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Dedicated to the memory of Academician Mitrofan M. Choban (1942-2021)

Existence of common fixed points of non-linear semigroups of Enriched Kannan contractive mappings

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ABSTRACT. In this article, we consider the non-linear semigroup of *enriched Kannan* contractive mapping and prove the existence of common fixed point on a non-empty closed convex subset $\mathcal{C}$ of a real Banach space $\mathscr{X}$, having uniformly normal structure.

1. INTRODUCTION

Metric fixed point theory begins with the classical Banach Contraction Principle [2] due to S. Banach in 1922. After that many famous fixed point theorems have been proved over complete metric spaces by several researchers [4, 7, 15, 10]. Recently, Berinde and Păcurar [5] have introduced a new contractive mapping, called *enriched Kannan mapping* and proved a fixed point theorem over a Banach space, which states below:

Theorem 1.1. [5] *In a Banach space* $(\mathscr{X}, \|.\|)$*, the enriched Kannan mapping defined as:*

$$\|k(x-y)+\Upsilon x-\Upsilon y\| \leq \alpha(\|x-\Upsilon x\|+\|y-\Upsilon y\|) \text{ for all } x,y\in\mathscr{X} \tag{1.1}$$

for some $\alpha\in[0,\frac{1}{2})$ *and* $k\in[0,\infty)$*; possesses a unique fixed point in* $\mathscr{X}$.

From the year 1960 onwards, common fixed point theorems for semigroups of non-linear self-mappings play an important roles in non-linear operator theory and its applications. Several researchers have investigated various types of non-linear self-mappings, viz., uniformly Lipshitzian semigroup ([8], [23],[17]), non-expansive semigroup ([18],[19], [22]), semigroups for pseudocontractions ([9],[1]), Lipschitzian pseudocontraction semigroups [25], uniformly continuous semigroups [20] etc.

In a real Banach space $\mathscr{X}$, let $\mathcal{C}\subset\mathscr{X}$ be non-empty closed and convex. A mapping $\Upsilon:\mathcal{C}\to\mathcal{C}$ is said to be Lipschitzian mapping if for each integer $n\geq 1$ there exists positive constants k_n such that,

$$\|\Upsilon^n x-\Upsilon^n y\|\leq k_n\|x-y\| \text{ for all } x,y\in\mathcal{C}. \tag{1.2}$$

A Lipschitzian mapping is said to be k-uniformly Lipschitzian mapping if $k_n\equiv k$ for all $n\geq 1$. In the year 1973, Goebel & Kirk [12] proved that every k-uniformly Lipschitzian self-mapping on a closed, convex, bounded subset $\mathcal{C}$ of a uniformly convex Banach space $\mathscr{X}$ possesses a fixed point if $k<\gamma$, where $\gamma>1$ is the unique solution of the equation $\left(1-\delta_{\mathscr{X}}\left(\frac{1}{\gamma}\right)\right)\gamma=1$; where $\delta_{\mathscr{X}}$ is the modulus of convexity of $\mathscr{X}$.

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In 2001, Zeng and Yang [24] proved a fixed point result of Lipschitzian semigroup, which states below.

Theorem 1.2. [24] *Let $\mathcal{C}(\neq \emptyset)$ be a bounded subset of a uniformly convex Banach space $\mathscr{X}$. Let, $\mathscr{F} : \{\Upsilon_s : s \in \mathcal{G}\}$ be a Lipschitzian semigroup of self-mappings on $\mathcal{C}$ satisfying $\lim_s \|\Upsilon_s\| < (\gamma_0 \mathcal{N}(\mathscr{X}))^{1/2}$, where $\gamma_0 = \inf\{\gamma : \gamma(1 - \delta_{\mathscr{X}}(1/\gamma)) \geq \frac{1}{2}\}$ and $\|\Upsilon_s\|$ is the exact Lipschitzian constant of Υ_s. Moreover assume that there exists a non-empty bounded closed convex subset $\mathcal{H}$ of $\mathcal{C}$ having the following two properties:*
($\mathcal{P}_1$) $x \in \mathcal{H}$ implies that $w_\omega(x) \subset \mathcal{H}$, where $w_\omega(x)$ is the weak ω-limit of $\mathscr{F}$ at x.
($\mathcal{P}_2$) $\mathscr{F}$ is asymptotic regular on $\mathcal{H}$. i.e., $\lim_t \|\Upsilon_{s+t}x - \Upsilon_t x\| = 0$ for all $s \in \mathcal{G}$ and $x \in \mathcal{H}$.
Then the semigroup $\mathscr{F}$ possesses a unique fixed point in $\mathcal{C}$.

Ceng et.al. [8] improved the result of Zeng and Yang [24] and proved a similar result by removing the condition of asymptotic regularity (*i.e.*, only under the assumption of the condition ($\mathcal{P}_1$) given above).

In the year 2010, Ceng et al. [8] proved the existence of common fixed point of k-uniformly Lipschitzian semigroup in Banach space having uniformly normal structure.

Theorem 1.3. [8] *Let $\{\Upsilon_s x_0 : s \in \mathcal{G}\}$ be a uniformly k-Lipschitzian semigroup on a non-empty closed, convex and bounded subset $\mathcal{C}$ of a real Banach space $\mathscr{X}$ with normal structure co-efficient $\mathcal{N}(\mathscr{X}) > \max\{1, \epsilon_0\}$, where ϵ_0 is the characteristic convexity of $\mathscr{X}$. If $\{\Upsilon_s x_0\}$ is bounded for some $x_0 \in \mathcal{C}$, then the semigroup $\{\Upsilon_s x_0 : s \in \mathcal{G}\}$ admits a common fixed point provided $k < \alpha_*$, where*

$$\alpha_* = \sup\left\{\alpha : \frac{\alpha^2}{\mathcal{N}(\mathscr{X})}\delta_{\mathscr{X}}^{-1}\left(1 - \frac{1}{\alpha}\right) \leq 1 \text{ and } 1 - \frac{1}{\alpha} \in \left(0, 1 - \frac{\epsilon_0}{2}\right)\right\}.$$

In 2017, Soliman et al.[21] introduced the uniformly generalized Kannan type semigroup and proved the existence and uniqueness theorem of common fixed point under certain condition in a Banach space.

