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Dedicated to the memory of Academician Mitrofan M. Choban (1942-2021)

Center problem for cubic differential systems with the line at infinity of multiplicity four

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ABSTRACT. In this paper the center problem for cubic differential systems with the line at infinity of multiplicity four is solved.

1. INTRODUCTION

Consider the real cubic system of differential equations

$$(1.1)\qquad \begin{cases} \dot{x} = y + ax^2 + cxy + fy^2 + kx^3 + mx^2y + pxy^2 + ry^3 \equiv p\,(x,y)\,, \\ \dot{y} = -(x + gx^2 + dxy + by^2 + sx^3 + qx^2y + nxy^2 + ly^3) \equiv q\,(x,y)\,, \\ \gcd(p,q) = 1,\ (k,l,m,n,p,q,r,s) \neq 0. \end{cases}$$

The critical point $(0,0)$ of the system (1.1) is either a focus or a center. The problem of distinguishing between a center and a focus is called *the center problem*. It is well known that $(0,0)$ is a center if and only if the Lyapunov quantities $L_1, L_2, ..., L_j, ...$ vanish (see, for example, [2], [6], [7], [8]). Also, the critical point $(0,0)$ is a center if the system (1.1) has an axis of symmetry ([7]) or an analytical integrating factor in a neighborhood of $(0,0)$.

We suppose that the infinity is non-degenerate for (1.1), i.e.

$$(1.2)\qquad sx^4 + (k+q)x^3y + (m+n)x^2y^2 + (l+p)xy^3 + ry^4 \not\equiv 0.$$

The homogeneous system associated to the system (1.1) has the form

$$\begin{cases} \dot{x} = yZ^2 + (ax^2 + cxy + fy^2)Z + kx^3 + mx^2y + pxy^2 + ry^3 \equiv P\,(x,y,Z)\,, \\ \dot{y} = -(xZ^2 + (gx^2 + dxy + by^2)Z + sx^3 + qx^2y + nxy^2 + ly^3) \equiv Q\,(x,y,Z)\,. \end{cases}$$

Denote $\mathbb{X} = p\,(x,y)\,\frac{\partial}{\partial x} + q\,(x,y)\,\frac{\partial}{\partial y}$, $\mathbb{X}_\infty = P\,(x,y,Z)\,\frac{\partial}{\partial x} + Q\,(x,y,Z)\,\frac{\partial}{\partial y}$ and $E_\infty = P \cdot \mathbb{X}_\infty(Q) - Q \cdot \mathbb{X}_\infty(P)$. The polynomial E_∞ has the form $E_\infty = C_2(x,y) + C_3(x,y)Z + C_4(x,y)Z^2 + \cdots + C_8(x,y)Z^6$, where $C_j(x,y)$, $j = 2,...,8$, are polynomial in x and y. We say that the line at infinity $Z = 0$ has *multiplicity* ν if $C_2(x,y) \equiv 0, ..., C_\nu(x,y) \equiv 0$, $C_{\nu+1}(x,y) \not\equiv 0$, i.e. $\nu - 1$ is the greatest positive integer such that $Z^{\nu-1}$ divides E_∞. In particular, $Z = 0$ has multiplicity four if the identity in Z:

$$(1.3)\qquad C_2(x,y) + C_3(x,y)Z + C_4(x,y)Z^2 \equiv 0$$

holds, i.e. $C_2(x,y) \equiv 0$, $C_3(x,y) \equiv 0$ and $C_4(x,y) \equiv 0$. If $C_2(x,y) \not\equiv 0$, then we say that $Z = 0$ has the multiplicity one.

The algebraic line $f(x,y) = 0$ is called *invariant* for (1.1) if there exists a polynomial $K \in \mathbb{C}[x,y]$ such that the identity $\mathbb{X}(f) \equiv f \cdot K(x,y)$ holds. Some notions on multiplicity (algebraic, integrable, infinitesimal, geometric) of an invariant algebraic line and its equivalence for polynomial differential systems are given in [1].

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The cubic differential systems with multiple invariant straight lines (including the line at infinity) was studied in [5], [11], [14], and the center problem for (1.1) with invariant straight lines was considered in [2], [3], [4], [9], [10], [12], [13].

In this paper the main result is following:

Theorem 1.1. *The cubic system* (1.1) *with the line at infinity of multiplicity four has at the origin a center if and only if the first three Lyapunov quantities vanish* $L_1 = L_2 = L_3 = 0$.

