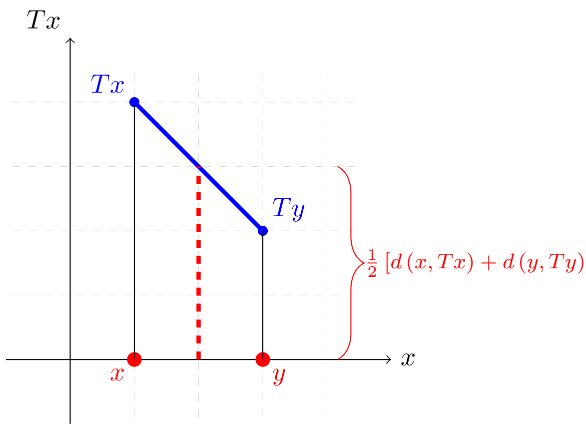

FIGURE 2. A geometric interpretation of Kannan contractive condition

In the case of Kannan contraction condition, the map $T : X \to X$ satisfies the inequality

$$d(Tx, Ty) \le a\,[\,d(x, Tx) + d(y, Ty)\,], \quad \forall x, y \in X, \tag{1.10}$$

where $a \in \left(0, \frac{1}{2}\right)$. So, if we consider the case in which the quadrilateral $x - Tx - Ty - y$ is a trapezoid, then (1.10) implies that the length of the side $Tx - Ty$ must be less than the length of the midsegment of the considered trapezoid.

The following fixed point theorem is due to Kannan [17].

Theorem 1.2 ([17]). *Let (X, d) be a complete metric space. If the mapping $T : X \to X$ satisfies* (1.10), *with $0 < a < \frac{1}{2}$, then T has a unique fixed point.*

A similar results is due to Berinde [5].

Theorem 1.3 ([5]). *Let (X, d) be a complete metric space and $T : X \to X$ be a mapping such that there exist $l \in (0, 1)$ and some $L \in [0, \infty)$ such that*

$$d(Tx, Ty) \le l \cdot d(x, y) + L \cdot d(x, Tx) \text{ for all } x, y \in X, \tag{1.11}$$

holds. Then T has a unique fixed point.

Most of metrical fixed point theorems deal with a large variety of mappings satisfying a certain metric condition that involves various functions of the displacements given by (1.9) and satisfy some properties such that the existence of fixed point is ensure. In this paper, we study the fixed points set via a transformation applied to the elements of X, not to the values given by (1.9). In order to do so, we first present in the following sections a few aspects and result related to (l, L)–Berinde operator (or almost contractions), and then, in Section 3, we shall give a new fixed point theorem. Similar approaches to the previous fixed point results can be found in the works [21, 22]. The case of contractive condition of integral type are considered, too.

2. Preliminaries

We shall need the following concept.

Definition 2.1. Let (X, d) be a metric space and the map $\flat : X \to X$. The mapping $T : X \to X$ is a $\left(\flat, l, L\right)$–Berinde operator if there is a constant $l \in (0,1)$ and some $L \in [0, \infty)$ such that

$$d\left(\flat Tx, \flat Ty\right) \leq l \cdot d\left(\flat x, \flat y\right) + L \cdot d\left(\flat y, \flat Tx\right), \text{ for all } x, y \in X. \tag{2.12}$$

Due to the symmetry of the metric d, the condition (2.12) can be satisfied only if the dual one

$$d\left(\flat Tx, \flat Ty\right) \leq l \cdot d\left(\flat x, \flat y\right) + L \cdot d\left(\flat x, \flat Ty\right), \text{ for all } x, y \in X \tag{2.13}$$

holds.

Remark 2.3. Let Let (X, d) be a metric space, $a \in X$ be an element of X and the map $\flat_c : X \to \{a\}$, i.e., $\flat_c x = a$ for any $x \in X$. Any mapping $T : X \to X$ is $\left(\flat_c, l, L\right)$–Berinde operator for any $l \in (0,1)$ and $L \geq 0$.

