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Dedicated to the memory of Academician Mitrofan M. Choban (1942-2021)

Convergence estimates for abstract second order differential equations with two small parameters and Lipschitzian nonlinearities

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ABSTRACT. In a real Hilbert space H we consider the following singularly perturbed Cauchy problem

$$\begin{cases} \varepsilon u''_{\varepsilon\delta}(t) + \delta\, u'_{\varepsilon\delta}(t) + Au_{\varepsilon\delta}(t) + B\big(u_{\varepsilon\delta}(t)\big) = f(t), \quad t \in (0,T), \\ u_{\varepsilon\delta}(0) = u_0, \quad u'_{\varepsilon\delta}(0) = u_1, \end{cases}$$

where $u_0, u_1 \in H$, $f : [0,T] \mapsto H$, ε, δ are two small parameters, A is a linear self-adjoint operator and B is a nonlinear $A^{1/2}$ Lipschitzian operator.

We study the behavior of solutions $u_{\varepsilon\delta}$ in two different cases: $\varepsilon \to 0$ and $\delta \geq \delta_0 > 0$; $\varepsilon \to 0$ and $\delta \to 0$, relative to solution to the corresponding unperturbed problem.

We obtain some *a priori* estimates of solutions to the perturbed problem, which are uniform with respect to parameters, and a relationship between solutions to both problems. We establish that the solution to the unperturbed problem has a singular behavior, relative to the parameters, in the neighbourhood of $t = 0$.

1. INTRODUCTION

Let H be a real Hilbert space endowed with the scalar product $(\cdot,\cdot)$ and the norm $|\cdot|$, and V be a real Hilbert space endowed with the norm $||\cdot||$. Let $A : V \subset H \to H$, be a linear self-adjoint operator and B is nonlinear $A^{1/2}$ Lipschitzian opeartor. Consider the following Cauchy problem:

$$\begin{cases} \varepsilon u''_{\varepsilon\delta}(t) + \delta\, u'_{\varepsilon\delta}(t) + Au_{\varepsilon\delta}(t) + B\big(u_{\varepsilon\delta}(t)\big) = f(t), \quad t \in (0,T), \\ u_{\varepsilon\delta}(0) = u_0, \quad u'_{\varepsilon\delta}(0) = u_1, \end{cases} \tag{$P_{\varepsilon\delta}$}$$

where $u_0, u_1, f : [0,T] \to H$ and ε, δ are two small parameters. We investigate the behavior of solutions $u_{\varepsilon\delta}$ to the problem $(P_{\varepsilon\delta})$ in two different cases:
(i) $\varepsilon \to 0$ and $\delta \geq \delta_0 > 0$, relative to the solutions to the following unperturbed system:

$$\begin{cases} \delta l'_\delta(t) + Al_\delta(t) + B\big(l_\delta(t)\big) = f(t), \quad t \in (0,T), \\ l_\delta(0) = u_0; \end{cases} \tag{P_δ}$$

(ii) $\varepsilon \to 0$ and $\delta \to 0$, relative to the solutions to the following unperturbed system:

$$Av(t) + B\big(v(t)\big) = f(t), \quad t \in [0,T), \tag{P_0}$$

The problem $(P_{\varepsilon\delta})$ is the abstract model of singularly perturbed problems of hyperbolic-parabolic type in the case (i) and of the hyperbolic-parabolic-elliptic type in the case (ii). Such kind of problems arise in the mathematical modeling of elasto-plasticity phenomena. These abstract results can be applied to singularly perturbed problems of hyperbolic-parabolic-elliptic type with stationary part defined by strongly elliptic operators.

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A large class of works is dedicated to the study of singularly perturbed Cauchy problems for linear or nonlinear differential equations of second order of type $(P_{\varepsilon\delta})$. Without pretending to make a complete analysis we will mention some of these works.

Under various restrictions, in [2]-[7], [12], [15], [16], [17], [33] convergence estimates for linear hyperbolic-parabolic singular perturbations were obtained.

The nonlinear problems of hyperbolic-parabolic type have been analyzed in [13], [32], [18]-[20].

Under some assumptions, closely related to those we use in this article, in [21] and [22] the author has been analyzed the behavior of solutions to the semi-linear second order Cauchy problems with a small parameter.

Some special cases of hyperbolic-parabolic singular perturbations for nonlinear equations of Kirchhoff type with a small parameter in front of the second-order time-derivative were studied in [8] -[11] and [14].

In most of the mentioned cases the results were obtained by using the theory of semigroups of linear operators. Different to other methods, our approach is based on two key points. The first one is the relationship between solutions to the Cauchy problem for the abstract linear second order differential equation and the corresponding problem for the first order equation. The second key point are *a priori* estimates of solutions, which are uniform with respect to the small parameter. Moreover, we study the problem $(P_{\varepsilon\delta})$ for a larger class of functions, i. e. $f \in W^{1,p}(0,T;H)$. Also we obtain the convergence rate, as $\varepsilon \to 0$, which depends on p. Using these specific techniques in our previews works we managed to obtain convergence estimates for abstract second order differential equations with one parameter and: linear operators [23], depending on time linear operators [27], Lipschitzian nonlinearities [25], monotone nonlinearities [26]; two small parameters and: linear operators [28], [29], depending on time linear operators [30], monotone nonlinearities [31].

The organization of this paper is as follows. In the next section the theorems of existence and uniqueness of solutions to the problems $(P_{\varepsilon\delta})$, (P_δ) and some *a priori* estimates of these solutions are presented. In the section 3 we present a relationship between solutions to the problem for the abstract linear second order differential equation and the corresponding solution to the problem for the first order equation. The main result of this paper is established in the section 4. More precisely, we prove the convergence estimates of the difference of solutions to the problems $(P_{\varepsilon\delta})$ and (P_δ) for $\varepsilon \to 0$, $\delta \geq \delta_0 > 0$ and also to the problems $(P_{\varepsilon\delta})$ and (P_0) for $\varepsilon \to 0$, $\delta \to 0$.

For $k \in N^*$, $1 \leq p \leq +\infty$, $(a,b) \subset (-\infty,+\infty)$ and the Banach space X, by $W^{k,p}(a,b;X)$ we denote the Banach space of vectorial distributions $u \in D'(a,b;X)$, $u^{(j)} \in L^p(a,b;X)$, $j = 0,1,\dots,k$, endowed with the norm

$$\|u\|_{W^{k,p}(a,b;X)} = \begin{cases} \left(\sum_{j=0}^{k} \|u^{(j)}\|^p_{L^p(a,b;X)}\right)^{\frac{1}{p}} & \text{for} \quad p \in [1,\infty), \\ \|u\|_{W^{k,\infty}(a,b;X)} = \max\limits_{0\leq j\leq k} \|u^{(j)}\|_{L^\infty(a,b;X)} & \text{for} \quad p = \infty. \end{cases}$$

In the particular case $p = 2$ we put $W^{k,2}(a,b;X) = H^k(a,b;X)$.

If X is a Hilbert space, then $H^k(a,b;X)$ is also a Hilbert space with the scalar product

$$(u,v)_{H^k(a,b;X)} = \sum_{j=0}^{k} \int_a^b \left(u^{(j)}(t), v^{(j)}(t)\right)_X dt.$$

The framework of our paper will be determined by the following conditions:

(HA) *$V \subset H$ densely and continuously, i. e. $|u|^2 \leq \omega_0||u||^2, \quad \forall u \in V$. The operator $A : V \subset H \to H$ is linear, self-adjoint and positive definite, i.e. there exists $\omega > 0$ such that*

$$(Au, u) \geq \omega\, |u|^2, \quad \forall u \in V;$$

(HB) *The operator $B : D(B) \subseteq H \mapsto H$ is $A^{1/2}$ lipschitzian, i. e. $D(A^{1/2}) \subset D(B)$ and there exists $L > 0$ such that*

$$|B(u) - B(v)| \leq L\, |A^{1/2}(u - v)|, \quad \forall u, v \in D(A^{1/2}).$$

2. Solvability of problems $P_{\varepsilon\delta}$ and P_δ and *a priori* estimates for their solutions

In this section we will remind the results about the solvability of problems $(P_{\varepsilon\delta})$ and (P_δ) and also about the regularity of their solutions. Also we will prove an a priori estimates for the solutions of these problems. The following theorems were inspired by the work [1] and are completely proved in the work [24]

Theorem 2.1. *Let $T \in (0, \infty]$. Let us assume that the operators A and B satisfy conditions* **(HA)**, **(HB)**. *If $u_0 \in D(A)$, $u_1 \in D(A^{1/2})$ and $f \in W^{1,1}(0, T; H)$, then there exists a unique function $u \in W^{2,\infty}_\gamma(0, T; H)$, $A^{1/2}u \in W^{1,\infty}_\gamma(0, T; H)$ and $Au \in L^\infty_\gamma(0, T; H)$, with some γ depending on L and ω, such that u satisfies the equation*

$$u''(t) + u'(t) + Au(t) + B\big(u(t)\big) = f(t), \quad t \in (0, T), \tag{2.1}$$

in the sense of distributions and also the initial conditions

$$u(0) = u_0, \quad u'(0) = u_1. \tag{2.2}$$

The function $t \in [0, T) \to u'(t)$ is differentiable on the right and

$$\frac{d^+u'}{dt}(t_0) = f(t_0) - Au(t_0) - B\big(u(t_0)\big) - u'(t_0), \quad t_0 \in [0, T).$$

The function $t \in [0, T] \to Au(t)$ is weakly continuous in H and

$$\frac{d}{dt}\big(Au(t), u(t)\big) = 2\big(Au(t), u'(t)\big), \quad t \in [0, T].$$

This function is called the strong solution to the problem (2.1), (2.2).

Theorem 2.2. *Let $T \in (0, \infty]$. Let us assume that the operators A and B satisfy conditions* **(HA)**, **(HB)**. *If $v_0 \in D(A)$, $f \in W^{1,1}(0, T; H)$, then there exists a unique function $v \in W^{1,2}_\gamma(0, T; V)$, $\quad A^{1/2}v \in W^{1,2}_\gamma(0, T; H)$, with some γ depending on L and ω, such that v satisfies the equation*

$$\frac{d^+v}{dt}(t) + Av(t) + B\big(v(t)\big) = f(t), \quad t \in [0, T), \quad in \quad H \tag{2.3}$$

and the initial condition

$$v(0) = v_0. \tag{2.4}$$

This function is called the strong solution to the problem (2.3), (2.4).

For the further consideration we rewrite the problems $(P_{\varepsilon\delta})$ and (P_δ) in the form:

$$\begin{cases} \mu\, U''_\mu(s) + U'_\mu(s) + AU_\mu(s) + B\big(U_\mu(s)\big) = F(s), \quad s \in (0, T/\delta), \\ U_\mu(0) = u_0, \quad U'_\mu(0) = \delta u_1, \end{cases} \tag{$\mathcal{P}_\mu$}$$

and

$$\begin{cases} \mathcal{L}'(s) + A\mathcal{L}(s) + B\big(\mathcal{L}(s)\big) = F(s), \quad s \in (0, T/\delta), \\ \mathcal{L}(0) = u_0, \end{cases} \tag{$\mathcal{P}_0$}$$

where $U_\mu(s) = u_{\varepsilon\delta}(\delta\, s)$, $\mathcal{L}(s) = l_\delta(s\delta)$, $F(s) = f(s\delta)$ and $\mu = \varepsilon/\delta^2$.

In what follows we will prove an *a priori* estimates for solutions to the problems $(\mathcal{P}_\mu)$ and $(\mathcal{P}_0)$. These estimates play a key role in determining the behavior of solutions to the problem $(P_{\varepsilon\delta})$ as $\varepsilon \to 0$ and $\delta \to 0$. To this end, we give two lemmas of the Gronwall-Belman type.