2. Classification of cubic systems with multiple line at infinity

Let $\mathcal{X} = (x, y)$, $\mathcal{A}_2 = (a, b, c, d, f, g)$, $\mathcal{A}_3 = (k, l, m, n, p, q, r, s)$, $\mathcal{B}_2 = (A, B, C, D, F, G)$, $\mathcal{U} = (u, v)$, $\mathcal{B}_3 = (K, L, M, N, P, Q, R, S)$ and $\mathcal{X} = 2^{-1}M_1\mathcal{U}$, $\mathcal{A}_2 = 2^{-3}\mathcal{M}_2\mathcal{B}_2$, $\mathcal{A}_3 = 2^{-4}\mathcal{M}_3\mathcal{B}_3$, where

$$\mathcal{M}_1 = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}, \quad \mathcal{M}_2 = \begin{pmatrix} -i & i & -i & i & -i & i \\ -1 & -1 & 1 & 1 & -1 & -1 \\ -2 & -2 & 0 & 0 & 2 & 2 \\ -2i & 2i & 0 & 0 & 2i & -2i \\ i & -i & -i & i & i & -i \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix},$$

$$\mathcal{M}_3 = \begin{pmatrix} -i & i & -i & i & -i & i & -i & i \\ i & -i & -i & i & i & -i & -i & i \\ -3 & -3 & -1 & -1 & 1 & 1 & 3 & 3 \\ -3 & -3 & 1 & 1 & 1 & 1 & -3 & -3 \\ 3i & -3i & -i & i & -i & i & 3i & -3i \\ -3i & 3i & -i & i & i & -i & 3i & -3i \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix},$$

$\det \mathcal{M}_1 = -2i$, $\det \mathcal{M}_2 = -2^9 i$, $\det \mathcal{M}_3 = 2^{16}$, $i^2 = -1$.

We remark that, in general, the elements of $\mathcal{U}$, $\mathcal{B}_2$, $\mathcal{B}_3$ are complex and $v = \overline{u}$, $B = \overline{A}$, $D = \overline{C}$, $G = \overline{F}$, $L = \overline{K}$, $N = \overline{M}$, $Q = \overline{P}$, $S = \overline{R}$.

In $u, v, A, B, ..., R, S$ the non-identity (1.2) and the identity (1.3), up to a non zero factor, look as

$$C_3(u, v) = Ku^4 + (M + S)u^3v + (P + Q)u^2v^2 + (N + R)uv^3 + Lv^4 \not\equiv 0$$

and

$$M_2(u, v) + M_3(u, v)Z + M_4(u, v)Z^2 \equiv 0,$$

respectively, where

$$M_2(u, v) = 2^{-10}C_3(u, v)(N_2(u, v) + \overline{N_2(u, v)}), M_j(u, v) = 2^{j-12}(N_j(u, v) + \overline{N_j(u, v)}), j = 3, 4,$$
$$N_2(u, v) = (KQ - MS)u^4 + 2(KN - PS)u^3v + (3KL + MN - PQ - 3RS)u^2v^2/2;$$
$$N_3(u, v) = (K(DK + 2AQ - CS) - GK(M - S) - AS(M + S))\,u^7 + (M(DK - GM) + Q(CK + AM) - S(GM + AQ) + 2K(BK + 2AN + G(Q - P) + S(D - F)) - 2S(AP + C(M + S)))u^6v + 3((M(AN - GP) + K(2AL + BM + CN + GN + DQ - GR + BS) - S(FM + CP + GP + CQ + AR + FS)))u^5v^2 + (2P(AN - GP) + Q(DM + AN - CP - 2GP - CQ - AR) + K(5CL + 4GL + 4DN + 2FN + 2BP + 4BQ - DR) + M(5AL + BM + 2CN + GN - DP - FQ - 4GR) + S(AL + BM - 2CN - 2DP - 4FP - 5FQ - 4CR - 5GR))u^4v^3;$$

$$N_4(u,v) = (A^2Q + K(G(G-C) - 2M + 2S) + A(2DK - CS - G(M+S)))u^6 \\ +(A(DM + 2AN - 2GP + CQ) - 2M(CG+M) + K(4AB + CD + 3DG - 2FG - 4P \\ +8Q) - S(C^2 + 2AF + 3CG + 4M + 6S))u^5v + (B(3C+4G)K + D(2DK + GM) \\ -G^2P - 2M(3P - Q) + 2K(7N - 3R) - 10S(P+Q) - FG(3M+5S) + A(3AL + 3BM \\ +3CN + GN + DQ - 3GR + BS) - C(3GP + 2GQ + 2DS + 3FS))u^4v^2 \\ +(10KL - 2P(2P+Q) + 4M(N-2R) + A(5CL + 2GL + 2DN + 2FN + BP \\ +BQ - DR) + C(CN - GN - DP - FQ - 4GR) - 8RS - 2F(2GP + FS))u^3v^3.$$