Remark 2.4. If $\flat : X \to X$ is the identity map 1_X, i.e. $\flat x = x$ for all $x \in X$, then (2.12) is (1.2), so any (l, L)–Berinde operator is $(1_X, l, L)$–Berinde operator.

Example 2.3. Let $a \in (1, \infty)$ be a real number and $X = [0, \infty)$ be endowed with the metric $d(t, s) = |t - s|$ for all $t, s \in [1, \infty)$. We consider $\flat : [0, \infty) \to [0, \infty)$ and $T_a : [0, \infty) \to [0, \infty)$ given by $T_a(t) = a \cdot t$ and

$$\flat(t) = \begin{cases} \dfrac{a}{t} & \text{if } t \in (0, \infty) \\ 0 & \text{if } x = 0. \end{cases}$$

We claim that is a $\left(\flat, l, L\right)$–Berinde operator. Indeed, for any $t, s \in (0, \infty)$ we have

$$d\left(\flat(T_a(t)), \flat(T_a(s))\right) = \left|\frac{1}{t} - \frac{1}{s}\right| = \frac{1}{a} \cdot \left|\frac{a}{t} - \frac{a}{s}\right| = \frac{1}{a} \cdot d\left(\flat(t), \flat(s)\right).$$

Therefore, (2.12) and (2.13) hold for any $t, s \in (0, \infty)$, with $l \in \left[\frac{1}{a}, 1\right)$ and for any $L \geq 0$. Since $T_a(0) = 0$, we have $\flat(T_a(0)) = \flat(0) = 0$. So, the equality

$$d\left(\flat(T_a(t)), \flat(T_a(0))\right) = d\left(\flat(T_a(t)), \flat(0)\right) = \frac{1}{t} \tag{2.14}$$

holds for any $t \in (0, \infty)$. Now, by $\frac{1}{t} \leq L \cdot \frac{1}{t}$ for any $L \geq 1$, results $d\left(\flat(T_a(t)), \flat(T_a(0))\right) \leq L \cdot d\left(\flat(0), \flat(T_a(t))\right)$. Hence, (2.12) holds for any $t \in (0, \infty)$, with $s = 0$, $l \in (0, 1)$ and $L \geq 1$. On the other hand, by $1 \leq (l + L) \cdot a$, we obtain the inequality $\frac{1}{t} \leq l \cdot \frac{a}{t} + L \cdot \frac{a}{t}$. This is equivalent with (2.13) for any $t \in (0, \infty)$ and $s = 0$, with $l \in (0, 1)$ and $L \geq 0$. Obviously, (2.12) and (2.13) hold for $t = s = 0$. In conclusion, T_a is a $\left(\flat, l, L\right)$–Berinde operator for any $L \geq 1$ and $l \in \left[\frac{1}{a}, 1\right)$, where $a \in (1, \infty)$.

Remark 2.5. By Example 1.2 and Example 2.3, we can say that the mapping

$$T_2 : [0, \infty) \to [0, \infty), \text{ given by } T_2(x) = 2x,$$

is not a (l, L)–Berinde operator, but is $\left(\flat, l, L\right)$–Berinde operator.

Example 2.4. Let a and b be two real numbers, $a, b \in \mathbb{R}$. Let $X = \{a, b\}$ be endowed with the usual metric, denoted by d. There are only four maps on X to itself, namely

$$f_1(a) = f_1(b) = a,$$

$$f_2(a) = f_2(b) = b,$$

$$f_3(a) = a, \text{ and } f_3(b) = b,$$

and

$$f_4(a) = b, \text{ and } f_4(b) = a.$$

Assume $T = f_4$. By Remark 2.3 and Remark 2.4, the mapping T is (f_i, l, L)–Berinde operator, for $i \in \{1, 2, 3\}$, $l \in (0, 1)$ and $L \geq 0$. On the other hand, we have

$$d(f_4(a), f_4(T(b))) = d(f_4(b), f_4(T(a))) = 0,$$

so

$$d(f_4(T(a)), f_4(T(b))) = d(a, b) > l \cdot d(f_4(a), f_4(b)) + L \cdot \delta,$$

holds for any $l \in (0, 1)$ and $L \geq 0$, where $\delta \in \{d(f_4(a), f_4(T(b))), d(f_4(b), f_4(T(a)))\}$. Hence, none of (2.12) and (2.13) are not satisfied, therefore it does not exists $l \in (0, 1)$ and $L \in [0, \infty)$ such that T is a (f_4, l, L)–Berinde operator.