Theorem 1.4. [21] *Let $\mathcal{C}$ be a non-empty closed, convex subset of a real Banach space $\mathscr{X}$ with $\mathcal{N}(\mathscr{X}) > \max\{1, \epsilon_0\}$, ϵ_0 being the characteristic convexity of $\mathscr{X}$. Let $\{\Upsilon_s : s \in \mathcal{G}\}$ be a generalized uniformly Kannan type semigroup of self mapping defined on $\mathcal{C}$. i.e.,*

$$\|\Upsilon_s x - \Upsilon_s y\| \leq \alpha\{\|x - \Upsilon_s x\| + \|y - \Upsilon_s y\|\} \text{ for all } x, y \in \mathcal{C} \tag{1.3}$$

for each $s \in \mathcal{G}$ and for some $\alpha \in [0, 1)$. If $\{\Upsilon_s x_0\}$ is bounded for some $x_0 \in \mathcal{C}$, then the semigroup $\{\Upsilon_s x_0 : s \in \mathcal{G}\}$ admits a common fixed point in $\mathcal{C}$, provided $\dfrac{3\xi\alpha}{\mathcal{N}(\mathscr{X})}\delta_{\mathscr{X}}^{-1}\left(1 - \dfrac{1}{\xi}\right) < 1$, where $\xi = \frac{3\alpha}{1-\alpha}$.

Very recently, in 2020, Kesahorm and Sintunavarat [16] have introduced the non-linear semigroup of weak-contraction due to Berinde and proved the existence and convergence theorem in real Banach space under certain condition, given below.

Theorem 1.5. [16] *Let $\mathcal{C}$ be a non-empty closed, convex subset of a real Banach space $\mathscr{X}$ with $\mathcal{N}(\mathscr{X}) > \max\{1, \epsilon_0\}$, ϵ_0 being the characteristic convexity of $\mathscr{X}$. Let $\{\Upsilon_s : s \in \mathcal{G}\}$ be a weak contraction semigroup of self mappings defined on $\mathcal{C}$. i.e.,*

$$\|\Upsilon_s x - \Upsilon_s y\| \leq \alpha\|x - y\| + K\|y - \Upsilon_s x\| \text{ for all } x, y \in \mathcal{C} \tag{1.4}$$

for each $s \in \mathcal{G}$ and for some $\alpha \in [0, 1)$ and $K \geq 0$. If $\{\Upsilon_s x_0 : s \in \mathcal{G}\}$ is bounded for some $x_0 \in \mathcal{C}$, then the semigroup $\{\Upsilon_s : s \in \mathcal{G}\}$ admits a common fixed point in $\mathcal{C}$, provided $K + 1 < \alpha_*$, where

$$\alpha_* = \sup\left\{\beta : \frac{\beta^2}{\mathcal{N}(\mathscr{X})}\delta_{\mathscr{X}}^{-1}\left(1 - \frac{1}{\beta}\right) \leq 1 \text{ and } 1 - \frac{1}{\beta} \in \left(0, 1 - \frac{\epsilon_0}{2}\right)\right\}.$$

Inspired by this way of research, in this paper we introduce a new non-linear semigroup of enriched Kannan type contraction and prove the existence of common fixed point on a closed, convex, bounded subset of a real Banach space having uniform normal structure.

2. Preliminaries

Let $\mathcal{C}$ be a closed, convex subset of a real Banach space $\mathscr{X}$ and $\mathcal{G}$ be an unbounded subset of $[0, \infty)$ satisfying the followings:
(i) $s + t \in \mathcal{G}$ for all $s, t \in \mathcal{G}$ ($i.e.$, $\mathcal{G}$ is closed under addition).
(ii) $s - t \in \mathcal{G}$ for all $s, t \in \mathcal{G}$ with $s > t$.
A collection $\{\Upsilon_s : s \in \mathcal{G}\}$ of self-mappings defined on $\mathcal{C}$ is said to be semigroup if:
(S1) for all $s, t \in \mathcal{G}$ and $x \in \mathcal{C}$, $\Upsilon_{s+t}x = \Upsilon_s(\Upsilon_t x)$.
(S2) for all $x \in \mathcal{C}$, the mapping $s \mapsto \Upsilon_s x$ is continuous.
Chebyshev's radius and diameter of $\mathcal{C}$ (see [11]) are denoted by $r_{\mathcal{C}}(\mathcal{C})$ and $\delta(\mathcal{C})$ respectively and defined as below:

$$r_{\mathcal{C}}(\mathcal{C}) := \inf_{x\in\mathcal{C}} \sup_{y\in\mathcal{C}} \|x - y\|$$
$$\delta(\mathcal{C}) := \sup_{x,y\in\mathcal{C}} \|x - y\|.$$

A Banach space $\mathscr{X}$ is said to have normal structure (see [12]) if each closed, convex, bounded subset $\mathcal{C}$ (consisting more than one point) of $\mathscr{X}$, we have $r_{\mathcal{C}}(\mathcal{C}) < \delta(\mathcal{C})$. If there exists a constant $k \in (0, 1)$ such that $r_{\mathcal{C}}(\mathcal{C}) < k\delta(\mathcal{C})$, whenever $\delta(\mathcal{C}) > 0$, then $\mathscr{X}$ is said to have uniform normal structure.