The identity $N_2(u,v) + \overline{N_2(u,v)} \equiv 0$ gives us the following three set of conditions:

(2.4) $$K = L = R = S = 0,\ P = \alpha M, Q = N/\alpha,\ MN \neq 0,\ \alpha \in \mathbb{C},\ \alpha\overline{\alpha} = 1;$$

(2.5) $$M = N = P = Q = 0,\ R = \beta K,\ S = L/\beta,\ KL \neq 0, \beta \in \mathbb{C},\ \beta\overline{\beta} = 1;$$

(2.6) $$P = \gamma N,\ Q = M/\gamma,\ R = \gamma L,\ S = K/\gamma,\ KLMN \neq 0,\ \gamma \in \mathbb{C},\ \gamma\overline{\gamma} = 1.$$

Lemma 2.1. *The line at infinity has the multiplicity at least two for cubic system* {(1.1),(1.2)} *if and only if the coefficients of* {(1.1),(1.2)} *verify one of the set of conditions* (2.4), (2.5) *and* (2.6).

Under the conditions (2.4), (2.5) and (2.6) we have respectively:

$$M_3(u,v) = -uv(u+\alpha v)((FN - \alpha BM)(2Nuv + \alpha Mu^2 + \alpha Nv^2)v^2 + (CN - \alpha DM)\cdot \\ (N - \alpha^2 M)u^2v^2 - \alpha(AN - \alpha GM)(Mu^2 + Nv^2 + 2\alpha Muv)u^2)/\alpha^2 \equiv 0 \ \Rightarrow$$

(2.7) $$K = L = R = S = 0, F = B/\alpha, G = \alpha A, N = \alpha^2 M, P = \alpha M, Q = \alpha M, M \neq 0, \alpha\overline{\alpha} = 1;$$

(2.8) $$K = L = R = S = 0,\ D = CN/(\alpha M),\ F = \alpha BM/N,\ G = AN/(\alpha M), \\ P = \alpha M, Q = N/\alpha, M(N - \alpha^2 M) \neq 0, \alpha\overline{\alpha} = 1;$$

$$M_3(u,v) = -(u^3 + \beta v^3)(v(2u^3 - \beta v^3)(CL^2 - \beta DKL + \beta FKL - \beta^2 BK^2 \\ +u(u^3 - 2\beta v^3)\cdot(AL^2 + \beta CKL - \beta GKL - \beta^2 DK^2) \\ +3u^2v^2(FL^2 - \beta BKL - \beta^2 AKL + \beta^3 GK^2))/\beta^2 \equiv 0 \ \Rightarrow$$

(2.9) $$M = N = P = Q = 0,\ C = \beta DK/L,\ F = \beta BK/L, \\ G = AL/(\beta K),\ R = \beta K,\ S = L/\beta,\ \beta\overline{\beta} = 1;$$

(2.10) $$M = N = P = Q = 0,\ F = D + (\beta^2 BK^2 - CL^2)/(\beta KL), \\ G = C + (AL^2 - \beta^2 DK^2)/(\beta KL),\ R = \beta K,\ S = L/\beta,\ L^3 - \beta^4 K^3 = 0,\ \beta\overline{\beta} = 1;$$

$$M_3(u,v) = (Ku^3 + Mu^2v + \gamma Nuv^2 + \gamma Lv^3)((\gamma K(\gamma D - C) - (A - \gamma G)(K - \gamma M))u^4 \\ -2(\gamma^2 N(\gamma G - A) + K(C - \gamma D + \gamma F - \gamma^2 B))u^3v + (\gamma(A - \gamma G)(N + 3\gamma L) \\ -(F - \gamma B)(3K + \gamma M) - (C - \gamma D)(M - \gamma^2 N))u^2v^2 - 2(M(F - \gamma B) - \gamma L(A \\ +\gamma C - \gamma G - \gamma^2 D))uv^3 + \gamma(CL - \gamma DL - (F - \gamma B)(N - \gamma L))v^4)/\gamma^2 \equiv 0 \ \Rightarrow$$