Lemma 2.1. *Let $\psi \in L^1(a,b)$ $(-\infty < a < b < \infty)$ with $\psi(s) \geq 0$ a. e. on (a,b) and let $f \in C([a,b])$ such that $|f|$ is non-decreasing function. If $h \in C([a,b])$, $h(x) \geq 0$ verifies*

$$h^2(t) \leq f^2(t) + 2\int_a^t \psi(s)h(s)ds, \quad \forall t \in [a,b],$$

then

$$|h(t)| \leq |f(t)| + \int_a^t \psi(s)ds, \quad \forall t \in [a,b]$$

also holds.

Lemma 2.2. *Let $\psi \in L^1(a,b)$ $(-\infty < a < b < \infty)$ with $\psi(s) \geq 0$ a. e. on (a,b) and let $f \in C([a,b])$ such that $f \geq 0$ is non-decreasing function. If $h \in C([a,b])$, $h(x) \geq 0$ verifies*

$$h(t) \leq f(t) + \int_a^t \psi(s)h(s)ds, \quad \forall t \in [a,b],$$

then

$$h(t) \leq f(t)\ exp\Big\{\int_a^t h(s)\, ds\Big\}, \quad \forall t \in [a,b]$$

also holds.

In what follows, we will need also the following lemma.

Lemma 2.3. *Suppose that $v,\, z,\, h : [a,b] \subset \mathbb{R} \to \mathbb{R}$, $v \in C([a,b])$, $z \in L^2(a,b)$, $h \in L^1(a,b)$, $v(t) \geq 0$ for $t \in [a,b]$ and $z(t) \geq 0$, $h(t) \geq 0$, a. e. $t \in (a,b)$. If*

$$v(t) + \Big(\int_{t_0}^t z^2(s)\, ds\Big)^{1/2} \leq$$

$$\leq c_0\Big(v(t_0) + \int_{t_0}^t h(s)\, ds\Big) + c_1\int_{t_0}^t z(s)\, ds, \quad \forall t_0,\, t \in [a,b], \quad t > t_0, \tag{2.5}$$

with $c_0 \geq 1,\ c_1 > 0$, then for any $\alpha \in (0,1)$ the inequality

$$v(t) + \big(1-\alpha\big)\Big(\int_a^t z^2(s)\, ds\Big)^{1/2} \leq c_0\, e^{\tilde{\gamma}\,(t-a)}\Big[v(a) + \int_a^t h(s)ds\Big], \quad t \in [a,b], \tag{2.6}$$

is true with $\tilde{\gamma} = c_1^2\,\alpha^{-2}\,\ln c_0$.

Proof. From (2.5) it follows that

$$v(t) + \big[1 - c_1\,(t-t_0)^{1/2}\big]\Big(\int_{t_0}^t z^2(s)\, ds\Big)^{1/2} \leq c_0\Big[v(t_0) + \int_{t_0}^t h(s)\, ds\Big], \quad t > t_0 \geq 0. \tag{2.7}$$

Let $q=\alpha^2\,c_1^{-2}$ with $a\in(0,1)$ and $t_k=a+k\,q,\ \ k\in\mathbb{N}$. Denoting by $m\big(t_k\big)=-(1-\alpha)\Big(\int\limits_{t_{k-1}}^{t_k} z^2(s)\,ds\Big)^{1/2}$ and putiing in (2.7) $t_0=t_{k-1}$ and $t=t_k$, we get

$$v(t_k)\le m\big(t_k\big)+c_0\Big[v(t_{k-1})+\int\limits_{t_{k-1}}^{t_k} h(s)\,ds\Big].$$

From this inequality, we deduce

$$v(t_k)\le c_0^k\,v(a)+\sum_{j=0}^{k-1}c_0^{k-j}\int\limits_{t_j}^{t_{j+1}} h(s)\,ds+\sum_{j=1}^{k}c_0^{k-j}\,m(t_j). \tag{2.8}$$

Let $t\in(a,b)$ be arbitrary. There exists $k\in\mathbb{N}$, such that $a+k\,q<t\le a+(k+1)\,q$. Putting in (2.7) $t_0=t_k$ and taking into account (2.8), we obtain

$$v(t)+(1-\alpha)\Big[\Big(\int\limits_{t_k}^{t} z^2(s)\,ds\Big)^{1/2}+\sum_{j=1}^{k}c_0^{k-j+1}\Big(\int\limits_{t_{j-1}}^{t_j} z^2(s)\,ds\Big)^{1/2}\Big]\le$$

$$\le c_0\Big[c_0^k\,v(a)+\int\limits_{t_k}^{t} h(s)\,ds+\sum_{j=0}^{k-1}c_0^{k-j}\int\limits_{t_j}^{t_{j+1}} h(s)\,ds\Big],\quad t\in(a+k\,q,a+(k+1)\,q].$$

Since $1-\alpha>0$ and $\sqrt{k+1}\,\Big(\sum\limits_{j=1}^{k+1}a_j\Big)^{1/2}\le\sum\limits_{j=1}^{k+1}\sqrt{a_j}$ for $a_j\ge0$, then

$$v(t)+\sqrt{k+1}\,\big(1-\alpha\big)\Big[\int\limits_{t_k}^{t} z^2(s)\,ds+\sum_{j=1}^{k}c_0^{2(k-j+1)}\int\limits_{t_{j-1}}^{t_j} z^2(s)\,ds\Big]^{1/2}\le$$

$$\le c_0\Big[c_0^k\,v(a)+\int\limits_{t_k}^{t} h(s)\,ds+\sum_{j=0}^{k-1}c_0^{k-j}\int\limits_{t_j}^{t_{j+1}} h(s)\,ds\Big],\quad t\in(a+k\,q,a+(k+1)\,q]. \tag{2.9}$$

As $c_0\ge1$, then (2.9) implies the inequality

$$v(t)+\big(1-\alpha\big)\Big(\int\limits_{a}^{t} z^2(s)\,ds\Big)^{1/2}\le c_0^{k+1}\Big[v(a)+\int\limits_{a}^{t} h(s)ds\Big],\quad t\in[a,b]. \tag{2.10}$$

Since $c_0^k=e^{k\,\ln c_0}\le e^{\big(c_1^2\,\alpha^{-2}\,\ln c_0\big)\,(t-a)}$, from (2.10) we obtain (2.6). □

Lemma 2.4. *Suppose that the operators A and B satisfy conditions* **(HA)**, **(HB)**, $u_0\in D(A)$, $u_1\in D(A^{1/2})$, $F\in W^{1,1}(0,\infty;H)$ *and* $\alpha\in(0,1)$. *Then for any strong solution U_μ to the problem ($\mathcal{P}_\mu$) the following estimate*

$$\mu\,\|U_\mu''\|_{L^\infty(0,s:H)}+\|U_\mu'\|_{C([0,s]:H)}+(1-\alpha)\,\|A^{1/2}U_\mu'\|_{L^2(0,s:H}\le$$

$$\le C(L,\omega)\,\mathcal{M}_0(s)\,e^{\gamma_0\,s},\quad s\ge0,\quad \mu,\delta\in(0,1], \tag{2.11}$$

holds with

$$\mathcal{M}_0(s)=|Au_0|+|A^{1/2}u_1|+|F(0)-B(0)|+||F||_{W^{1,1}(0,s;H)},\gamma_0=\frac{L^2\,\ln2}{\alpha^2}. \tag{2.12}$$

If $L < \sqrt{\omega}$ *and* $0 < \mu \le \mu_0 = \dfrac{\sqrt{\omega} - L}{2\sqrt{\omega}\, L^2}$, *then the estimate*

$$\mu \, \|U''_\mu\|_{L^\infty(0,s:H)} + \|U'_\mu\|_{C([0,s]:H)} + \|A^{1/2} U'_\mu\|_{L^2(0,s:H)} \le C(L,\omega)\, \mathcal{M}_0(s), s \ge 0, \tag{2.13}$$

is true for $\mu, \delta \in (0,1]$.

Proof. Proof of the estimate (2.11). Let $U_{\mu h}(s) = U_\mu(s+h) - U_\mu(s)$ and denote by

$$E\big(U_{\mu h}, s\big) = \big|U_{\mu h}(s)\big|^2 + \big|U_{\mu h}(s) + 2\,\mu\, U'_{\mu h}(s)\big|^2 + 4\,\mu\, \big(AU_{\mu h}(s), U_{\mu h}(s)\big) +$$

$$+4\,\mu \int_{s_0}^{s} \big|U'_{\mu h}(\tau)\big|^2 \, d\tau + 4 \int_{s_0}^{s} \big(AU_{\mu h}(\tau), U_\mu(\tau)\big)\, d\tau. \tag{2.14}$$

If U_μ is strong solution to the problem $(\mathcal{P}_\mu)$, then

$$\frac{d}{ds} E\big(U_{\mu h}, s\big) = 4 \Big(F_h(s) - \Big(B\big(U_\mu(s)\big)\Big)_h, U_{\mu h}(s) + 2\,\mu\, U'_{\mu h}(s) \Big), \quad s \ge 0.$$

Since, according to condition **(HB)**, $\big|\big(B(U_\mu(s)\big)_h\big| \le L\, \big|A^{1/2} U_{\mu h}(s)\big|$, then integrating the last equality on $[s_0, s) \subset (0, \infty)$, we get

$$E(U_{\mu h}, s) \le E(U_{\mu h}, s_0) + 4 \int_{s_0}^{s} \Big(\big|F_h(\tau\big| + L\, \big|A^{1/2} U_{\mu h}(\tau)\big| \Big) \big|U_{\mu h}(\tau) + 2\,\mu\, U'_{\mu h}(\tau)\big|\, d\tau \le$$

$$\le E(U_{\mu h}, s_0) + 4 \int_{s_0}^{s} \Big(\big|F_h(\tau\big| + L\, \big|A^{1/2} U_{\mu h}(\tau)\big| \Big) E^{1/2}(U_{\mu h}, \tau)\, d\tau, s > s_0 \ge 0. \tag{2.15}$$

Denoting by

$v(s) = \big[\big|U_{\mu h}(s)\big|^2 + \big|U_{\mu h}(s) + 2\,\mu\, U'_{\mu h}(s)\big|^2 + 4\,\mu\, \big(AU_{\mu h}(s), U_{\mu h}(s)\big)\big]^{1/2}, z(s) = 2\, \big|A^{1/2} U_{\mu h}(s)\big|$

and using Lemma 2.1, from (2.15) we obtain the inequality

$$\Big(v^2(s) + \int_{s_0}^{s} z^2(\tau)\, d\tau \Big)^{1/2} \le v(s_0) + \int_{s_0}^{s} \big[2\, |F_h(\tau)| + L\, |z(\tau)| \big]\, d\tau, \quad s > s_0 \ge 0.$$

As $\sqrt{a} + \sqrt{b} \le \sqrt{2}\, \sqrt{a+b}$ for $a, b \ge 0$, then

$$v(s) + \Big(\int_{s_0}^{s} z^2(\tau)\, d\tau \Big)^{1/2} \le \sqrt{2}\, \Big[v(s_0) + \int_{s_0}^{s} \big[2\, |F_h(\tau)| + L\, |z(\tau)| \big]\, d\tau \Big], \quad s > s_0 \ge 0.$$

Taking $c_0 = \sqrt{2}$, $c_1 = \sqrt{2}\, L, a = 0$, $h(s) = 2\, |F_h(s)|$ and applying Lemma 2.3 to the last inequality, we get

$$\big|U_{\mu h}(s)\big| + \big|U_{\mu h}(s) + 2\,\mu\, U'_{\mu h}(s)\big| + (1-\alpha) \Big(\int_0^s \big|A^{1/2} U_{\mu h}(\tau)\big|^2\, d\tau \Big)^{1/2} \le$$

$$\le C\, e^{\gamma_0 s} \Big[E^{1/2}(U_{\mu h}, 0) + \int_0^s |F_h(\tau)|\, d\tau \Big], \quad s \ge 0. \tag{2.16}$$

with γ_0 from (2.12).