Normal structure co-efficient (see [6]) of a Banach space $\mathscr{X}$ is defined as:

$$\mathcal{N}(\mathscr{X}) := \inf_{\substack{\mathcal{C}\subset\mathscr{X} \\ r_{\mathcal{C}}(\mathcal{C})>0}} \frac{\delta(\mathcal{C})}{r_{\mathcal{C}}(\mathcal{C})}.$$

Modulus of convexity (see [13]) of a Banach space $\mathscr{X}$ is the function $\delta_{\mathscr{X}} : [0, 2] \to [0, 1]$ defined by

$$\delta_{\mathscr{X}}(\epsilon) := \inf\left\{1 - \left\|\frac{x + y}{2}\right\| : \|x\| \leq 1, \|y\| \leq 1 \text{ and } \|x - y\| \geq \epsilon\right\}$$

and the characteristic of convexity (see [13]) of $\mathscr{X}$ is defined as

$$\epsilon_0(\mathscr{X}) := \sup\{\epsilon : \delta_{\mathscr{X}}(\epsilon) > 0\}.$$

Now we list up some properties of modulus of convexity $\delta_{\mathscr{X}}$ (see [13]):
(M1) $\delta_{\mathscr{X}}$ is monotone increasing on $[0, 2]$ and strictly increasing on $[\epsilon_0, 2]$.
(M2) $\delta_{\mathscr{X}}$ is continuous on $[0, 2)$.
(M3) $\delta_{\mathscr{X}}(0) = 0$ and $\lim_{\epsilon\to 2^-} \delta_{\mathscr{X}}(\epsilon) = 1 - \frac{\epsilon_0}{2}$.
(M4) $\left\|a - \frac{x + y}{2}\right\| \leq r\left(1 - \delta_{\mathscr{X}}\left(\frac{\epsilon}{r}\right)\right)$ whenever $\|a - x\| \leq r$, $\|a - y\| \leq r$ and $\|x - y\| \geq \epsilon$.

Definition 2.1. [11] For a non-empty closed, convex and bounded subset $\mathcal{C}$ of a real Banach space $\mathscr{X}$, let $\{x_s : s \in \mathcal{G}\}$ be a bounded net of elements of $\mathscr{X}$, where $\mathcal{G}$ is an unbounded subset of $[0, \infty)$. Then we define

- Asymptotic radius of $\{x_s : s \in \mathcal{G}\}$ with respect to $\mathcal{C}$ by

$$r_{\mathcal{C}}(\{x_s\}) := \inf_{y\in\mathcal{C}} \limsup_{s\to\infty} \|x_s - y\|.$$

- Asymptotic center of $\{x_s : s \in \mathcal{G}\}$ with respect to $\mathcal{C}$ by

$$A_{\mathcal{C}}(\{x_s\}) := \left\{ y \in \mathcal{C} : \limsup_{s\to\infty} \|x_s - y\| = r_{\mathcal{C}}(\{x_s\}) \right\}.$$

Finally, we give the statements of the following two lemmas from [23] which are crucial in proving our main results.

Lemma 2.1. [23] *Let $\mathscr{X}$ be a reflexive Banach space and $\mathcal{C}$ be a non-empty closed, convex subset of $\mathscr{X}$. Then $\mathcal{A}_{\mathcal{C}}(\{x_t\})$ forms a non-empty closed, convex and bounded subset of $\mathcal{C}$, for every bounded net $\{x_t\}_{t\in\mathcal{G}}$ of elements of $\mathscr{X}$.*

Lemma 2.2. [23] *Let $\mathscr{X}$ be a Banach space having uniformly structure. Then for every bounded net $\{x_t\}_{t\in\mathcal{G}}$ of elements of $\mathscr{X}$, there exists $y \in \overline{co}(\{x_t : t \in \mathcal{G}\})$ such that $\limsup_{t\to\infty} \|x_t - y\| \leq \frac{1}{\mathcal{N}(\mathscr{X})}\mathcal{D}(\{x_t\})$, where $\overline{co}(M)$ is the closure of the convex hull of $M \subset \mathscr{X}$.*

3. Main results

Definition 3.2. Let $\mathcal{C}$ be a non-empty closed, convex and bounded subset of a real Banach space $\mathscr{X}$. Then the family $\{\Upsilon_s : s \in \mathcal{G}\}$ of self-mappings defined on $\mathcal{C}$ is said to be enriched Kannan type semigroup if the followings hold:
(a) for all $s, t \in \mathcal{G}$ and $x \in \mathcal{C}$, $\Upsilon_{s+t}x = \Upsilon_s(\Upsilon_t x)$.
(b) for all $x \in \mathcal{C}$, the mapping $s \mapsto \Upsilon_s x$ is continuous.
(c) for each $t \in \mathcal{G}$, $\Upsilon_s : \mathcal{C} \to \mathcal{C}$ satisfies

$$\|k(x-y) + \Upsilon x - \Upsilon y\| \leq \alpha(\|x - \Upsilon x\| + \|y - \Upsilon y\|) \text{ for all } x, y \in \mathcal{C} \tag{3.5}$$

for some $\alpha \in [0, 1)$ and for some $k \geq 0$.

Theorem 3.6. Let $\mathscr{X}$ be a real Banach space with normal structure coefficient $\mathcal{N}(\mathscr{X}) > \max\{1, \epsilon_0\}$, where ϵ_0 is the characteristic convexity of $\mathscr{X}$. Let $\mathcal{C}$ be a non-empty closed, convex subset of $\mathscr{X}$ and $\mathscr{F} := \{\Upsilon_s : s \in \mathcal{G}\}$ be the enriched Kannan type semigroup of self mappings defined on $\mathcal{C}$ with $0 \leq \alpha < 1$ and $k \geq 0$ satisfying $\frac{k+3\alpha}{1-\alpha} < \hat{\alpha}$, where

$$\hat{\alpha} = \sup\left\{ \beta : \frac{\beta^2}{\mathcal{N}(\mathscr{X})}\delta_{\mathscr{X}}^{-1}\left(1 - \frac{1}{\beta}\right) \leq 1 \text{ and } 1 - \frac{1}{\beta} \in \left(0, 1 - \frac{\epsilon_0}{2}\right) \right\}.$$

If $\{\Upsilon_s x_0 : s \in \mathcal{G}\}$ is bounded for some $x_0 \in \mathcal{C}$, then the semigroup $\mathscr{F}$ possesses a common fixed point in $\mathcal{C}$.