(2.11) $$D = C/\gamma, F = B\gamma, G = A/\gamma, P = \gamma N, R = \gamma L, \\ Q = M/\gamma, S = K/\gamma, KM \neq 0, \gamma\overline{\gamma} = 1;$$

(2.12) $$D = (CL\gamma^3 + (F - B\gamma)(K - M\gamma))/(L\gamma^4), G = (K(B\gamma - F) + AL\gamma^2)/(L\gamma^3), \\ N = (-K + M\gamma + L\gamma^4)/\gamma^3, P = (-K + M\gamma + L\gamma^4)/\gamma^2, R = L\gamma, \\ Q = M/\gamma, S = K/\gamma, M(F - B\gamma) \neq 0, \gamma\overline{\gamma} = 1.$$

Substituting (2.7), (2.8) and (2.9) in the polynomial $M_4(u,v)$ we obtain, respectively,

$$M_4(u,v) = Muv(u^3\alpha(AD - 2M - AC\alpha)(u + 2v\alpha) - v^3\alpha(2u + v\alpha)(BD - BC\alpha + 2M\alpha^3) \\ -u^2v^2(4M\alpha^3 + (D - C\alpha)(B - D\alpha + C\alpha^2 - A\alpha^3))/\alpha \not\equiv 0;$$

$$M_4(u,v) = -2uv(u + v\alpha)(N^2v^3\alpha + Nuv^2(2N - M\alpha^2) - Mu^2v\alpha(N - 2M\alpha^2) \\ +M^2u^3\alpha^2)/\alpha^2 \not\equiv 0;$$

$$M_4(u,v) = 2(u^3 + v^3\beta)(KLu^3\beta - 3L^2u^2v - 3K^2uv^2\beta^3 + KLv^3\beta^2)/\beta^2 \not\equiv 0.$$

Similarly, it can be verified that the polynomial $M_4(u,v)$ is not identical zero in the case (2.10). For this it is sufficient to examine separately the identity $M_4(u,v) \equiv 0$ in cases $DL^2 - CK^2\beta^3 = 0$ and $DL^2 - CK^2\beta^3 \neq 0$.

In this way we have proved the following Lemma.

Lemma 2.2. *The line at infinity has the multiplicity at least three for cubic system* {(1.1),(1.2)} *if and only if the coefficients of* {(1.1),(1.2)} *verify one of the set of conditions* (2.7) – (2.12). *In the cases* (2.7) − (2.10) *the multiplicity is exactly three.*

To obtain the cubic systems (1.1) which have the line at infinity of multiplicity four we will investigate the identity $M_4(u,v) \equiv 0$ in each of the series of conditions (2.11) and (2.12):

$$M_4(u,v)\Big|_{(2.11)} = 2(Ku^3 + Mu^2v + Nuv^2\gamma + Lv^3\gamma)(u^3\gamma(K - M\gamma) - u^2v(3K - M\gamma + 2N\gamma^3) - uv^2(2M - N\gamma^2 + 3L\gamma^3) - v^3\gamma(N - L\gamma))/\gamma^2 \equiv 0 \;\Rightarrow$$

$$\text{(2.13)} \qquad \begin{array}{l} D = CS/K, F = BK/S, G = AS/K, L = -S^4/K^3, \\ M = S, N = R = -S^3/K^2, Q = -P = S^2/K; \end{array}$$

$$M_4(u,v)\Big|_{(2.12)} = (Ku^2\gamma - Kuv + Muv\gamma + Lv^2\gamma^3)(u^4\gamma^2(K^2(F - B\gamma)^2 + (AKL\gamma^2 + CKL\gamma^3 - ALM\gamma^3)(F - B\gamma) + 2L^2\gamma^5(K - M\gamma)) - 2u^3v\gamma(2L^2\gamma^6(M + L\gamma^3) + (F - B\gamma)(K(F - B\gamma)(K - M\gamma) - KL\gamma^3(C + F\gamma) - AL\gamma^2(K - M\gamma - L\gamma^4))) + u^2v^2((K^2 - 2KM\gamma + M^2\gamma^2 + 5KL\gamma^4 + LM\gamma^5)(F - B\gamma)^2 + L\gamma^2(AK + CK\gamma - AM\gamma + 3BK\gamma^3 - 4AL\gamma^4 + BM\gamma^4 - CL\gamma^5)(F - B\gamma) - 4L^2\gamma^5(K + M\gamma + 2L\gamma^4)) - 2Luv^3\gamma^3((FK - BK\gamma - 2FM\gamma + AL\gamma^2 + BM\gamma^2 + CL\gamma^3)(F - B\gamma) + 2L\gamma^2(M + L\gamma^3)) - Lv^4\gamma^2((FK - FM\gamma + CL\gamma^3 - FL\gamma^4 + BL\gamma^5)(F - B\gamma) - 2L\gamma(K - M\gamma))))/(L^2\gamma^9) \equiv 0 \;\Rightarrow$$