Definition 2.2. Let (X, d) be a metric space. We say that the map $T : X \to X$ is s-admissible if for any sequences $\{x_n\}_{n=0}^{\infty}$, the implication

$$\lim_{n\to\infty} Tx_n = \Lambda \Longrightarrow \lim_{n\to\infty} x_n = \lambda, \text{ with } T\lambda = \Lambda. \tag{2.15}$$

holds.

Remark 2.6. If T maps continuously the compact space X into itself, then T is s-admissible.

3. Fixed point theorem for (l, L)-Berinde operator via the equivalent metric space

We now state and prove one of the main result in this paper.

Theorem 3.4. *Let (X, d) be a metric space and $\hbar : X \to X$ be an one-to-one, s-admissible map. If $T : X \to X$ is a $\hbar$–Berinde operator, i.e. a map satisfying* (2.12) *with $0 < \delta < 1$ and some $L \geq 0$, then T has at least one fixed point in X.*

Proof. This proof is based on an argument similar to the one used by Berinde in [5].
For any $x \in X$, the inequality

$$d\left(\hbar Tx, \hbar T^2x\right) \leq l \cdot \left(\hbar x, \hbar Tx\right) + L \cdot d\left(\hbar Tx, \hbar Tx\right) = l \cdot d\left(\hbar x, \hbar Tx\right). \tag{3.16}$$

holds.

Let $x, y \in X$. By (2.12), we have

$$\begin{aligned} d\left(\hbar x, \hbar y\right) &\leq d\left(\hbar x, \hbar Tx\right) + d\left(\hbar Tx, \hbar Ty\right) + d\left(\hbar y, \hbar Ty\right) \\ &\leq d\left(\hbar x, \hbar Tx\right) + l \cdot d\left(\hbar x, \hbar y\right) + L \cdot d\left(\hbar y, \hbar Tx\right) + d\left(y, \hbar Ty\right) \end{aligned}$$

Hence, the inequality

$$d\left(\hbar x, \hbar y\right) \leq \frac{1}{1-l}\left[d\left(\hbar x, \hbar Tx\right) + d\left(\hbar y, \hbar Ty\right)\right] + \frac{L}{1-l} \cdot d\left(\hbar y, \hbar Tx\right) \tag{3.17}$$

holds for all $x, y \in X$.

Let $x_0 \in X$ be arbitray and let $\{x_n\}_{n=0}^{\infty}$ be the Picard iteration defined by (1.6). Now, (3.16) is

$$d\left(\mathfrak{h}x_n, \mathfrak{h}x_{n+1}\right) \leq l \cdot d\left(\mathfrak{h}x_{n-1}, \mathfrak{h}x_n\right), \quad n = 1, 2, 3, \ldots$$

and, by induction, we obtain

$$d\left(\mathfrak{h}x_n, \mathfrak{h}x_{n+1}\right) \leq l^n \cdot d\left(\mathfrak{h}x_0, \mathfrak{h}x_1\right), \quad n = 1, 2, 3, \ldots \tag{3.18}$$

and

$$\begin{aligned} d\left(\mathfrak{h}x_n, \mathfrak{h}x_{n+p}\right) &\leq \sum_{k=1}^{p} d\left(\mathfrak{h}x_n, \mathfrak{h}x_{n+k}\right) \\ &\leq l^n \cdot \left(\sum_{k=1}^{p} l^{p-1}\right) \cdot d\left(\mathfrak{h}x_0, \mathfrak{h}x_1\right) \\ &= \frac{l^n\left(1 - l^p\right)}{1 - l} \cdot d\left(\mathfrak{h}x_0, \mathfrak{h}x_1\right), \quad n = 1, 2, 3, \ldots \end{aligned} \tag{3.19}$$