Proof. Since $\mathscr{X}$ has a uniform normal structure, it is reflexive. Since $\{\Upsilon_s x_0 : s \in \mathcal{G}\}$ is bounded so by Lemma 2.1, $\mathcal{A}_{\mathcal{C}}(\{x_0\})$ is a non-empty closed, convex and bounded subset of $\mathcal{C}$. Choose $x_1 \in \mathcal{A}_{\mathcal{C}}(\{x_0\})$. Then by definition of asymptotic center,

$$\limsup_{t\to\infty} \|\Upsilon_t x_0 - x_1\| = \inf_{y\in\mathcal{C}} \limsup_{t\to\infty} \|\Upsilon_t x_0 - y\|.$$

Since $\mathscr{F}$ is enriched Kannan type semigroup, we have

$$\begin{aligned}
\|\Upsilon_t x_1\| &\leq \|\Upsilon_t x_1 - \Upsilon_t x_0\| + \|\Upsilon_t x_0\| \\
&\leq k.\|x_1 - x_0\| + \alpha.\{\|x_1 - \Upsilon_t x_1\| + \|x_0 - \Upsilon_t x_0\|\} + \|\Upsilon_t x_0\| \\
&\leq k.\|x_1 - x_0\| + \alpha.\{\|x_1\| + \|\Upsilon_t x_1\| + \|x_0\| + \|\Upsilon_t x_0\|\} + \|\Upsilon_t x_0\|
\end{aligned}$$

which implies $\|\Upsilon_t x_1\| \leq \frac{k}{1-\alpha}\|x_1 - x_0\| + \frac{\alpha}{1-\alpha}\{\|x_1\| + \|x_0\| + \|\Upsilon_t x_0\|\} + \frac{1}{1-\alpha}\|\Upsilon_t x_0\|$.

Thus we get, $\{\Upsilon_t x_1 : t \in \mathcal{G}\}$ is bounded. Therefore we can choose $x_2 \in \mathcal{A}_{\mathcal{C}}(\{x_1\})$ such that

$$\limsup_{t\to\infty} \|\Upsilon_t x_1 - x_2\| = \inf_{y\in\mathcal{C}} \limsup_{t\to\infty} \|\Upsilon_t x_1 - y\|.$$

Continuing this process we can construct a sequence $\{x_n\}$ in $\mathcal{C}$ such that,
A. for each $n \geq 0$, $\{\Upsilon_t x_n\}_{t\in\mathcal{G}}$ is bounded.
B. for each $n \geq 0$, $x_{n+1} \in \mathcal{A}_{\mathcal{C}}(\{x_n\})$ such that

$\limsup_{t\to\infty}\|\Upsilon_t x_n - x_{n+1}\| = \inf_{y\in\mathcal{C}} \limsup_{t\to\infty}\|\Upsilon_t x_n - y\| = r_{\mathcal{C}}\left(\{\Upsilon_t x_n\}\right).$

Claim 1: $\{x_n\}$ is a Cauchy sequence.

Let us denote $r_{\mathcal{C}}\left(\{\Upsilon_t x_n\}\right)$ simply by r_n. Then by Lemma 2.2,

$$\begin{aligned}
r_n &= \limsup_{t\to\infty}\|\Upsilon_t x_n - x_{n+1}\| \\
&\leq \tfrac{1}{\mathcal{N}(\mathscr{X})}\mathcal{D}(\{\Upsilon_t x_n\}) \\
&= \tfrac{1}{\mathcal{N}(\mathscr{X})}\lim_{t\to\infty}\left(\sup_{t\leq i,j\in\mathcal{G}}\|\Upsilon_i x_n - \Upsilon_j x_n\|\right) \\
&= \tfrac{1}{\mathcal{N}(\mathscr{X})}\lim_{t\to\infty}\left(\sup_{t\leq i,j\in\mathcal{G}}\|\Upsilon_i x_n - \Upsilon_i\Upsilon_{j-i} x_n\|\right) \\
&\leq \tfrac{1}{\mathcal{N}(\mathscr{X})}\lim_{t\to\infty}\left(\sup_{t\leq i,j\in\mathcal{G}}\{k.\|x_n - \Upsilon_{j-i}x_n\| + \alpha\|x_n - \Upsilon_i x_n\| + \alpha\|\Upsilon_j x_n - \Upsilon_{j-i}x_n\|\}\right) \\
&\leq \tfrac{1}{\mathcal{N}(\mathscr{X})}\left[k.d(x_n) + \alpha\{d(x_n) + d(x_n) + d(x_n)\}\right] \\
&= \tfrac{1}{\mathcal{N}(\mathscr{X})}(k+3\alpha)d(x_n)
\end{aligned}$$

where $d(x_n) = \sup_{t\in\mathcal{G}}\|x_n - \Upsilon_t x_n\|$.

Now if $d(x_n) = 0$ for some $n \in \mathbb{N}$, then x_n is a common fixed point of the semigroup $\mathscr{F}$ and hence we are done. So suppose, $d(x_n) > 0$, for all $n \geq 0$. Let $\epsilon > 0$ be arbitrary and $n \geq 0$ be fixed.