Under the conditions of Lemma, due to the Theorem 2.1, we have that $U'_\mu \in C([0,T];H)$, $|A^{1/2}U_\mu| \in C([0,T])$. Therefore, the following relations

$$\begin{cases} |h^{-1}U_{\mu h}(s)| \to |U'_\mu(s)|, \quad h \to 0, \\ |h^{-1}U_{\mu h}(s) + 2\,\mu\, h^{-1}U'_{\mu h}| \to |U'_\mu(s) + 2\,\mu\, U''_\mu(s)|, \quad h \downarrow 0, \\ \big(h^{-1}\, AU_{\mu h}(s), h^{-1}\, U_{\mu,h}(s)\big) \to \big|A^{1/2}U'_\mu(s)\big|^2, \quad h \downarrow 0, \\ h^{-2}E(U_{\mu h}, 0) \to |\delta\, u_1|^2 + \\ +\big|2\,\big(F(0) - Au_0 - B(u_0)\big) - \delta\, u_1\big|^2 + 4\,\mu\,\delta^2\big|A^{1/2}u_1\big|^2, h \downarrow 0 \end{cases} \tag{2.17}$$

hold. Taking into account the relations (2.17), we divide (2.16) by h and then pass to the limit in the obtained inequality, to get the estimate (2.11).

Proof of the estimate (2.13). Let $L < \omega$. In this case

$$\mu\,\big|U'_{\mu h}\big|^2 + \big|A^{1/2}U_{\mu h}\big|^2 - L\,\big|A^{1/2}U_{\mu h}\big|\,\big|U_{\mu h} + 2\,\mu\, U'_{\mu h}\big| \geq$$

$$\geq \mu\,\big|U'_{\mu h}\big|^2 + \Big[1 - \frac{L}{\sqrt{\omega}}\Big]\,\big|A^{1/2}U_{\mu h}\big|^2 - 2\,\mu\, L\,\big|A^{1/2}U_{\mu h}\big|\,\big|U'_{\mu h}\big| =$$

$$= \mu\,\big[\big|U'_{\mu h}\big| - L\,\big|A^{1/2}U_{\mu h}\big|\big]^2 + \nu_0\,\big|A^{1/2}U_{\mu h}\big|^2, \quad \nu_0 = 1 - L\,\big(\omega^{-1/2} + \mu\, L\big), \quad \mu \in (0,\mu_0],$$

Because $\nu_0 \geq q_0 = \dfrac{1}{2}\Big(1 - \dfrac{L}{\sqrt{\omega}}\Big) > 0$ for $\mu \in (0,\mu_0]$, then from (2.15) we have that

$$\big|U_{\mu h}(s)\big|^2 + \big|U_{\mu h}(s) + 2\,\mu\, U'_{\mu h}(s)\big|^2 + 4\,\mu\big(AU_{\mu h}(s), U_{\mu h}(s)\big) + 4\,q_0 \int\limits_0^s \big(AU_{\mu h}(\tau), U_\mu(\tau)\big)\, d\tau \leq$$

$$\leq E(U_{\mu h}, 0) + 4 \int\limits_0^s \big[|F_h(\tau)|\,\big|U_{\mu h}(\tau) + 2\,\mu\, U'_{\mu h}(\tau)|\big]\, d\tau, \quad s \geq 0.$$

Consequently, taking

$$H_h^2(s) = \big|U_{\mu h}(s)\big|^2 + \big|U_{\mu h}(s) + 2\,\mu\, U'_{\mu h}(s)\big|^2 + 4\,q_0 \int\limits_0^s \big(AU_{\mu h}(\tau), U_{\mu h}(\tau)\big)\, d\tau$$

and applying Lemma 2.1 to the last inequality, we get the estimate

$$H_h(s) \leq E^{1/2}(U_{\mu h}, 0) + 2 \int\limits_0^s |F_h(\tau)|\, d\tau, \quad s \geq 0, \mu \in (0,\mu_0]. \tag{2.18}$$

In such thatrtue of relations (2.17), dividing the inequality (2.18) by h and then passing to the limit in the obtained inequality as $h \to 0$ we get the estimate (2.13). □

3. Relationship between solutions to the problems $(\mathcal{P}_\mu)$ and $(\mathcal{P}_0)$ in the linear case.

In what follows for $\mu > 0$ denote by

$$K(t,\tau,\mu) = \frac{1}{2\sqrt{\pi\mu}}\Big(K_1(t,\tau,\mu) + 3K_2(t,\tau,\mu) - 2K_3(t,\tau,\mu)\Big), \quad \forall \mu > 0,$$

where

$$K_1(t,\tau,\mu) = \exp\Big\{\frac{3t - 2\tau}{4\mu}\Big\}\lambda\Big(\frac{2t - \tau}{2\sqrt{\mu t}}\Big), K_2(t,\tau,\mu) = \exp\Big\{\frac{3t + 6\tau}{4\mu}\Big\}\lambda\Big(\frac{2t + \tau}{2\sqrt{\mu t}}\Big),$$

$$K_3(t,\tau,\mu) = \exp\Big\{\frac{\tau}{\mu}\Big\}\lambda\Big(\frac{t + \tau}{2\sqrt{\mu t}}\Big), \quad \lambda(s) = \int_s^\infty e^{-\eta^2}\, d\eta.$$

The properties of kernel $K(t,\tau,\mu)$ are collected in the following lemma.

Lemma 3.5. [23] *The function $K(t,\tau,\mu)$ possesses the following properties:*

(i) $K(t,\tau,\mu)>0, \forall t\geq 0, \forall \tau \geq 0;$

(ii) $\displaystyle\int_0^\infty K(t,\tau,\mu)\,d\tau = 1, \forall t\geq 0;$

(iii) *Let $q\in[0,1]$. Then* $\displaystyle\int_0^\infty K(t,\tau,\mu)\,|t-\tau|^q\,d\tau \leq C\left(\mu+\sqrt{\mu\, t}\right)^q,$ $\forall \mu>0, \forall t\geq 0;$

(iv) *Let $p\in(1,\infty]$ and $f:\,[0,\,\infty)\rightarrow H$, $f(t)\in W^{1,p}(0,\infty;H)$. Then*

$$\Big|f(t)-\int_0^\infty K(t,\tau,\mu)f(\tau)d\tau\Big| \leq C(p)\,\|f'\|_{L^p(0,\infty;H)}\left(\mu+\sqrt{\mu\, t}\right)^{\frac{p-1}{p}}, \forall \mu>0, \forall t\geq 0;$$

(v) *If $0<2\,\gamma\,\mu<1$, then*

$$\int_0^\infty K(t,\tau,\mu)\,\tau^k\,e^{\gamma\,\tau}\,d\tau \leq \frac{C(\gamma)\,(\mu+t)}{(1-2\,\gamma\mu)^{k+1}}\,e^{\gamma(1+\gamma\mu)t}, \quad t\geq 0, \quad k=1,2.$$

Lemma 3.6. [23] *Let $B=0$. Assume that $A:D(A)\subset H\rightarrow H$ is a linear, self-adjoint, positive definite operator and $F\in L^\infty_\gamma(0,\infty;H)$ for some $\gamma\geq 0$. If U_μ is strong solution to the problem $(\mathcal{P}_\mu)$ with $U_\mu\in W^{2,\infty}_\gamma(0,\infty;H)\cap L^\infty_\gamma(0,\infty;H)$, $AU_\mu\in L^\infty_\gamma(0,\infty;H)$, then for every $0<\mu<(2\gamma)^{-1}$ the function W_μ, defined by $W_\mu(s)=\int_0^\infty K(s,\tau,\mu)\,U_\mu(\tau)\,d\tau$, is the strong solution in H to the problem*

$$\begin{cases} W_\mu'(s)+AW_\mu(s)=F_0(s,\mu), \quad \text{a.e.} \quad s>0, \quad in \quad H,\\ W_\mu(0)=\varphi_\mu, \end{cases}$$

$$F_0(s,\mu)=\frac{1}{\sqrt{\pi}}\Big[2\exp\Big\{\frac{3s}{4\mu}\Big\}\lambda\Big(\sqrt{\frac{s}{\mu}}\Big)-\lambda\Big(\frac{1}{2}\sqrt{\frac{s}{\mu}}\Big)\Big]\delta u_1+\int_0^\infty K(s,\tau,\mu)\,F(\tau)\,d\tau,$$

$$\varphi_\mu=\int_0^\infty e^{-\tau}\,U_\mu(2\,\mu\,\tau)\,d\tau.$$

4. Behaviour of solutions to the problem $(P_{\varepsilon\delta})$

In this section, in the framework of conditions **(HA)** and **(HB)**, we wiil prove the main result about the behavior of the solutions to the problem $(P_{\varepsilon\delta})$, in both cases: $\varepsilon\rightarrow 0$ and $\delta\geq\delta_0>0$; $\varepsilon\rightarrow 0$ and $\delta\rightarrow 0$, relative to solution to the corresponding unperturbed problem.

Theorem 4.3. *Let $T>0$ and $p>1$. Let us assume that the operators A and B satisfy conditions* **(HA)** *and* **(HB)**. *If $u_0\in D(A)$, $u_1\in D(A^{1/2})$ and $f\in W^{1,p}(0,T;H)$, then there exists constant $C=C(T,p,\omega,L)>0$ such that*

(4.19)
$$||u_{\varepsilon\delta}-l_\delta||_{C([0,T];H)}\leq\frac{C\,\mathcal{M}\,e^{\gamma/\delta}\;\Theta(\varepsilon,\delta)}{(1-\alpha)\,(\delta^2-4\,\gamma_0\,\varepsilon)^{3/2}}, \quad \delta\in(0,1],\;\alpha\in(0,1),\;\varepsilon\in\Big(0,\frac{\delta^2}{4\,\gamma_0}\Big),$$

where $u_{\varepsilon\delta}$ abd l_δ are strong solutions to the problems $(P_{\varepsilon\delta})$ and (P_δ), respectively,

$$\mathcal{M}=|Au_0|+|A^{1/2}u_1|+|B(0)|+||f||_{W^{1,p}(0,T;H)},\gamma_0=\frac{L^2\,\ln 2}{\alpha^2},\gamma=\frac{49\,L^2\,T\,\ln 2}{16\,\alpha^2}, \tag{4.20}$$

$$(4.21) \qquad \Theta(\varepsilon,\delta)=\begin{cases}\dfrac{\varepsilon^{1/4}}{\delta^{(p+1)/p}} & if \quad p\geq 2,\\[2ex] \dfrac{\varepsilon^{(p-1)/2p}}{\delta^{3/2}} & in \quad p\in(1,2).\end{cases}$$

If $f=0$*, then* $\Theta(\delta,\varepsilon)=\dfrac{\varepsilon^{1/4}}{\delta^{(p+1)/p}}$*. If* $B=0$*, then* $\gamma=0$*.*

Proof. During the proof, we will agree to denote by C all constants $C(T,p,\omega,L)$. For any $f\in W^{1,p}(0,T;H)$ let us define the function
$\widetilde{f}:[0,\infty)\mapsto H$ as follows:

$$\widetilde{f}(t)=\begin{cases}f(t), & 0\leq t\leq T;\\ \dfrac{2T-t}{T}f(T), & T<t\leq 2T;\\ 0, & t>2T.\end{cases}$$

Then $\widetilde{f}(t)\in W^{1,p}(0,T;H)$ and

$$(4.22) \qquad \int_0^{2T}\big[|\widetilde{f}(t)|^p+|\widetilde{f}'(t)|^p\big]\,dt=\int_0^{T}\big[|f(t)|^p+|f'(t)|^p\big]\,dt+\frac{|f(T)|^p}{T^{p-1}}\Big[1+\frac{T^p}{p+1}\Big].$$

Since $W^{1,p}(0,T;H)\hookrightarrow C([0,T];H)$ continuously and

$$(4.23) \qquad ||f||_{C([0,T];H)}\leq C(p)\,\frac{\max\{1,T\}}{T^{1/p}}\,||f||_{W^{1,p}(0,T;H)},$$

then from (4.22) we get

$$(4.24) \qquad ||\widetilde{f}||_{W^{1,p}(0,\infty;H)}\leq C(p,T)\,||f||_{W^{1,p}(0,T;H)},\quad C(p,T)=C(p)\,\max\Big\{T,\frac{1}{T}\Big\}.$$

If we denote by $\widetilde{U}_\mu$ the unique strong solution to the problem $(\mathcal{P}_\mu)$, defined on $(0,\infty)$ instead of $(0,S)$ with $S=T/\delta$ and $\widetilde{f}$ instead of f, then, from Theorem 2.1 and Lemma 2.4, it follows that $\tilde{U}_\mu\in W^{2,\infty}_{\gamma_0}(0,\infty;H)\cap W^{1,2}_{\gamma_0}(0,\infty;V)$, $A^{1/2}\tilde{U}_\mu\in L^{\infty}_{\gamma_0}(0,\infty;H)$, $A\tilde{U}_\mu\in L^{\infty}_{\gamma_0}(0,\infty;H)$ with γ_0 from (2.12).