Since $l \in (0, 1)$, (3.19) shows that $\{\mathfrak{h}x_n\}_{n=0}^{\infty}$ is Cauchy sequence in the complete metric space (X, d) and hence is convergent. Denote

$$\mathfrak{h}x_n \to h_1, \text{ as } n \to \infty.$$

Since $\{\mathfrak{h}x_n\}_{n=0}^{\infty}$ is convergent and $\mathfrak{h}$ is s-admissible, it results that $\{x_n\}_{n=0}^{\infty}$ has a convergent subsequence. So, there are $\{x_{n_k}\}_{k=0}^{\infty}$ and $h_2 \in X$ such that $x_{n_k} \to h_2$, as $k \to \infty$. Without loss the generality, for the simplicity, we denote this subsequence by $\{x_k\}_{k=0}^{\infty}$, so $x_k \to h_2$, as $k \to \infty$.

Now, we estimate the distance between $\mathfrak{h}h_2$ and $\mathfrak{h}Th_2$. We have

$$d\left(\mathfrak{h}h_2, \mathfrak{h}Th_2\right) \leq d\left(\mathfrak{h}h_2, \mathfrak{h}x_{k+1}\right) + d\left(\mathfrak{h}x_{k+1}, \mathfrak{h}Th_1\right) = d\left(\mathfrak{h}h_2, \mathfrak{h}x_{k+1}\right) + d\left(\mathfrak{h}Tx_k, \mathfrak{h}Th_2\right).$$

By (2.12) and (3.17) we obtain

$$\begin{aligned} d\left(\mathfrak{h}h_2, \mathfrak{h}Th_2\right) &\leq d\left(\mathfrak{h}h_2, \mathfrak{h}x_{k+1}\right) + l \cdot d\left(\mathfrak{h}h_2, \mathfrak{h}x_k\right) + L \cdot d\left(\mathfrak{h}h_2, \mathfrak{h}Tx_k\right) \\ &= d\left(\mathfrak{h}h_2, \mathfrak{h}x_{k+1}\right) + l \cdot d\left(\mathfrak{h}h_2, \mathfrak{h}x_k\right) + L \cdot d\left(\mathfrak{h}h_2, \mathfrak{h}x_{k+1}\right) \\ &= (1 + L) \cdot d\left(\mathfrak{h}h_2, \mathfrak{h}x_{k+1}\right) + l \cdot d\left(\mathfrak{h}h_2, \mathfrak{h}x_k\right) \end{aligned} \tag{3.20}$$

Letting $k \to \infty$, we obtain

$$d\left(\mathfrak{h}h_2, \mathfrak{h}Th_2\right) \leq (1 + l + L) \cdot d\left(\mathfrak{h}h_2, h_1\right). \tag{3.21}$$

Now, because $\mathfrak{h}$ is s-admissible, by (2.15) we have $\mathfrak{h}h_2 = h_1$, and by (3.21) we obtain $d\left(\mathfrak{h}h_2, \mathfrak{h}Th_2\right) = 0$, i.e., $\mathfrak{h}h_2 = \mathfrak{h}Th_2$. Since $\mathfrak{h}$ is one-to-one and onto X, it results

$$h_2 = Th_2,$$

which shows that h_2 is a fixed point of T. □

4. Fixed point theorem for mappings satisfying a general contractive condition of integral type via the equivalent metric space

From the remarks presented in Section 1, it can be concluded that the conditions which ensure the existence of a fixed point are related to some lengths between specific elements of the considered metric space. One tool to measure the distances is the notion of integral. So, there is a variant of contractive condition expressed by integral. The general condition of integral type was introduced by Rhoades, B. E. [26] and Branciari, A. [11]. In the last decade, the literature records many papers in which this type of contractive condition are studied, see for example [14, 16, 30, 23, 33] and reference therein.