Since $d(x_{n+1}) = \sup_{t\in\mathcal{G}}\|\Upsilon_t x_{n+1} - x_{n+1}\|$, by definition of supremum, there exists $j \in \mathcal{G}$ such that

$$\|\Upsilon_j x_{n+1} - x_{n+1}\| > d(x_{n+1}) - \epsilon. \tag{3.6}$$

Since $r_n = \limsup_{t\to\infty}\|T_t x_n - x_{n+1}\|$, so by definition of limit superior, there exists $h \in \mathcal{G}$ such that

$$\sup_{s\geq h}\|\Upsilon_s x_n - x_{n+1}\| < r_n + \epsilon.$$

So for all $s \geq h$,

$$\|\Upsilon_s x_n - x_{n+1}\| < r_n + \epsilon. \tag{3.7}$$

Now for $s \geq h + j$,

$$\begin{aligned}
\|\Upsilon_s x_n - \Upsilon_j x_{n+1}\| &= \|\Upsilon_{j+(s-j)} x_n - \Upsilon_j x_{n+1}\| \\
&\leq k.\|\Upsilon_{s-j} x_n - x_{n+1}\| + \alpha.\{\|\Upsilon_s x_n - \Upsilon_{s-j} x_n\| + \|x_{n+1} - \Upsilon_j x_{n+1}\|\} \\
&< (k+3\alpha).(r_n+\epsilon) + \alpha.\|\Upsilon_s x_n - \Upsilon_j x_{n+1}\|
\end{aligned}$$

implying that

$$\|\Upsilon_s x_n - \Upsilon_j x_{n+1}\| < \gamma(r_n+\epsilon)\text{ , where } \gamma = \frac{k+3\alpha}{1-\alpha}. \tag{3.8}$$

Now using (3.6), (3.7), (3.8) and property (M4) we have for $s \geq h + j$,

$$\|\Upsilon_s x_n - \frac{1}{2}(x_{n+1} + \Upsilon_j x_{n+1})\| \leq \gamma(r_n+\epsilon)\left(1 - \delta_{\mathscr{X}}\left(\frac{d(x_{n+1}) - \epsilon}{\gamma(r_n+\epsilon)}\right)\right). \tag{3.9}$$

Therefore,

$$\begin{aligned}
r_n &\leq \limsup_{s\to\infty}\|\Upsilon_s x_n - \frac{1}{2}(x_{n+1} + \Upsilon_j x_{n+1})\| \\
&\leq \gamma(r_n+\epsilon)\left(1 - \delta_{\mathscr{X}}\left(\frac{d(x_{n+1}) - \epsilon}{\gamma(r_n+\epsilon)}\right)\right).
\end{aligned}$$

Letting $\epsilon \to 0$ we get, $r_n \leq \gamma r_n \left(1 - \delta_{\mathscr{X}}\left(\frac{d(x_{n+1})}{\gamma r_n}\right)\right)$ which yields,

$$\delta_{\mathscr{X}}\left(\frac{d(x_{n+1})}{\gamma r_n}\right) \leq 1 - \frac{1}{\gamma}. \tag{3.10}$$

Claim 2: $d(x_{n+1}) \leq \gamma r_n \delta_{\mathscr{X}}^{-1}\left(1 - \frac{1}{\gamma}\right)$.

If $\frac{d(x_{n+1})}{\gamma r_n} \in [\epsilon_0, 2)$, then as $\delta_{\mathscr{X}}^{-1}$ exists so, $\frac{d(x_{n+1})}{\gamma r_n} \leq \delta_{\mathscr{X}}^{-1}\left(1 - \frac{1}{\gamma}\right)$ and we are done.

Now consider, $\frac{d(x_{n+1})}{\gamma r_n} \in [0, \epsilon_0)$. Since $\delta_{\mathscr{X}} : [\epsilon_0, 2) \to \delta_{\mathscr{X}}([\epsilon_0, 2)) = [0, 1-\frac{\epsilon_0}{2})$ is a bijection and $1 - \frac{1}{\gamma} \in [0, 1 - \frac{\epsilon_0}{2})$, from the hypothesis $\gamma = \frac{k+3\alpha}{1-\alpha} < \hat{\alpha}$, we get $\delta_{\mathscr{X}}^{-1}(1 - \frac{1}{\gamma}) \geq \epsilon_0$, which implies that $\frac{d(x_{n+1})}{\gamma r_n} \leq \delta_{\mathscr{X}}^{-1}(1 - \frac{1}{\gamma})$. Therefore we have,

$$\begin{aligned} d(x_{n+1}) &\leq \gamma r_n \delta_{\mathscr{X}}^{-1}(1 - \frac{1}{\gamma}) \text{ (Claim 2 is justified)} \\ &\leq \frac{\gamma}{\mathcal{N}(\mathscr{X})}(k + 3\alpha) d(x_n) \delta_{\mathscr{X}}^{-1}(1 - \frac{1}{\gamma}) \\ &= \mathcal{A} d(x_n) \end{aligned}$$

where, $\mathcal{A} = \frac{\gamma}{\mathcal{N}(\mathscr{X})}(k + 3\alpha)\delta_{\mathscr{X}}^{-1}(1 - \frac{1}{\gamma}) < \frac{\gamma^2}{\mathcal{N}(\mathscr{X})}\delta_{\mathscr{X}}^{-1}(1 - \frac{1}{\gamma}) < 1$.

Then by induction,

$$d(x_n) \leq \mathcal{A} d(x_{n-1}) \leq \mathcal{A}^2 d(x_{n-2}) \leq \cdots \leq \mathcal{A}^n d(x_0). \tag{3.11}$$

Now for each $n \in \mathbb{N}$,

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq \limsup_{t\to\infty} \|\Upsilon_t x_n - x_{n+1}\| + \limsup_{t\to\infty} \|\Upsilon_t x_n - x_n\| \\ &\leq r_n + d(x_n) \\ &\leq \left(\frac{k + 3\alpha}{\mathcal{N}(\mathscr{X})} + 1\right) \mathcal{A}^n d(x_0). \qquad (3.12) \end{aligned}$$

Since $\mathcal{A} < 1$, so $\sum_{n=1}^{\infty} \|x_{n+1} - x_n\|$ is convergent and consequently $\{x_n\}$ is a Cauchy sequence. Thus Claim 1 is justified.