Moreover, for $p\in(1,\infty)$ and $\forall\delta\in(0,1]$, the estimate (4.24) implies

$$(4.25) \qquad ||\widetilde{F}||_{W^{1,p}(0,\infty;H)}\leq C(p,T)\,\delta^{-1/p}\,||f||_{W^{1,p}(0,T;H)}.$$

Due to the estimates (4.24), (4.25) and Lemma 2.4, we obtain the following estimates

$$\big|\big|\tilde{U}_\mu'\big|\big|_{C([0,s];H)}+(1-\alpha)\,\big|\big|A^{1/2}\tilde{U}_\mu'\big|\big|_{L^2(0,s;H)}\leq$$

$$(4.26) \qquad \leq C\,e^{\gamma_0\,s}\,\delta^{-1/p}\,\mathcal{M},\quad s\geq 0,\quad \mu,\,\delta\in(0,1],\quad \alpha\in(0,1).$$

holds.

By Lemma 3.6, the function W_μ, defined by $W_\mu(s)=\int_0^\infty K(s,\tau,\mu)\,\widetilde{U}_\mu(\tau)\,d\tau$, is strong solution to the problem

$$(4.27) \qquad \begin{cases}W_\mu'(s)+AW_\mu(s)=\widetilde{F}_0(s,\mu), & \text{a.e.}\quad s>0,\quad in\quad H,\\ W_\mu(0)=\varphi_\mu,\end{cases}$$

for every $\mu\in(0,(2\,\gamma_0)^{-1})$, where

$$\widetilde{F}_0(s,\mu)=\delta f_0(s,\mu)u_1+\int_0^\infty K(s,\tau,\mu)\,\widetilde{F}(\tau)\,d\tau-\int_0^\infty K(s,\tau,\mu)\,B(\widetilde{U}_\mu(\tau))\,d\tau,$$

$$f_0(s,\mu)=\frac{1}{\sqrt{\pi}}\Big[2\exp\Big\{\frac{3s}{4\mu}\Big\}\lambda\Big(\sqrt{\frac{s}{\mu}}\Big)-\lambda\Big(\frac{1}{2}\sqrt{\frac{s}{\mu}}\Big)\Big],\varphi_\mu=\int_0^\infty e^{-\tau}\,\widetilde{U}_\mu(2\mu\tau)\,d\tau. \tag{4.28}$$

Denote by $R(s,\mu)=\widetilde{\mathcal{L}}(s)-W_\mu(s)$, where $\widetilde{\mathcal{L}}$ is the strong solution to the problem $(\mathcal{P}_0)$ with $\tilde{F}$ instead of F, $T=\infty$ and W_μ is the strong solution to the problem (4.27). Then, due to Theorem 2.2, $R(\cdot,\mu)\in W^{1,\infty}_{loc}(0,\infty;\,H)$ and R is strong solution in H to the problem

$$\begin{cases} R'(s,\mu)+AR(s,\mu)+B(\widetilde{\mathcal{L}}(s))-B(W_\mu(s))=\mathcal{F}(s,\mu), \ \text{a.e. } s>0,\\ R(0,\mu)=u_0-W_\mu(0),\end{cases} \tag{4.29}$$

where

$$\mathcal{F}(s,\mu)=\tilde{F}(s)-\int_0^\infty K(s,\tau,\mu)\tilde{F}(\tau)\,d\tau-\delta\,f_0(s,\mu)\,u_1-$$

$$-B(W_\mu(s))+\int_0^\infty K(s,\tau,\mu)\,B(\widetilde{U}_\mu(\tau))\,d\tau. \tag{4.30}$$

In what follows we need the following two Lemmas, which will be proved after the proof of the Theorem 4.3.

Lemma 4.7. *Assume the conditions of Theorem 4.3 are fulfilled. Then for any $\delta\in(0,1]$ and any $\alpha\in(0,1)$ the following estimates:*

$$|\widetilde{U}_\mu(s)-W_\mu(s)|\le\frac{C\,\mathcal{M}\,\mu^{1/2}}{\delta^{1/p}\,(1-2\,\gamma_0\,\mu)}\,(1+\sqrt{s})\,e^{\gamma_0(1+\mu\gamma_0)s},\,s\ge0,\mu\in\Big(0,\frac{1}{2\,\gamma_0}\Big), \tag{4.31}$$

$$\int_0^\infty K(s,\tau,\mu)\,\big|A^{1/2}\big(\widetilde{U}_\mu(s)-\widetilde{U}_\mu(\tau)\big)\big|\,d\tau\le$$

$$\le\frac{C\,\mathcal{M}\,\mu^{1/4}}{\delta^{1/p}\,(1-\alpha)\,(1-4\,\gamma_0\,\mu)^{1/2}}\,(1+\sqrt{s})^{1/2}\,e^{\gamma_0(1+2\,\mu\gamma_0)s},\,s\ge0,\quad\mu\in\Big(0,\frac{1}{4\,\gamma_0}\Big), \tag{4.32}$$

$$|A^{1/2}\big(\widetilde{U}_\mu(s)-W_\mu(s)\big)|\le$$

$$\le\frac{C\,\mathcal{M}\,\mu^{1/4}}{\delta^{1/p}\,(1-\alpha)\,(1-4\,\gamma_0\,\mu)^{1/2}}\,(1+\sqrt{s})^{1/2}\,e^{\gamma_0(1+2\,\mu\gamma_0)s},\,s\ge0,\mu\in\Big(0,\frac{1}{4\,\gamma_0}\Big), \tag{4.33}$$

$$\int_0^\infty K(s,\tau,\mu)\Big|A^{1/2}\big(\widetilde{U}_\mu(\tau)-W_\mu(\tau)\big)\Big|d\tau\le$$

$$\le\frac{C\mathcal{M}\mu^{1/4}}{\delta^{1/p}\,(1-\alpha)\,(1-4\,\gamma_0\,\mu)^{3/2}}(1+\sqrt{s})^{1/2}e^{\gamma_0\,(1+17\,\mu\gamma_0/4)s},\ s\ge0,\quad\mu\in\Big(0,\frac{1}{4\,\gamma_0}\Big). \tag{4.34}$$

are valid.

Lemma 4.8. *Let the conditions of Theorem 4.3 are fulfilled. Then for the strong solution to the problem (4.29) the following estimate*

$$||R||_{C([0,\,s];\,H)}\le\frac{C\,\mathcal{M}\,(1+\sqrt{s})\,e^{49\,\gamma_0 s/16}\,\mu^\beta}{\delta^{1/p}\,(1-\alpha)\,(1-4\,\gamma_0\mu)^{3/2}},\ s\ge0,\ \delta\in(0,1],\ \alpha\in(0,1),\ \mu\in\Big(0,\frac{1}{4\,\gamma_0}\Big), \tag{4.35}$$

is true with $\beta=\min\Big\{\frac{1}{4},\frac{p-1}{2p}\Big\}$.

Consequently, from (4.31) and (4.35), we deduce
(4.36)

$$||\tilde U_\mu-\tilde L||_{C([0,s];H)}\leq ||\tilde U_\mu-W_\mu||_{C([0,s];H)}+||R||_{C([0,s];H)}\leq \frac{C\,\mathcal M\,(1+\sqrt s)\,e^{49\,\gamma_0 s/16}\,\mu^\beta}{\delta^{1/p}\,(1-\alpha)\,(1-4\,\gamma_0\mu)^{3/2}},$$

$s\geq 0,\ \delta\in(0,1],\ \alpha\in(0,1),\ \mu\in\Big(0,\frac{1}{4\,\gamma_0}\Big)$, Since $U_\mu(s)=\tilde U_\mu(s)$, $L(s)=\tilde L(s)$, for all $s\in[0,T/\delta]$, $U_\mu(s)=u_{\varepsilon\delta}(\delta\,s)$ and $L(s)=l_\delta(\delta\,s)$, then for $t\in[0,T]$ we have

$$|u_{\varepsilon\delta}(t)-l_\delta(t)|=|u_{\varepsilon\delta}(\delta s)-l_\delta(\delta s)|=\big|\tilde U_\mu(s)-\tilde L(s)\big|,\quad s\in\Big[0,\frac{T}{\delta}\Big].$$

Concequently, from (4.36) follows the estimate (4.19). □

Proof of Lemma 4.7. Proof of the estimate (4.31). Using properties (**i**), (**ii**), (**v**) from Lemma 3.5 and (4.26), we get

$$|\widetilde U_\mu(s)-W_\mu(s)|\leq\int_0^\infty K(s,\tau,\mu)\,\big|\widetilde U_\mu(s)-\widetilde U_\mu(\tau)\big|\,d\tau\leq$$

$$\leq\int_0^\infty K(s,\tau,\mu)\Big|\int_\tau^s\big|\widetilde U'_\mu(\xi)\big|\,d\xi\Big|\,d\tau\leq\frac{C\,\mathcal M}{\gamma_0\,\delta^{1/p}}\int_0^\infty K(s,\tau,\mu)\Big|e^{\gamma_0\,\tau}-e^{\gamma_0\,s}\Big|\,d\tau\leq$$

$$\leq\frac{C\,\mathcal M\,\mu^{1/2}}{\delta^{1/p}\,(1-2\,\gamma_0\,\mu)}\,(1+\sqrt s)\,e^{\gamma_0(1+\mu\gamma_0)s},\quad s\geq 0,\quad \delta\in(0,1],\ \mu\in\Big(0,\frac{1}{2\,\gamma_0}\Big).$$

Thus, the estimate (4.31) is proved.

Proof of the estimate (4.32). Using $H\ddot{o}lder's$ inequality, we have

$$\int_0^\infty K(s,\tau,\mu)\,\big|A^{1/2}\big(\widetilde U_\mu(s)-\widetilde U_\mu(\tau)\big)\big|\,d\tau\leq\int_0^\infty K(s,\tau,\mu)\,\Big|\int_\tau^s\big|A^{1/2}\widetilde U'_\mu(\xi)\big|\,d\xi\Big|\,d\tau\leq$$

$$\leq\int_0^\infty K(s,\tau,\mu)\,|s-\tau|^{1/2}\,\Big|\int_\tau^s|A^{1/2}\widetilde U'_\mu(\xi)|^2\,d\xi\Big|^{1/2}\,d\tau\leq$$

$$\leq\Big(\int_0^\infty K(s,\tau,\mu)\,|s-\tau|\,d\tau\Big)^{1/2}\times\Big(\int_0^\infty K(s,\tau,\mu)\,\Big|\int_\tau^s|A^{1/2}\widetilde U'_\mu(\xi)|^2\,d\xi\Big|\Big)^{1/2}\leq$$

$$\leq C\,\mu^{1/4}(1+\sqrt s)^{1/2}\,Q_\mu^{1/2}(s),\tag{4.37}$$

where, due to (4.26) and property **(v)** from Lemma 3.5,

$$Q_\mu(s)=\int_0^\infty K(s,\tau,\mu)\,\Big|\int_\tau^s|A^{1/2}\widetilde U'_\mu(\xi)|^2\,d\xi\Big|\leq$$

$$\leq\int_0^\infty K(s,\tau,\mu)\,\Big[\int_0^s|A^{1/2}\widetilde U'_\mu(\xi)|^2\,d\xi+\int_0^\tau|A^{1/2}\widetilde U'_\mu(\xi)|^2\,d\xi\Big]\,d\tau\leq$$

$$\leq\frac{C\,\mathcal M^2}{\delta^{2/p}\,(1-\alpha)^2}\,\Big[e^{2\,\gamma_0 s}+\int_0^\infty K(s,\mu,\tau)\,e^{2\,\gamma_0\tau}\,d\tau\Big]\leq$$

$$\leq\frac{C\,\mathcal M^2}{\delta^{2/p}\,(1-\alpha)^2}\,\Big[e^{2\,\gamma_0 s}+\frac{1}{1-4\,\gamma_0\mu}\,e^{2\,\gamma_0(1+2\,\gamma_0\mu)s}\Big]\leq\frac{C\,\mathcal M^2\,e^{2\,\gamma_0(1+2\,\gamma_0\mu)s}}{\delta^{2/p}\,(1-\alpha)^2\,(1-4\,\gamma_0\mu)},\tag{4.38}$$

$s\geq 0,\delta\in(0,1],\ \alpha\in(0,1),\ \mu\in\Big(0,\frac{1}{4\,\gamma_0}\Big)$.