We remind here a fixed point theorem for mappings satisfying a general contractive condition of integral type.

Theorem 4.5 ([11]). *Let (X,d) be a complete metric space, $k \in (0,1)$ and $T : X \to X$ be a mapping such that for each $x, y \in X$ one has*

$$\int_0^{d(Tx,Ty)} f(t)dt \leq k \int_0^{d(x,y)} f(t)dt$$

where $f : [0,\infty) \to [0,\infty]$ is a Lebesque-integrable mapping which is summable on each compact subset of $[0,\infty)$, non-negative and such that $\int\limits_0^t f(s)ds > 0$ for each $\varepsilon > 0$. Then T has a unique fixed point $x^* \in X$ such that $\lim\limits_{n\to\infty} T^n x_0 = x^*$ for each $x_0 \in X$.

We shell need the following concept.

Definition 4.3. Let (X,d) be a metric space. Let $\mathfrak{h} : X \to X$ be a mapping of X to itself and $f : [0,\infty) \to [0,\infty]$ be a function such that

i) the map f is Lebesque-integrable, i.e. f is summable on each compact subset of $[0,\infty)$,

ii) $f(x) \geq 0$ for all $x \in [0,\infty)$,

iii) $\int\limits_0^t f(s)ds > 0$ for each $\varepsilon > 0$.

The mapping $T : X \to X$ is a $\mathfrak{h}$-contraction of integral type if the inequality

$$\int_0^{d(\mathfrak{h}Tx,\mathfrak{h}Ty)} f(t)dt \leq k \int_0^{d(\mathfrak{h}x,\mathfrak{h}y)} f(t)dt \tag{4.22}$$

holds for all $x, y \in X$.

In that follows, based on an argument similar to the one used by Branciari, A. [11], we state and prove another fixed point theorem in terms of equivalent metric space.

Theorem 4.6. *Let (X,d) be a metric space and $\mathfrak{h} : X \to X$ be an one-to-one and onto X, s-admissible map. If $T : X \to X$ is a $\mathfrak{h}$-contraction, then T has at least one fixed point in X.*

Proof. For the simplicity, we use the notations $\left[0, d\left(\mathfrak{h}x, \mathfrak{h}y\right)\right] := \mathfrak{d}(x,y)$ and

$$\int_a^b f(t)\,dt := \int\limits_{[a,b]} f d\mu,$$

so (4.22) can rewrite as

$$\int_{\mathsf{d}(Tx,Ty)} f d\mu \leq k \cdot \int_{\mathsf{d}(x,y)} f d\mu. \tag{4.23}$$

Let $x_0 \in X$ and $x_n = T^n x_0$, $n = 0, 1, 2, \ldots$ be the Picard iteration. We claim that $\{\mathfrak{h}x_n\}_{n=0}^{\infty}$ is Cauchy sequence. First, we have

$$\int_{\mathsf{d}(x_n,x_{n+1})} f d\mu \leq k \cdot \int_{\mathsf{d}(x_{n-1},x_n)} f d\mu \leq \ldots \leq k^n \cdot \int_{\mathsf{d}(x_0,x_1)} f d\mu \text{ for all } n \geq 1,$$

and, since $k \in (0, 1)$ we obtain

$$\int_{\mathsf{d}(x_n,x_{n+1})} f d\mu \longrightarrow 0, \text{ as } n \to \infty. \tag{4.24}$$

By iii) from Definition 4.3 and (4.24), we can conclude that

$$d\left(\mathfrak{h}x_n, \mathfrak{h}x_{n+1}\right) \longrightarrow 0, \text{ as } n \to \infty. \tag{4.25}$$

In fact, hypothesis iii) from Definition 4.3 ensure that for any $y \in X$ the implication

$$\int_{\mathsf{d}(x_n,y)} f d\mu \longrightarrow 0 \text{ as } n \to \infty \Longrightarrow d\left(\mathfrak{h}x_n, y\right) \longrightarrow 0 \text{ as } n \to \infty \tag{4.26}$$

holds.