By completeness of $\mathcal{C}$, sequence $\{x_n\}$ is convergent and let $x_n \to x_* \in \mathcal{C}$ as $n \to \infty$. Finally for each $t \in \mathcal{G}$ we have,

$$\begin{aligned} \|x_* - \Upsilon_t x_*\| &\leq \|x_* - x_n\| + \|x_n - \Upsilon_t x_n\| + \|\Upsilon_t x_n - \Upsilon_t x_*\| \\ &\leq \|x_* - x_n\| + \|x_n - \Upsilon_t x_n\| + k\|x_n - x_*\| + \alpha\left(\|x_n - \Upsilon_t x_n\| + \|x_* - \Upsilon_t x_*\|\right) \end{aligned}$$

which yields,

$$(1 - \alpha)\|x_* - \Upsilon_t x_*\| \leq (1 + k)\|x_n - x_*\| + (1 + \alpha) d(x_n) \to 0 \text{ as } n \to \infty.$$

Consequently, $\Upsilon_t x_* = x_*$ for all $t \in \mathcal{G}$. □

Now we are going to present an example corresponding to the Theorem 3.6. Before that, we note down some results which are needed to establish the example.

- (See [14]) For the space L^p with $p \geq 2$, we have the modulus of convexity $\delta_{\mathscr{X}}(\epsilon) = 1 - \left(1 - \left(\frac{\epsilon}{2}\right)^p\right)^{1/p}$.
- (See [12]) If $\delta_{\mathscr{X}}(1) > 0$ (*i.e.*, if $\epsilon_0(\mathscr{X}) < 1$), then $\mathscr{X}$ has normal structure.

• (See [3]) If Ω is a σ-finite measure space, then the normal structure co-efficient of the space $L^p(\Omega)$, $1 \leq p < \infty$ is given by $\mathcal{N}(L^p(\Omega)) = \min\{2^{1-1/p}, 2^{1/p}\}$.

Example 3.1. Consider, $\mathscr{X} = L^2([-10,10])$ and let $\mathcal{C} = \mathscr{X}$. Then $\mathscr{X}$ is a real Banach space. Since $[-10,10]$ is a σ-finite measured space, so $\delta_{\mathscr{X}}(1) = 1 - \frac{\sqrt{3}}{4} > 0$. Then $\epsilon_0(\mathscr{X}) < 1$ and $\mathscr{X}$ has normal structure with normal structure co-efficient $\mathcal{N}(\mathscr{X}) = \min\{2^{1-\frac{1}{2}}, 2^{\frac{1}{2}}\} = \sqrt{2} > \max\{1, \epsilon_0\}$.

Now,

$$\begin{aligned}\hat{\alpha} &= \sup\left\{\beta : \frac{\beta^2}{\mathcal{N}(\mathscr{X})}\delta_{\mathscr{X}}^{-1}\left(1-\frac{1}{\beta}\right) \leq 1 \text{ and } 1-\frac{1}{\beta} \in \left(0, 1-\frac{\epsilon_0}{2}\right)\right\}\\ &= \sup\left\{\beta : \delta_{\mathscr{X}}\left(\frac{\sqrt{2}}{\beta^2}\right) \geq 1-\frac{1}{\beta} \text{ and } 1-\frac{1}{\beta} \in \left(0, 1-\frac{\epsilon_0}{2}\right)\right\}.\end{aligned}$$

By doing some elementary calculation we get, $\hat{\alpha} = \left(\frac{\sqrt{3}}{2} + \frac{1}{2}\right)^{1/2}$.

Now we consider the collection of mappings $\mathscr{F} := \{\Upsilon_s : \mathcal{C} \to \mathcal{C} | s \in \mathcal{G}\}$ such that

$$(\Upsilon_s f)(x) = 5^{-\frac{s}{3}} f(x) \text{ for all } f \in \mathcal{C},$$

where $\mathcal{G} := \{5n : n \in \mathbb{N}\}$, an unbounded subset of $[0,\infty)$ satisfying $s+t \in \mathcal{G}$ for all $s,t \in \mathcal{G}$ and $s-t \in \mathcal{G}$ for all $s,t \in \mathcal{G}$ with $s > t$.

It is clearly seen that the collection $\mathscr{F}$ satisfies the property (S2). Again for all $s,t \in \mathcal{G}$ and for all $f \in \mathcal{C}$ we have, $\Upsilon_{s+t} f = 5^{-\frac{s+t}{3}} f = 5^{-\frac{s}{3}}(5^{-\frac{t}{3}} f) = 5^{-\frac{s}{3}}(\Upsilon_t f) = \Upsilon_s(\Upsilon_t f)$. Thus property (S1) is satisfied.

Moreover, the function $s \mapsto 5^{-s/3}$ is bounded measurable function on $[-10,10]$ and hence the collection $\mathscr{F}$ forms a semigroup of self mappings defined on $\mathcal{C}$. Now we will verify that Υ_s is the *enriched Kannan type mapping* with $\alpha = \frac{9}{40}$ and $k = \frac{1}{10}$.

For all $f, g \in \mathcal{C}$ and for all $s \in \mathcal{G}$ we have,

$$\|k(f-g) + \Upsilon_s f - \Upsilon_s g\| = \|k(f-g) + 5^{-\frac{s}{3}} f - 5^{-\frac{s}{3}} g\| = \left(\frac{1}{10} + 5^{-\frac{s}{3}}\right)\|f-g\| \tag{3.13}$$

and

$$\begin{aligned}\alpha\left(\|f-\Upsilon_s f\| + \|g-\Upsilon_s g\|\right) &= \frac{9}{40}\left(\|f-5^{-\frac{s}{3}} f\| + \|g-5^{-\frac{s}{3}} g\|\right)\\ &= \frac{9}{40}(1-5^{-\frac{s}{3}})(\|f\|+\|g\|).\end{aligned} \tag{3.14}$$

Now by the triangle inequality, $\|f-g\| \leq \|f\|+\|g\|$ and it is easy to see that for all $s \in \mathcal{G}$, $\left(\frac{1}{10} + 5^{-\frac{s}{3}}\right) \leq \frac{9}{40}(1-5^{-\frac{s}{3}})$.