Now, (4.37) and (4.38) imply the estimate (4.32).

Proof of the estimate (4.33). Because A is self-adjoint, the operator $A^{1/2}$ is also self-adjoint and, consequently, $A^{1/2}$ is a closed operator. Thus,

$$|A^{1/2}\big(\widetilde U_\mu(s)-W_\mu(s)\big)|\leq\int_0^\infty K(s,\tau,\mu)\,\big|A^{1/2}\big(\widetilde U_\mu(s)-\widetilde U_\mu(\tau)\big)\big|\,d\tau\tag{4.39}$$

and the estimate (4.33) follows from (4.32).

Proof of the estimate (4.34). Using the estimate (4.33), we deduce that

$$\int_0^\infty K(s,\tau,\mu)\left|A^{1/2}\big(\widetilde{U}_\mu(\tau)-W_\mu(\tau)\big)\right| d\tau \leq$$

$$\leq \frac{C\,\mathcal{M}\,\mu^{1/4}}{\delta^{1/p}\,(1-\alpha)\,(1-4\,\gamma_0\mu)^{1/2}}\int_0^\infty K(s,\tau,\mu)\,e^{\gamma_0(1+2\,\gamma_0\mu)\,\tau}\,(1+\sqrt{\tau})^{1/2}\,d\tau, \tag{4.40}$$

for $\delta\in(0,1],\ \mu\in\Big(0,\dfrac{1}{4\,\gamma_0}\Big),\ \alpha\in(0,1).$

Denote by

$$J_k(s,\mu)=\int_0^\infty K(s,\tau,\mu)\,\tau^k\,e^{\gamma_0(1+2\,\gamma_0\mu)\,\tau}\,d\tau,\quad k=0,1.$$

Then, using $H\ddot{o}lder$'s inequality and property $(\mathbf{v})$ from Lemma 3.5, we have

$$\int_0^\infty K(s,\tau,\mu)\,e^{\gamma_0(1+2\,\gamma_0\mu)\tau}\,(1+\sqrt{\tau})^{1/2}\,d\tau\leq$$

$$\leq\Big(\int_0^\infty K(s,\tau,\mu)\,e^{\gamma_0(1+2\,\gamma_0\mu)\tau}\,d\tau\Big)^{1/2}\times\Big(\int_0^\infty K(s,\tau,\mu)\,e^{\gamma_0(1+2\,\gamma_0\mu)\tau}\,(1+\sqrt{\tau})\,d\tau\Big)^{1/2}\leq$$

$$\leq J_0^{1/2}\Big[J_0+J_0^{1/2}J_1^{1/2}\Big]^{1/2}\leq J_0+J_0^{3/4}J_1^{1/4}\leq$$

$$\leq\frac{C\,(1+\sqrt{s})^{1/2}}{(1-4\,\gamma_0\mu)}\,e^{\gamma_0\,(1+17\,\gamma_0\,\mu/4)\,s},\quad s\geq0,\quad \mu\in\Big(0,\frac{1}{4\,\gamma_0}\Big). \tag{4.41}$$

Thus, the estimates (4.40) and (4.41) imply (4.34). Lemma 4.7 is proved.

Proof of Lemma 4.8. In virtue of condition **(HB)**, multiplying scalarly in H the equation (4.29) with R and then integrating on (s_0,s), we obtain

$$|R(s,\mu)|^2+2\int_{s_0}^s\left|A^{1/2}R(\xi,\mu)\right|^2 d\xi\leq$$

$$\leq|R(s_0,\mu)|^2+2\int_{s_0}^s\big[|\mathcal{F}(\xi,\mu)|+L\left|A^{1/2}R(\xi,\mu)\right|\big]\,|R(\xi,\mu)|\,d\xi, \tag{4.42}$$

$\forall s>s_0\geq0$. Applying Lemma 2.1 to the last inequality, we get

$$\Big(|R(s,\mu)|^2+2\int_{s_0}^s\left|A^{1/2}R(\xi,\mu)\right|^2 d\xi\Big)^{1/2}\leq$$

$$\leq|R(s_0,\mu)|+\int_{s_0}^s\big[|\mathcal{F}(\xi,\mu)|+L\left|A^{1/2}R(\xi,\mu)\right|\big]\,d\xi,\quad\forall s>s_0\geq0,$$

or

$$|R(s,\mu)|+\Big(\int_{s_0}^s\left|A^{1/2}R(\xi,\mu)\right|^2 d\xi\Big)^{1/2}\leq$$

$$\leq\sqrt{2}\,\Big[|R(s_0,\mu)|+\int_{s_0}^s\big[|\mathcal{F}(\xi,\mu)|+L\left|A^{1/2}R(\xi,\mu)\right|\big]\Big],\quad\forall s>s_0\geq0.$$

Using Lemma 2.3, we obtain the inequality

$$|R(s,\mu)|+(1-\alpha)\Big(\int_0^s\left|A^{1/2}R(\xi,\mu)\right|^2 d\xi\Big)^{1/2}\leq$$

$$\leq C\,e^{\gamma_0\,s}\Big[|R(0,\mu)|+\int_0^s|\mathcal{F}(\xi,\mu)|\,d\xi\Big],\quad\forall s\geq0,\quad\alpha\in(0,1). \tag{4.43}$$

In what follows, we will estimate the right side of (4.43). Using (4.26), we get

$$\big|R(0,\mu)\big| \le \int_0^\infty e^{-\tau}\Big|\tilde U_\mu(2\mu\tau)-u_0\Big|\,d\tau \le \int_0^\infty e^{-\tau}\int_0^{2\mu\tau}\big|\tilde U'_\mu(\xi)\big|\,d\xi\,d\tau \le$$

$$\le \int\limits_0^\infty e^{-\tau}\,(2\,\mu\,\tau)^{1/2}\Big(\int\limits_0^\infty \big|\tilde U'_\mu(\xi)\big|^2\,d\xi\Big)^{1/2} d\tau \le \frac{C\,\mathcal M\,\mu^{1/2}}{\delta^{1/p}}\int\limits_0^\infty \tau^{1/2}\,e^{(\gamma_0\,\mu-1)\,\tau}\,d\tau =$$

$$= \frac{C\,\mathcal M\,\mu^{1/2}}{\delta^{1/p}\,(1-\gamma_0\,\mu)},\quad \delta\in(0,1],\quad \mu\in\Big(0,\frac{1}{\gamma_0}\Big). \tag{4.44}$$

Let us estimate $\big|\mathcal F(t,\mu)\big|$. Using the property (**iv**) from Lemma 3.5 and (4.25), we have

$$\Big|\tilde F(s)-\int_0^\infty K(s,\tau,\mu)\tilde F(\tau)\,d\tau\Big| \le C(p)\|\tilde F'\|_{L^p(0,\infty\,;\,H)}\,(\mu+\sqrt{\mu\,s})^{(p-1)/p}\le$$

$$\le C(p)\|\tilde f'\|_{L^p(0,T\,;\,H)}\,\big(\delta\,(\mu+\sqrt{\mu\,s})\big)^{(p-1)/p},\quad s\ge 0,\quad \delta\in(0,1]. \tag{4.45}$$

Since $e^{\xi}\lambda(\sqrt{\xi})\le C,\quad \forall \xi\ge 0$, the following estimates

$$\int_0^s \exp\Big\{\frac{3\xi}{4\mu}\Big\}\lambda\Big(\sqrt{\frac{\xi}{\mu}}\Big)d\xi \le C\,\mu\int_0^\infty e^{-\xi/4}\,d\xi \le C\mu,\quad \forall s\ge 0,$$

$$\int_0^s \lambda\Big(\frac12\sqrt{\frac{\xi}{\mu}}\Big)\,d\xi \le \mu\int_0^\infty \lambda\Big(\frac12\sqrt{\xi}\Big)\,d\xi \le C\,\mu,\quad \forall s\ge 0,$$

hold. Then

$$\Big|\delta\int_0^s f_0(\xi,\mu)\,u_1 d\xi\Big| \le C\,\delta\,\mu|u_1|,\quad s\ge 0. \tag{4.46}$$

Further, we estimate the difference

$$I(s,\mu)=\int_0^\infty K(s,\tau,\mu)\,B(\widetilde U_\mu(\tau))\,d\tau - B(W_\mu(s)) = I_1(s,\mu)+I_2(s,\mu), \tag{4.47}$$

where, due to the property (**ii**) from Lemma 3.5, we have

$$I_1(s,\mu)=\int_0^\infty K(s,\tau,\mu)\,\Big(B(\widetilde U_\mu(\tau))-B(W_\mu(\tau))\Big)\,d\tau,$$

$$I_2(s,\mu)=\int_0^\infty K(s,\tau,\mu)\,\Big(B(W_\mu(\tau))-B(W_\mu(s))\Big)\,d\tau.$$

Since $A^{1/2}$ is a closed operator, using properties (**i**), (**iii**) from Lemma 3.5, condition **(HB)** and estimate (4.34), for $I_1(s,\mu)$ we deduce the following estimate

$$|I_1(s,\mu)| \le L\int_0^\infty K(s,\tau,\mu)\,\Big|A^{1/2}\big(\widetilde U_\mu(\tau)-W_\mu(\tau)\big)\Big|\,d\tau \le$$

$$\le \frac{C\,\mathcal M\,\mu^{1/4}(1+\sqrt{s})^{1/2}\,e^{\gamma_0(1+17\,\gamma_0\,\mu/4)\,s}}{\delta^{1/p}\,(1-\alpha)\,(1-4\,\gamma_0\,\mu)^{3/2}},. \tag{4.48}$$

$s\ge 0,\ \ \delta\in(0,1],\ \ \alpha\in(0,1),\mu\in\Big(0,\frac{1}{4\,\gamma_0}\Big)$.

Next, we evaluate $I_2(s,\mu)$. Because

$$\big|B(W_\mu(s))-B(W_\mu(\tau))\big| \le L\Big[\big|A^{1/2}\big(W_\mu(s)-\widetilde U_\mu(s)\big)\big| + \big|A^{1/2}\big(\widetilde U_\mu(\tau)-W_\mu(\tau)\big)\big|+$$

$$+\big|A^{1/2}\big(\widetilde U_\mu(s)-\widetilde U_\mu(\tau)\big)\big|\Big], \tag{4.49}$$

then using property **(ii)** from Lemma 3.5, (4.31), (4.32) and (4.34), we get

$$|I_2(s,\mu)| \le L\Big[\big|A^{1/2}\big(W_\mu(s)-\widetilde{U}_\mu(s)\big)\big| + \int_0^\infty K(s,\tau,\mu)\,\big|A^{1/2}\big(\widetilde{U}_\mu(\tau)-W_\mu(\tau)\big)\big|\,d\tau +$$

(4.50)

$$+\int_0^\infty K(s,\tau,\mu)\,\big|A^{1/2}\big(\widetilde{U}_\mu(\tau)-\widetilde{U}_\mu(s)\big)\big|\,d\tau\Big] \le \frac{C\,\mathcal{M}\,\mu^{1/4}\,(1+\sqrt{s})}{\delta^{1/p}\,(1-\alpha)\,(1-4\,\gamma_0\,\mu)^{3/2}}\, e^{\gamma_0\,(1+17\,\mu\gamma_0/4)s},$$

$s\ge 0,\ \delta\in(0,1],\ \alpha\in(0,1),\ \mu\in\Big(0,\frac{1}{4\,\gamma_0}\Big)$.