From (4.25) we can prove that $\{\mathfrak{h}x_n\}_{n=0}^{\infty}$ is Cauchy sequence in the complete metric space (X, d) and hence is convergent. Hence, there is $h_1 \in X$ such that

$$\mathfrak{h}x_n \to h_1, \text{ as } n \to \infty.$$

Since $\{\mathfrak{h}x_n\}_{n=0}^{\infty}$ is convergent and $\mathfrak{h}$ is s-admissible map, results that $\{x_n\}_{n=0}^{\infty}$ has a convergent subsequence. So, there are $\{x_{n_k}\}_{k=0}^{\infty}$ and $h_2 \in X$ such that $x_{n_k} \to h_2$, as $k \to \infty$. Without loss the generality, for the simplicity, we denote this subsequence by $\{x_k\}_{k=0}^{\infty}$, so $x_k \to h_2$, as $k \to \infty$ and $\mathfrak{h}h_2 = h_1$.

Since

$$\int_{\mathsf{d}(h_2,x_k)} f d\mu \longrightarrow 0, \text{ as } k \to \infty$$

and

$$\int_{\mathsf{d}(Th_2,x_k)} f d\mu \longrightarrow 0, \text{ as } k \to \infty$$

by (4.26) we obtain

$$d\left(\mathfrak{h}h_2, \mathfrak{h}x_k\right) \longrightarrow 0, \text{ as } k \to \infty, \tag{4.27}$$

respectively

$$d\left(\mathfrak{h}Th_2, \mathfrak{h}x_k\right) \longrightarrow 0, \text{ as } k \to \infty. \tag{4.28}$$

By triangle inequality, we have

$$d\left(\mathfrak{h}h_2, \mathfrak{h}Th_2\right) \leq d\left(\mathfrak{h}h_2, \mathfrak{h}x_{k+1}\right) + d\left(\mathfrak{h}x_{k+1}, \mathfrak{h}Th_1\right) \tag{4.29}$$

Now, letting $k \to \infty$, by (4.27) and (4.28), results $d\left(\mathfrak{h}h_2, \mathfrak{h}Th_2\right) = 0$, i.e. $\mathfrak{h}h_2 = \mathfrak{h}Th_2$. Since $\mathfrak{h}$ is one-to-one and onto X, results $h_2 = Th_2$, which shows that h_2 is a fixed point of T. □

5. Conclusion

It is well known that two metric spaces (X_1, d_1) and (X_2, d_2) are equivalent if there is a function $h : X_1 \to X_2$ which is one-to-one and onto, such that $\tilde{D} : X_1 \to X_1$ defined by $\tilde{D}(x, y) = d_2(h(x), h(y))$, for all $x, y \in X_1$ is a metric equivalent with d_1, i.e., the metrics d_1 and $\tilde{D}$ induce the same topology on X_1. In the light of this remark, Theorem 3.4 establishes a set of hypotheses which ensure the existence of a fixed point of the map T not by its properties in the metric space (X, d), but by its properties in the equivalent metric space $\left(X, \tilde{D}\right)$. Hence, the fixed point theorems can be studied as homotopies, for details see [25, 32]. In a similar way, a contractive condition of integral type can be rewrite with respect to the itself equivalent metric space, hence we can obtain new theorems related to results from [11, 14, 16, 23, 26, 30, 33].

Mathematicians have studied fixed point results in different spaces using various contractive conditions. One of the significant results from the several new contractive conditions which have been developed in an attempt to obtain more refined fixed point results is the concept of (l, L)-Berinde contractions. Some related fixed point results can be found in [7, 18, 31] etc. For all these results it is important to give some examples which involve the fixed point theory as a must-have tool in the study of the solutions of differential and integral equations, see for example [8, 12, 13, 15, 18]. So, as further study, we propose studying the numerous examples from literature to reveal a way to combine the methods to solve some nonlinear differential and/or integral equations with the techniques from metrical fixed point theory, not only by new contractive conditions but also via an equivalent metric space.

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