Thus from (3.13) and (3.14), it follows that,

$$\|k(f-g) + \Upsilon_s f - \Upsilon_s g\| \leq \alpha\left(\|f-\Upsilon_s f\| + \|g-\Upsilon_s g\|\right). \tag{3.15}$$

Therefore the family $\mathscr{F} = \{\Upsilon_s : s \in \mathcal{G}\}$ forms the *enriched Kannan type semigroup* of self mappings defined on $\mathcal{C}$ with $\alpha = \frac{9}{40}$, $k = \frac{1}{10}$ and then $\frac{k+3\alpha}{1-\alpha} = 1 < \left(\frac{\sqrt{3}}{2} + \frac{1}{2}\right)^{1/2} = \hat{\alpha}$. Moreover, $\{\Upsilon_s 0_{function} : s \in \mathcal{G}\} = \{0_{function}\}$, which is bounded trivially.

Thus all the conditions of Theorem 3.6 are satisfied and it is seen that $f \equiv 0_{function} \in \mathcal{C}$ is the common fixed point of the semigroup $\mathscr{F}$.

Corollary 3.1. If $k = 0$, then the enriched Kannan type mapping (3.5) reduces to the generalized uniformly Kannan mapping (1.3) and Theorem 3.6 generalizes Theorem 1.4.

Corollary 3.2. If $\alpha = 0$, then the enriched Kannan type mapping (3.5) reduces to the uniformly k-Lipschitzian mapping (1.2) and Theorem 3.6 generalizes Theorem 1.3.

Theorem 3.7. Let $\mathcal{C}(\neq \emptyset)$ be a bounded subset of a uniformly convex Banach space $\mathscr{X}$. Let $\mathscr{F} := \{\Upsilon_s : s \in \mathcal{G}\}$ be the enriched Kannan type semigroup of self mappings defined on $\mathcal{C}$ satisfying $\frac{k+3\alpha}{1-\alpha} < \mu_0 \, \mathcal{N}(\mathscr{X})$, where $\mu_0 = \inf\{\mu \geq 1 : \mu \left(1 - \delta_{\mathscr{X}}(\frac{1}{\mu})\right) \geq \frac{1}{2}\}$. Also suppose that there exists a non-empty, closed, convex and bounded subset $\mathcal{H}$ of $\mathcal{C}$ satisfying the following property:
($\mathscr{P}$) $x \in \mathcal{H}$ implies $w_\omega(x) \subset \mathcal{H}$ where,
$w_\omega(x) := \{y \in \mathscr{X} : y = weak - \lim_{s_\alpha} \Upsilon_{s_\alpha} x$ for some subnet $\{s_\alpha\} \subset \mathcal{G}\}$.
Then there exists $x_* \in \mathcal{H}$ such that $\Upsilon_s x_* = x_*$.

Proof. Let $x_0 \in \mathcal{H}$. For each $t \in \mathcal{G}$ consider the bounded net $\{\Upsilon_s x_0 : t \leq s \in \mathcal{G}\}$. From Lemma 2.2, there exists $y_t \in \overline{co}\{\Upsilon_s x_0 : t \leq s \in \mathcal{G}\}$ such that

$$\limsup_{s\to\infty} \|\Upsilon_s x_0 - y_t\| \leq \frac{1}{\mathcal{N}(\mathscr{X})} \mathcal{D}(\{\Upsilon_s x_0 : t \leq s \in \mathcal{G}\}). \tag{3.16}$$

Since $\mathscr{X}$ is reflexive, $\{y_t\}$ has a subnet $\{y_{t_l}\}$ which converges weakly to some x_1 for some $x_1 \in \mathscr{X}$. From (3.16) and due to weakly lower semi-continuity of the functional $\limsup_{t\to\infty} \|\Upsilon_t x_0 - y\|$, we obtain

$$\limsup_{t\to\infty} \|\Upsilon_t x_0 - x_1\| \leq \frac{1}{\mathcal{N}(\mathscr{X})} \mathcal{D}(\{\Upsilon_s x_0 : t \leq s \in \mathcal{G}\}). \tag{3.17}$$

Also, it can be easily seen that $x_1 \in \cap_{t\in\mathcal{G}} \overline{co}\{\Upsilon_s x_0 : t \leq s \in \mathcal{G}\}$ such that for all $z \in \mathscr{X}$,

$$\|z - x_1\| \leq \limsup_{t\to\infty} \|z - \Upsilon_t x_0\|. \tag{3.18}$$

Now, using the property ($\mathscr{P}$) and the fact that $\cap_{t\in\mathcal{G}} \overline{co}\{\Upsilon_s x_0 : t \leq s \in \mathcal{G}\} = \overline{co}\{w_w(x_0)\}$ we can conclude that $x_1 \in \mathcal{H}$. On repeating this process we obtain a sequence $\{x_n\}$ in $\mathcal{H}$ such that

$$\textbf{a. } \limsup_{t\to\infty} \|\Upsilon_t x_n - x_{n+1}\| \leq \frac{1}{\mathcal{N}(\mathscr{X})} \mathcal{D}(\{\Upsilon_t x_n : t \in \mathcal{G}\}) \tag{3.19}$$

and

$$\textbf{b. } \|z - x_{n+1}\| \leq \limsup_{t\to\infty} \|z - \Upsilon_t x_n\|. \tag{3.20}$$

Letting $r_n := \limsup_{t\to\infty} \|\Upsilon_t x_n - x_{n+1}\|$, along the same line of proof as in Theorem 3.6 we can get

$$r_n \leq \frac{1}{\mathcal{N}(\mathscr{X})} (k + 3\alpha) d(x_n) \text{ for all } n = 0, 1, 2, \cdots \tag{3.21}$$

and for all $s \geq h + j$,

$$\|\Upsilon_s x_n - \frac{1}{2}(x_{n+1} + \Upsilon_j x_{n+1})\| \leq \gamma(r_n + \epsilon) \left(1 - \delta_{\mathscr{X}} \left(\frac{d(x_{n+1}) - \epsilon}{\gamma(r_n + \epsilon)}\right)\right)$$

where, $\gamma = \frac{k+3\alpha}{1-\alpha}$.