From (4.47), using (4.48) and (4.50), for $I(t,\varepsilon)$ we get the estimate

$$|I(t,\varepsilon)| \le \frac{C\,\mathcal{M}\,\mu^{1/4}\,(1+\sqrt{s})^{1/2}\ e^{\gamma_0\,(1+17\,\mu\gamma_0/4)\,s}}{\delta^{1/p}\,(1-\alpha)\,(1-4\,\gamma_0\,\mu)^{3/2}},\ \ s\ge 0, \tag{4.51}$$

for $\delta\in(0,1],\ \alpha\in(0,1),\ \mu\in\Big(0,\dfrac{1}{4\,\gamma_0}\Big)$.

Using (4.45), (4.46) and (4.51), from (4.30) we obtain

$$\big|\mathcal{F}(\tau,\mu)\big| \le \frac{C\,\mathcal{M}}{\delta^{1/p}\,(1-\alpha)}\Big[\frac{\mu^{1/4}\,(1+\sqrt{s})\,e^{\gamma_0(1+17\,\gamma_0\,\mu/4)\,s}}{(1-4\,\gamma_0\,\mu)^{3/2}}+$$

$$+\delta\,\big(\mu+\sqrt{\mu s}\big)^{(p-1)/p}+\delta^{(p+1)/p}\mu\Big] \le$$

$$\le \frac{C\,\mathcal{M}\,(1+\sqrt{s})}{\delta^{1/p}\,(1-\alpha)\,(1-4\,\gamma_0\,\mu)^{3/2}}\Big[\mu^{1/4}\,e^{\gamma_0(1+17\,\gamma_0\,\mu/4)\,s}+\delta\,\mu^{(p-1)/2p}+\delta^{(p+1)/p}\mu\Big] \le$$

$$\le \frac{C\,\mathcal{M}\,(1+\sqrt{s})\,e^{\gamma_0\,(1+17\,\gamma_0\,\mu/4)\,s}}{\delta^{1/p}\,(1-\alpha)\,(1-4\,\gamma_0\,\mu)^{3/2}}\Big[\mu^{1/4}+\mu^{(p-1)/2p}\,\big(\delta+\delta^{(p+1)/p}\big)\Big] \le$$

$$\le \frac{C\,\mathcal{M}\,(1+\sqrt{s})\,e^{\gamma_0\,(1+17\,\gamma_0\,\mu/4)\,s}}{\delta^{1/p}\,(1-\alpha)\,(1-4\,\gamma_0\,\mu)^{3/2}}\Big[\mu^{1/4}+\mu^{(p-1)/2p}\,\delta\Big] \le \frac{C\,\mathcal{M}\,(1+\sqrt{s})\,e^{\gamma_0\,(1+17\,\gamma_0\,\mu/4)\,s}\,\mu^\beta}{\delta^{1/p}\,(1-\alpha)\,(1-4\,\gamma_0\,\mu)^{3/2}},$$

$s\ge 0,\ \ \delta\in(0,1],\ \ \alpha\in(0,1),\ \ \mu\in\Big(0,\frac{1}{4\,\gamma_0}\Big)$. Consequently,

$$\int_0^s \big|\mathcal{F}(\tau,\mu)\big|\,d\tau \le \frac{C\,\mathcal{M}\,(1+\sqrt{s})\,e^{\gamma_0\,(1+17\,\gamma_0\,\mu/4)\,s}\,\mu^\beta}{\delta^{1/p}\,(1-\alpha)\,(1-4\,\gamma_0\,\mu)^{3/2}},\ \ s\ge 0, \tag{4.52}$$

for $\delta\in(0,1],\ \ \alpha\in(0,1),\ \ \mu\in\Big(0,\dfrac{1}{4\,\gamma_0}\Big)$.

From (4.43), using (4.44) and (4.52) we get the estimate (4.35). Lemma 4.8 is proved.

In what follows we will investigate the special case when $L<\sqrt{\omega}$.

Theorem 4.4. *Let $T>0$ and $p>1$. Let us assume that the operators A and B satisfy conditions* **(HA)**, **(HB)** *and $L<\sqrt{\omega}$. If $u_0\in D(A)$, $u_1\in D(A^{1/2})$ and $f\in W^{1,p}(0,T;H)$, then there exists constant $C=C(T,p,\omega,L)>0$ such that*

$$||u_{\varepsilon\delta}-l_\delta||_{C([0,T];H)} \le C\,\mathcal{M}\,\delta^{-1}\,\Theta(\varepsilon,\delta), \tag{4.53}$$

$\delta\in(0,1],\ \varepsilon\in\big(0,\mu_0\delta^2\big), \mu_0=\frac{\sqrt{\omega}-L}{2\,\sqrt{\omega}\,L^2}$, where $u_{\varepsilon\delta}$ and l_δ are strong solutions to the problems $(P_{\varepsilon\delta})$ and (P_δ), respectively, and $\mathcal{M}$ and $\Theta(\varepsilon,\delta)$ are from (4.20) and (4.21). If $f=0$, then $\Theta(\delta,\varepsilon)=\dfrac{\varepsilon^{1/4}}{\delta^{(p+1)/p}}$.

Proof. In the proof of this theorem, we will denote by C all constants $C(T,p,\omega,L)$ and also we will keep the same value for $\widetilde{U}_\mu$, $\widetilde{f}$, $\widetilde{F}$, W_μ, $\widetilde{F}_0$, f_0, φ_μ, $\widetilde{\mathcal{L}}$ and R as in the proof of Theorem 4.3. In addition, it is easy to see that: for $\widetilde{F}$ the estimates (4.25) are valid, the function W_μ is the strong solution to the problem (4.27) for $\mu \in (0,\infty)$. But in this special case ($L < \sqrt{\omega}$), by virtue of Lemma 2.4, the functions $\widetilde{U}_\mu$ will satisfy the following estimates

$$\|\widetilde{U}'_\mu\|_{C([0,s];H)} + \|A^{1/2}\widetilde{U}'_\mu\|_{L^2(0,s;H)} \leq C\,\mathcal{M}\,\delta^{-1/p}, \tag{4.54}$$

$s \geq 0, \quad \delta \in (0,1], \quad \mu \in (0,\mu_0]$, with $\mathcal{M}$ from (4.20) and μ_0 from (4.53).

In what follows, we need the following two Lemmas, which will be proved after the proof of the Theorem 4.4.

Lemma 4.9. *Assume the conditions of Theorem 4.4 are fulfilled. Then the following estimates:*

$$|\widetilde{U}_\mu(s) - W_\mu(s)| \leq C\,\mathcal{M}\,\delta^{-1/p}\,\mu^{1/2}\,(1+\sqrt{s}), \tag{4.55}$$

$s \geq 0, \quad \delta \in (0,1], \quad \mu \in (0,\mu_0]$,

$$\int_0^\infty K(s,\tau,\mu)\,\big|A^{1/2}\big(\widetilde{U}_\mu(s) - \widetilde{U}_\mu(\tau)\big)\big|\,d\tau \leq$$

$$\leq C\,\mathcal{M}\,\delta^{-1/p}\,\mu^{1/4}\,(1+\sqrt{s})^{1/2}, \quad s \geq 0, \quad \delta \in (0,1], \quad \mu \in (0,\mu_0], \tag{4.56}$$

$$|A^{1/2}\big(\widetilde{U}_\mu(s) - W_\mu(s)\big)| \leq C\,\mathcal{M}\,\delta^{-1/p}\,\mu^{1/4}\,(1+\sqrt{s})^{1/2}, \tag{4.57}$$

$s \geq 0, \quad \delta \in (0,1], \quad \mu \in (0,\mu_0]$,

$$\int_0^\infty K(s,\tau,\mu)\Big|A^{1/2}\big(\widetilde{U}_\mu(\tau) - W_\mu(\tau)\big)\Big|d\tau \leq$$

$$\leq C\mathcal{M}\delta^{-1/p}\mu^{1/4}(1+\sqrt{s})^{1/2}, \quad s \geq 0, \quad \delta \in (0,1], \quad \mu \in (0,\mu_0], \tag{4.58}$$

hold with $\mathcal{M}$ and μ_0 from (4.20) and (4.54).

Lemma 4.10. *Let the conditions of Theorem 4.4 are fulfilled. Then for the strong solution to the problem (4.29) the following estimate*

$$||R||_{C([0,\,s];\,H)} + ||A^{1/2}R||_{L^2(0,s;H)} \leq$$

$$\leq C\,\mathcal{M}\,\delta^{-1/p}\,\mu^{\beta}\,(1+s)^{3/2}, \quad s \geq 0,\ \delta \in (0,1], \quad \mu \in \big(0,\mu_0\big]. \tag{4.59}$$

is true with $\beta = \min\Big\{\dfrac{1}{4}, \dfrac{p-1}{2p}\Big\}$, $\mathcal{M}$ and μ_0 from (4.20) and (4.54).

Consequently, from (4.55) and (4.59), we deduce

$$||\tilde{U}_\mu - \tilde{L}||_{C([0,s];H)} \leq ||\tilde{U}_\mu - W_\mu||_{C([0,s];H)} + ||R||_{C([0,s];H)} \leq$$

$$\leq C\,\mathcal{M}\,\delta^{-1/p}\,\mu^{\beta}\,(1+s)^{3/2}, \quad s \geq 0,\ \delta \in (0,1], \quad \mu \in \big(0,\mu_0\big]. \tag{4.60}$$

Since $U_\mu(s) = \tilde{U}_\mu(s)$, $L(s) = \tilde{L}(s)$, for all $s \in [0,T/\delta]$, $U_\mu(s) = u_{\varepsilon\delta}(\delta\, s)$ and $L(s) = l_\delta(\delta\, s)$, then for $t \in [0,T]$ we have

$$|u_{\varepsilon\delta}(t) - l_\delta(t)| = |u_{\varepsilon\delta}(\delta s) - l_\delta(\delta s)| = \big|\tilde{U}_\mu(s) - \tilde{L}(s)\big|, \quad s \in \Big[0, \frac{T}{\delta}\Big].$$

Concequently, from (4.60) the estimate (4.53) follows. □

Proof of Lemma 4.9. Proof of the estimate (4.55). Using properties (**i**), (**iii**) from Lemma 3.5 and (4.54), we get

$$|\widetilde{U}_\mu(s) - W_\mu(s)| \leq \int_0^\infty K(s,\tau,\mu)\, \big|\widetilde{U}_\mu(s) - \widetilde{U}_\mu(\tau)\big|\, d\tau \leq$$

$$\leq \int_0^\infty K(s,\tau,\mu) \Big| \int_\tau^s \big|\widetilde{U}'_\mu(\xi)\big|\, d\xi \Big|\, d\tau \leq \frac{C\,\mathcal{M}}{\delta^{1/p}} \int_0^\infty K(s,\tau,\mu)\, \big|\tau - s\big|\, d\tau \leq$$

$$\leq C\,\mathcal{M}\,\delta^{-1/p}\,\mu^{1/2}\,(1+\sqrt{s}), \quad s \geq 0, \quad \delta \in (0,1],\ \mu \in (0,\mu_0].$$

Thus, the estimate (4.55) is proved.