For $z = \frac{1}{2}(x_{n+1} + \Upsilon_j x_{n+1}) \in \mathscr{X}$ we have from equation (3.20),

$$\begin{aligned} \tfrac{1}{2}(d(x_{n+1}) - \epsilon) &< \left\|\tfrac{1}{2}(x_{n+1} + \Upsilon_j x_{n+1})\right\| \\ &\leq \limsup_{t\to\infty} \left\|\Upsilon_t x_n - \tfrac{1}{2}(x_{n+1} + \Upsilon_j x_{n+1})\right\| \\ &\leq \gamma(r_n + \epsilon) \left(1 - \delta_{\mathscr{X}} \left(\tfrac{d(x_{n+1}) - \epsilon}{\gamma(r_n + \epsilon)}\right)\right). \end{aligned}$$

Passing the limit $\epsilon \to 0$ we get,

$$\frac{1}{2} d(x_{n+1}) \leq \gamma r_n \left(1 - \delta_{\mathscr{X}} \left(\frac{d(x_{n+1})}{\gamma r_n}\right)\right). \tag{3.22}$$

Again from equation (3.18), it can be easily seen that for each $j \in \mathcal{G}$,

$$\|\Upsilon_j x_{n+1} - x_{n+1}\| \le \limsup_{t\to\infty} \|\Upsilon_j x_{n+1} - \Upsilon_t x_{n+1}\| \le \gamma r_n$$

which yields

$$d(x_{n+1}) \le \gamma r_n. \tag{3.23}$$

Now combining equations (3.22) and (3.23) and using the definition of μ_0, we derive that $\dfrac{\gamma r_n}{d(x_{n+1})} \ge \mu_0$. Hence from (3.21),

$$d(x_{n+1}) \le \frac{\gamma}{\mu_0} r_n \le \frac{\gamma^2}{\mu_0 \mathcal{N}(\mathscr{X})} d(x_n).$$

Setting $\mathcal{A} = \dfrac{\gamma^2}{\mu_0 \mathcal{N}(\mathscr{X})} < 1$(by assumption) we deduce that

$$d(x_n) \le \mathcal{A} d(x_{n-1}) \le \cdots \le \mathcal{A}^n d(x_0)$$

and from the fact that $\|x_{n+1} - x_n\| \le \left(\dfrac{k+3\alpha}{\mathcal{N}(\mathscr{X})} + 1\right) \mathcal{A}^n d(x_0)$ [see (3.12)] which implies that $\sum_{n=1}^{\infty} \|x_{n+1} - x_n\|$ is convergent which in turns $\{x_n\}$ converges strongly and let $\lim_{n\to\infty} x_n = x_*$, for some $x_* \in \mathcal{H}$. Now for each $t \in \mathcal{G}$ we have,

$$\begin{aligned} \|x_* - \Upsilon_t x_*\| &\le \|x_* - x_n\| + \|x_n - \Upsilon_t x_n\| + \|\Upsilon_t x_n - \Upsilon_t x_*\| \\ &\le \|x_* - x_n\| + \|x_n - \Upsilon_t x_n\| + k\|x_n - x_*\| + \alpha\left(\|x_n - \Upsilon_t x_n\| + \|x_* - \Upsilon_t x_*\|\right) \end{aligned}$$

which yields,

$$(1-\alpha)\|x_* - \Upsilon_t x_*\| \le (1+k)\|x_n - x_*\| + (1+\alpha) d(x_n) \to 0 \text{ as } n \to \infty. \tag{3.24}$$

Consequently, $\Upsilon_t x_* = x_*$ for all $t \in \mathcal{G}$. □

Example 3.2. Considering the example as in Example 3.1, and letting $\mathcal{H} = \mathcal{C}$, we have $\mathscr{X}$ is the enriched Kannan type semigroup defined on $\mathcal{C}$.

Now,

$$\begin{aligned} \mu_0 &= \inf\{\mu \ge 1 : \mu\left(1 - \delta_{\mathscr{X}}(\tfrac{1}{\mu})\right) \ge \tfrac{1}{2}\} \\ &= \inf\{\mu \ge 1 : \mu\left(1 - \tfrac{1}{4\mu^2}\right)^{1/2} \ge \tfrac{1}{2}\} \\ &= 1. \end{aligned}$$

Then, $\frac{k+3\alpha}{1-\alpha} = 1 < \sqrt{2} = \mu_0 \mathcal{N}(\mathscr{X})$. Also it is easy to see that, weak ω-limit of $\mathscr{F}$ at f is $w_\omega(f) = \{0_{function}\} \subset \mathcal{H}$. Thus all the conditions of Theorem 3.7 are satisfied and $0_{function}$ is the common fixed point of the semigroup $\mathscr{F}$.

Conclusion. In this paper, we consider the non-linear semigroup of *enriched Kannan type* contractive mappings over real Banach spaces and prove the existence of common fixed point of this semigroup under certain assumptions on the underlying space and the mappings. Examples are given to illustrate the feasibility of our fixed point theorems. In future, one can find several results in this connection by considering different types of contractive and non-expansive mappings like *enriched Chatterjea type* mappings. Also one can try to prove our theorems by using some weaker conditions.

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