Proof of the estimate (4.56). Using $H\ddot{o}lder's$ inequality, we have

$$\int_0^\infty K(s,\tau,\mu)\, \big|A^{1/2}\big(\widetilde{U}_\mu(s) - \widetilde{U}_\mu(\tau)\big)\big|\, d\tau \leq \int_0^\infty K(s,\tau,\mu) \Big| \int_\tau^s \big|A^{1/2}\widetilde{U}'_\mu(\xi)\big|\, d\xi \Big|\, d\tau \leq$$

$$\leq \int_0^\infty K(s,\tau,\mu)\, |s-\tau|^{1/2} \Big| \int_\tau^s |A^{1/2}\widetilde{U}'_\mu(\xi)|^2\, d\xi \Big|^{1/2} d\tau \leq$$

$$\leq \Big(\int_0^\infty K(s,\tau,\mu)\, |s-\tau|\, d\tau \Big)^{1/2} \times \Big(\int_0^\infty K(s,\tau,\mu) \Big| \int_\tau^s |A^{1/2}\widetilde{U}'_\mu(\xi)|^2\, d\xi \Big| \Big)^{1/2} \leq$$

$$\leq C\,\mu^{1/4}(1+\sqrt{s})^{1/2}\, Q_\mu^{1/2}(s), \tag{4.61}$$

where, due to (4.54), we have

$$Q_\mu(s) = \int_0^\infty K(s,\tau,\mu) \Big| \int_\tau^s |A^{1/2}\widetilde{U}'_\mu(\xi)|^2\, d\xi \Big| \leq$$

$$\leq \int_0^\infty K(s,\tau,\mu) \Big[\int_0^s |A^{1/2}\widetilde{U}'_\mu(\xi)|^2\, d\xi + \int_0^\tau |A^{1/2}\widetilde{U}'_\mu(\xi)|^2\, d\xi \Big]\, d\tau \leq$$

$$\leq C\,\mathcal{M}^2\,\delta^{-2/p}, \quad s \geq 0, \quad \delta \in (0,1], \quad \mu \in \big(0,\mu_0\big]. \tag{4.62}$$

Now, (4.61) and (4.62) imply the estimate (4.56).

Proof of the estimate (4.57). The proof follows from (4.39) and (4.56).

Proof of the estimate (4.58). Using the estimate (4.57), we deduce that

$$\int_0^\infty K(s,\tau,\mu) \Big| A^{1/2}\big(\widetilde{U}_\mu(\tau) - W_\mu(\tau)\big) \Big|\, d\tau \leq$$

$$\leq C\,\mathcal{M}\,\mu^{1/4}\,\delta^{-1/p} \int_0^\infty K(s,\tau,\mu)\,(1+\sqrt{\tau})^{1/2}\, d\tau, \tag{4.63}$$

$s \geq 0, \quad \delta \in (0,1], \quad \mu \in \big(0,\mu_0\big].$

Using $H\ddot{o}lder's$ inequality and properties (**ii**) and (**v**) from Lemma 3.5, we get

$$\int_0^\infty K(s,\tau,\mu)\,(1+\sqrt{\tau})^{1/2}\, d\tau \leq$$

$$\leq \Big(\int_0^\infty K(s,\tau,\mu)\, d\tau \Big)^{1/2} \times \Big(\int_0^\infty K(s,\tau,\mu)\,(1+\sqrt{\tau})\, d\tau \Big)^{1/2} \leq$$

$$\leq \Big(1 + \int_0^\infty K(s,\tau,\mu)\,\sqrt{\tau}\, d\tau \Big)^{1/2} \leq \Big(1 + \Big(\int_0^\infty K(s,\tau,\mu)\,\tau\, d\tau \Big)^{1/2} \Big)^{1/2} \leq$$

$$\leq C\,(1+\sqrt{s})^{1/2}, \quad s \geq 0, \quad \mu \in \big(0,\mu_0\big]. \tag{4.64}$$

Thus, the estimates (4.63) and (4.64) imply (4.58). Lemma 4.9 is proved.

Proof of Lemma 4.10. In virtue of conditions **(HB)** and $L < \sqrt{\omega}$, from (4.42) we obtain

$$|R(s,\mu)|^2 + 2\,\frac{\sqrt{\omega}-L}{\sqrt{\omega}} \int_0^s \left|A^{1/2}R(\xi,\mu)\right|^2 d\xi \le$$
$$\le |R(0,\mu)|^2 + 2\int_0^s |\mathcal{F}(\xi,\mu)|\,|R(\xi,\mu)|\,d\xi, \quad \forall s \ge 0,$$

where $\mathcal{F}(\xi,\mu)$ is defined by (4.30). Applying Lemma 2.1 to the last inequality, we get

$$|R(s,\mu)| \le |R(0,\mu)| + \int_0^s |\mathcal{F}(\xi,\mu)|\,d\xi, \quad \forall s \ge 0. \tag{4.65}$$

In what follows, we will estimate the right side of (4.65). Using (4.54), we get

$$\big|R(0,\mu)\big| \le \int_0^\infty e^{-\tau}\Big|\tilde{U}_\mu(2\mu\tau) - u_0\Big|\,d\tau \le \int_0^\infty e^{-\tau}\int_0^{2\mu\tau} \big|\tilde{U}_\mu'(\xi)\big|\,d\xi\,d\tau \le$$
$$\le C\,\mathcal{M}\,\mu\,\delta^{-1/p}\int_0^\infty \tau\,e^{-\tau}\,d\tau = C\,\mathcal{M}\,\mu^{1/2}\,\delta^{-1/p}, \tag{4.66}$$

$\delta \in (0,1], \mu \in \big(0,\mu_0\big]$. Let us estimate $\big|\mathcal{F}(t,\mu)\big|$. First of all let us mention that in this case the estimates (4.45) and (4.46) remain true.

Now we will estimate the integrals $I_1(s,\mu)$ and $I_2(s,\mu)$ from (4.47) in this case.

Using properties (**i**), (**iii**) from Lemma 3.5, condition **(HB)** and estimate (4.58), for $I_1(s,\mu)$ we deduce the following estimate

$$|I_1(s,\mu)| \le L\int_0^\infty K(s,\tau,\mu)\left|A^{1/2}\big(\widetilde{U}_\mu(\tau) - W_\mu(\tau)\big)\right|d\tau \le$$
$$\le C\,\mathcal{M}\,\delta^{-1/p}\,\mu^{1/4}(1+\sqrt{s})^{1/2}, \quad s \ge 0, \quad \delta \in (0,1], \quad \mu \in (0,\mu_0]. \tag{4.67}$$

Using (4.49), property **(ii)** from Lemma 3.5, (4.55), (4.56) and (4.58), for $I_2(s,\mu)$ we get

$$|I_2(s,\mu)| \le L\Big[\big|A^{1/2}\big(W_\mu(s) - \widetilde{U}_\mu(s)\big)\big| + \int_0^\infty K(s,\tau,\mu)\,\big|A^{1/2}\big(\widetilde{U}_\mu(\tau) - W_\mu(\tau)\big)\big|\,d\tau +$$
$$+\int_0^\infty K(s,\tau,\mu)\,\big|A^{1/2}\big(\widetilde{U}_\mu(\tau) - \widetilde{U}_\mu(s)\big)\big|\,d\tau\Big] \le$$
$$\le C\,\mathcal{M}\,\delta^{-1/p}\,\mu^{1/4}\,(1+\sqrt{s}), \quad s \ge 0,\ \delta \in (0,1], \quad \mu \in \big(0,\mu_0\big]. \tag{4.68}$$

Then (4.67) and (4.68) imply the estimate

$$|I(t,\varepsilon)| \le C\,\mathcal{M}\,\delta^{-1/p}\,\mu^{1/4}\,(1+\sqrt{s}), \quad s \ge 0,\ \delta \in (0,1], \quad \mu \in \big(0,\mu_0\big]. \tag{4.69}$$

Using (4.45), (4.46) and (4.69), from (4.30) we obtain

$$\big|\mathcal{F}(\tau,\mu)\big| \le C\,\mathcal{M}\,\delta^{-1/p}\Big[\mu^{1/4}\,(1+\sqrt{s}) + \delta\,\big(\mu+\sqrt{\mu s}\big)^{(p-1)/p} + \delta^{(p+1)/p}\mu\Big] \le$$
$$\le C\,\mathcal{M}\,\delta^{-1/p}\,\mu^{\beta}\,(1+\sqrt{s}), \quad s \ge 0,\ \delta \in (0,1], \quad \mu \in \big(0,\mu_0\big].$$

Consequently,

$$\int_0^s \big|\mathcal{F}(\tau,\mu)\big|\,d\tau \le C\,\mathcal{M}\,\delta^{-1/p}\,\mu^{\beta}\,s\,(1+\sqrt{s}), \tag{4.70}$$

$s \ge 0,\ \delta \in (0,1], \quad \mu \in \big(0,\mu_0\big]$.

From (4.65), using (4.66) and (4.70) we get the estimate (4.59). Lemma 4.10 is proved.

Remark 4.1. Suppose that A, B, u_0, u_1 and f satisfy conditions of Theorem 4.3 or Theorem 4.4. If $\delta \geq \delta_0 > 0$, then there exist $C = C(L, \omega, p, T, \delta_0)$ and $\varepsilon_0 = \varepsilon_0(\delta_0, L)$, such that the estimate

$$||u_{\varepsilon\delta} - l_\delta||_{C([0,T];H)} \leq C\,\mathcal{M}\,\varepsilon^{\beta}, \quad \varepsilon \in \big(0, \varepsilon_0\big).$$

is true with $\mathcal{M}$ and $\beta = \min\Big\{\dfrac{1}{4}, \dfrac{p-1}{2p}\Big\}$.

In what follows we will investigate the special case when $\varepsilon \to 0$ and $\delta \to 0$ simultaneously.

Theorem 4.5. *Let $T > 0$ and $p > 1$. Let us assume that the operators A and B satisfy conditions* **(HA)**, **(HB)** *and $\sqrt{\omega} > L$. If $u_0 \in D(A)$ and $f \in W^{1,p}(0,T;H)$, then there exists constant $C = C(p, \omega, L) > 0$ such that*

$$|l_\delta(t - v(t)| \leq \big|u_0 - (A+B)^{-1} f(0)|e^{-\omega_0 t/\delta} + C\,\delta^{(p-1)/p}\,||f||_{W^{1,p}(0,T;H)}, \tag{4.71}$$

$t \in (0,T]$, where $\omega_0 = \sqrt{\omega}(\sqrt{\omega} - L)$, l_δ and v are strong solutions to the problems (P_δ) and (P_0), respectively.

To prove the theorem, we need the following lemma, which will be proved after the proof of the theorem.

Lemma 4.11. *Let $T > 0$ and $p > 1$. Suppose the operators A and B satisfy conditions* **(HA)**, **(HB)** *and $\sqrt{\omega} > L$. If $f \in W^{1,p}(0,T;H)$, then the equation $Av + B(v) = f$ has a unique solution $v \in W^{1,p}(0,T;H)$ and*

$$||v||_{W^{1,p}(0,T;H)} \leq C(\omega, L, T, p)\,||f||_{W^{1,p}(0,T;H)}. \tag{4.72}$$

Proof of Theorem 4.5. Denote by $R_1(t,\delta) = l_\delta(t) - v(t)$, where l_δ is strong solution to the problem (P_δ) and v is strong solution to the problem (P_0). Then $R_1(t,\delta)$ is the strong solution to the problem

$$\begin{cases} \delta\, R_1'(t,\delta) + AR_1(t,\delta) = -\delta\, v'(t) - B(l_\delta(t)) + B\big(v(t)\big), t \in (0,T), \\ R_1(0,\delta) = u_0 - (A+B)^{-1} f(0), \end{cases} \tag{4.73}$$

Multiplying equation from (4.73) scalarly in H by R_1, we obtain the equality

$$\delta\,\frac{d}{dt}|R_1(t,\delta)|^2 + 2\left(AR_1(t,\delta), R_1(t,\delta)\right) =$$

$$= -2\,\delta\left(v'(t), R_1(t,\delta)\right) + \big(B(v(t)) - B(l_\delta(t)), R_1(t,\delta)\big), \quad t \in (0,T).$$

Then, using conditions **(HA)**, **(HB)** and $\sqrt{\omega} > L$, we get

$$\delta\,\frac{d}{dt}|R_1(t,\delta)|^2 + 2\,\omega_0\,|R_1(t,\delta)|^2 \leq 2\,\delta\,|v'(t)|\,|R_1(t,\delta)|, \quad t \in (0,T).$$

From the last inequality we get

$$\frac{d}{dt}|R_1(t,\delta)\,e^{\omega_0 t/\delta}|^2 \leq 2\,|v'(t)\,e^{\omega_0 t/\delta}|\,|R_1(t,\delta)\,e^{\omega_0 t/\delta}|, \quad t \in (0,T).$$

Integrating this inequality on $(0,t)$, we obtain

$$|R_1(t,\delta)\,e^{\omega_0 t/\delta}|^2 \leq |R_1(0,\delta)|^2 + 2\int_0^t |v'(\tau)\,e^{\omega_0 \tau/\delta}|\,|R_1(\tau,\delta)\,e^{\omega_0 \tau/\delta}|\,d\tau, t \in [0,T].$$

Applying Lemma 2.1 to the last inequality and using (4.72), we get the estimate

$$|R_1(t,\delta)| \leq |R_1(0,\delta)|\,e^{-\omega_0 t/\delta} + \int_0^t e^{-\omega_0\,(t-\tau)/\delta}\,|v'(\tau)|\,d\tau \leq$$

$$\leq |R_1(0,\delta)|\, e^{-\omega_0 t/\delta} + \left(\frac{\delta}{\omega_0}\right)^{(p-1)/p} ||v'||_{L^p(0,T;H)} \leq$$

$$\leq |R_1(0,\delta)|\, e^{-\omega_0 t/\delta} + C\left(\frac{\delta}{\omega_0}\right)^{(p-1)/p} ||f||_{W^{1,p}(0,T;H)}, t \in [0,T],$$

from which (4.71) follows. Theorem 4.5 is proved.

Proof of Lemma 4.11. Denote by $\mathcal{A} = A + B : D(A) \mapsto H$. Operator $\mathcal{A}$ is monotone. Indeed, in virtue of conditions **(HA)**, **(HB)** and $\sqrt{\omega} > L$ we have

$$\Big(\mathcal{A}u_1 - \mathcal{A}u_2, u_1 - u_2\Big) = |A^{1/2}\,(u_1 - u_2)|^2 + \Big(B(u_1) - B(u_2), u_1 - u_2\Big) \geq$$

$$\geq |A^{1/2}\,(u_1 - u_2)|^2 - L\,|A^{1/2}\,(u_1 - u_2)|\,|u_1 - u_2| \geq$$

(4.74) $$\geq \sqrt{\omega}\,(\sqrt{\omega} - L)\,|u_1 - u_2|^2 \geq 0, \quad \forall u_1, u_2 \in D(A).$$

We will show that the operator $\mathcal{A}$ is even maximal monotone in H. For this purpose we consider the equation

(4.75) $$\big(\lambda I + \mathcal{A}\big)u = f,$$

in H with $f \in H$ and $\lambda > 0$.

Due to the condition **(HA)**, there exists $(\lambda I + A)^{-1}) : D((\lambda I + A)^{-1}) = H \mapsto R((\lambda I + A)^{-1}) \subseteq D(A)$ for any $\lambda > 0$ and

$$||(\lambda I + A)^{-1}||_{H \to H} \leq (\lambda + \omega)^{-1}.$$

Then the equation (4.75) is equivalent to the equation

(4.76) $$u = \big(\lambda I + A\big)^{-1}\big(f - B(u)\big).$$

Let us examine the equation (4.76) in the real Hilbert space $V_0 = D(A^{1/2})$, endowed with the scalar product

$$(u,v)_{V_0} = (A^{1/2}\,u, A^{1/2}\,v), \quad u, v \in V_0.$$

The equation (4.76) can be written in the form

(4.77) $$\mathcal{B}(u) = u,$$

where $\mathcal{B}(u) = (\lambda I + A)^{-1}(f - B(u))$. The equality

$$\big((A + \lambda I)\,A^{-1}\,u, u\big) = |u|^2 + \lambda\,\big(A^{-1}\,u, u\big), \quad \forall u \in H,$$

implies

$$\big(A\,(A + \lambda I)^{-1}\,u, u\big) = |A\,(A + \lambda I)^{-1}\,u|^2 + \lambda\,\big|A^{1/2}\,(A + \lambda I)^{-1}\,u\big|^2, \quad \forall u \in H,$$

from which it follows that

$$|A\,(A + \lambda\,I)^{-1}\,u| \leq |u|, \quad \forall u \in H.$$

Consequently,

$$||A^{1/2}(\lambda I + A)^{-1}||_{H \to H} \leq \lambda^{-1/2}.$$

From condition **(HB)** follows that operator $\mathcal{B}$ is a contraction on V_0 for every $f \in H$ and every $\lambda > L^2$. Indeed,

$$||\mathcal{B}(u_1) - \mathcal{B}(u_2)||_{V_0} = |A^{1/2}(\lambda I + A)^{-1}\big(B(u_1) - B(u_2)\big)| \leq$$

$$\leq \lambda^{-1/2}|B(u_1) - B(u_2)| \leq L\,\lambda^{-1/2}\,|A^{1/2}(u_1 - u_2)| = L\,\lambda^{-1/2}\,||u_1 - u_2||_{V_0}, \quad \forall u_1, u_2 \in V_0.$$

According to the fix point Banach's Theorem, the equation (4.77) has a unique solution $u \in V_0$ for each $f \in H$. From (4.76) it follows that $u \in D(A)$. Thus, $R(\lambda\,I + \mathcal{A}) \supseteq H$ for $\lambda > L^2$. Therefore, according to Minty's Theorem, $\mathcal{A}$ is a maximal monotone operator if $\sqrt{\omega} > L$. Since, by virtue of (4.74), the operator $A + B$ is coercive, it is surjective i.e. $R(A + B) = H$. Hence there exists $(A + B)^{-1} : H \mapsto D(A)$. Then the inequality

$$\big(\mathcal{A}(u_1) - \mathcal{A}(u_2), u_1 - u_2\big) \geq \omega_0 \, |u_1 - u_2|^2, \quad u_1, u_2 \in D(A),$$

implies

$$\big|\mathcal{A}^{-1}(f_1) - \mathcal{A}^{-1}(f_2)\big| \leq \frac{1}{\omega_0} \, |f_1 - f_2|, \quad f_1, f_2 \in H. \tag{4.78}$$

It means that the equation $Av(t) + B(v(t)) = f(t)$ has a unique solution $v(t) \in D(A)$ for every $f(t) \in H$ and

$$|v(t)| \leq |v(0)| + \frac{1}{\omega_0} \, \big|f(t) - f(0)\big| \leq \big|\mathcal{A}^{-1}(f(0))\big| + \frac{1}{\omega_0} \, |f(t) - f(0)|$$

Thus, if $f \in W^{1,p}(0, T; H)$, then $v \in W^{1,p}(0, T; H)$. Finally, using (4.78) and $H\ddot{o}lder's$ inequality, we get

$$|v(t+h) - v(t)|^p \leq \frac{1}{\omega_0^p} \, |f(t+h) - f(t)|^p = \frac{1}{\omega_0^p} \Big| \int_t^{t+h} f'(\tau) \, d\tau \Big|^p \leq$$

$$\leq \frac{h^{p-1}}{\omega_0^p} \int_t^{t+h} |f'(\tau)|^p \, d\tau \quad t \in [0, T-h],$$

or

$$\int_0^{T-h} |v(t+h) - v(t)|^p \, dt \leq$$

$$\leq \frac{h^{p-1}}{\omega_0^p} \Big[\int_0^h \tau \, |f'(\tau)|^p \, d\tau + \int_h^{T-h} h \, |f'(t)|^p \, d\tau + \int_{T-h}^T (T - \tau) \, |f'(\tau)|^p \, d\tau \Big] \leq$$

$$\leq \frac{h^p}{\omega_0^p} \int_0^T |f'(\tau)|^p \, d\tau.$$

In this case (see Theorem 1.18 [1]) $v \in W^{1,p}(0, T; H)$ and

$$||v'||_{L^p(0,T;H)} \leq \frac{1}{\omega_0} \, ||f'||_{L^p(0,T;H)}. \tag{4.79}$$

From (4.78) and (4.79) follows (4.72). Lemma 4.11 is proved.

Remark 4.2. Under the conditions of Lemma 4.11, if $f = 0$, then from (4.79) follows that $v'(t) \equiv 0$. Concequently, in this case

$$|l_\delta(t) - v(t)| \leq \big|u_0 - (A + B)^{-1}(0)\big| \, e^{-\omega_0 t/\delta}, \quad t \in (0, T].$$

Remark 4.3. Under the conditions of Theorem 4.5 it follows that for every t_0 and T,

$$l_\delta(t) \to v(t) \quad in \quad C([t_0, T]; H) \quad as \ \delta \to 0, T > t_0 > 0. \tag{4.80}$$

If the concordance condition $u_0 = (A+B)^{-1} f(0)$ is satisfied, then $l_\delta(t) \to v(t)$ in $C([0, T]; H)$, as $\delta \to 0$.

Remark 4.4. The following simple example shows that it should not be expected that the relationship (4.80) is fulfilled in the case when $L \geq \sqrt{\omega}$.

Example 4.1. Consider the Cauchy problem for the ordinary differential equation

$$\begin{cases} \delta y'(t) + y(t) + \alpha \, |y(t)| = f(t), \quad t > 0, \\ y(0) = y_0. \end{cases}$$

If $y_0 > 0$, $f \in C([0,T])$, $f(t) > 0$ for $t \in [0,T]$ and $1+\alpha < 0$, then for every $t > 0$ we have

$$y(t) = e^{-(1+\alpha)t/\delta}\, y_0 + \int_0^t e^{-(1+\alpha)(t-\tau)/\delta}\, f(\tau)\, d\tau \geq C\, e^{-(1+\alpha)t/\delta} \to \infty, as\ \delta \to 0.$$

Theorems 4.4 and 4.5 imply the following theorem.

Theorem 4.6. *Let $T > 0$ and $p > 1$. Let us assume that the operators A and B satisfy conditions* **(HA)**, **(HB)** *and $L < \sqrt{\omega}$. If $u_0 \in D(A)$, $u_1 \in D(A^{1/2})$ and $f \in W^{1,p}(0,T;H)$, then there exists constant $C = C(T,p,\omega,L) > 0$ such that*

$$||u_{\varepsilon\delta} - v||_{C([0,T];H)} \leq$$

$$\leq h_0\, e^{-\omega_0 t/\delta} + C\,\mathcal{M}\left[\delta^{-1}\Theta(\varepsilon,\delta) + \delta^{(p-1)/p}\right], t \in [0,T], \delta \in (0,1], \varepsilon \in \left(0, \mu_0\, \delta^2\right),$$

$u_{\varepsilon\delta}$ and v are strong solutions to the problems $(P_{\varepsilon\delta})$ and (P_0), respectively, $\omega_0 = \sqrt{\omega}(\sqrt{\omega} - L)$, $h_0 = \left|u_0 - (A+B)^{-1} f(0)\right|$ and $\mathcal{M}$ and $\Theta(\varepsilon,\delta)$ are from (4.20) and (4.21), respectively, and μ_0 is defined in (4.53). If $f = 0$, then

$$||u_{\varepsilon\delta} - v||_{C([0,T];H)} \leq h_0\, e^{-\omega_0 t/\delta} + C\,\mathcal{M}\,\frac{\varepsilon^{1/4}}{\delta^{(2p+1)/p}},$$

$t \in [0,T]$, $\delta \in (0,1]$, $\varepsilon \in \left(0, \mu_0\, \delta^2\right)$.

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