$$\text{(2.14)} \qquad \begin{array}{l} A = 2(K^3L + S^4)/(S^2(BK - FS)) - S(BK - 2FS)/(KL), R = KL/S, \\ C = 2(K^3L + S^4)/(KS(BK - FS)) - (BK^4L - 2FK^3LS - FS^5)/(K^2LS^2), \\ D = (FK^2L + BS^3)/(K^2L) + 2(K^3L + S^4)/(K^2(BK - FS)), \\ G = FS^3/(K^2L) + 2(K^3L + S^4)/(KS(BK - FS)), M = (K^3L + 2S^4)/S^3, \\ N = (2K^3L + S^4)/(K^2S), P = (2K^3L + S^4)/(KS^2), Q = (K^3L + 2S^4)/(KS^2). \end{array}$$

Lemma 2.3. *The line at infinity has the multiplicity at least four for cubic system* {(1.1),(1.2)} *if and only if the coefficients of* {(1.1),(1.2)} *verify one of the set of conditions* (2.13) *and* (2.14).

3. Proof of the Theorem 1.1

Lemma 3.4. *The following four sets of conditions are sufficient conditions for the origin* $(0,0)$ *to be a center for system* (1.1)*:*

$$\text{(3.15)} \qquad \begin{array}{l} C{=}4KS^3/(AS^3{-}BK^3), D{=}4S^4/(AS^3{-}BK^3), F = BK/S, G = AS/K, M = S, \\ L{=}{-}S^4/K^3, N{=}R{=}{-}S^3/K^2, Q = -P = S^2/K, B^2K^6 + 4K^2S^5 - A^2S^6 = 0; \end{array}$$

$$\text{(3.16)} \qquad \begin{array}{l} A = 2(K^3L + S^4)/(S^2(BK - FS)) - S(BK - 2FS)/(KL), R = KL/S, \\ C = 2(K^3L + S^4)/(KS(BK - FS)) - (BK^4L - 2FK^3LS - FS^5)/(K^2LS^2), \\ D = (FK^2L + BS^3)/(K^2L) + 2(K^3L + S^4)/(K^2(BK - FS)), \\ G = FS^3/(K^2L) + 2(K^3L + S^4)/(KS(BK - FS)), M = (K^3L + 2S^4)/S^3, \\ N = (2K^3L + S^4)/(K^2S), P = (2K^3L + S^4)/(KS^2), Q = (K^3L + 2S^4)/(KS^2), \\ (BK^6L^2 - 2FK^5L^2S + BS^8)(BK - FS) + 2LS(K^3L + S^4)^2 = 0; \end{array}$$

$$(3.17)\quad \begin{array}{l} A=(2S^3-F^2K^2)/(FS^2),\ B=S(F^2K^2-2S^3)/(FK^3),\ C=2S^2/(FK),\\ D=-2S^3/(FK^2),\ G=-FK/S,\ L=S^4/K^3,\ M=3S,N=3S^3/K^2,\\ P=3S^2/K,\ Q=3S^2/K,\ R=S^3/K^2; \end{array}$$

$$(3.18)\quad \begin{array}{l} A=(2S^3-F^2K^2)/(FS^2),\ B=S(F^2K^2+2S^3)/(FK^3),\ C=-2S^2/(FK),\\ D=-2S^3/(FK^2),\ G=-FK/S,\ L=-S^4/K^3,\ M=S,\ N=-S^3/K^2,\\ P=-S^2/K,\ Q=S^2/K,\ R=-S^3/K^2. \end{array}$$

Proof. When one of the condition (3.15), (3.16) holds the system (1.1) has an affine invariant straight line l_1 and a Darboux integrating factor of the form $\mu(x,y)=1/l_1$.

In the case (3.15): $l_1=2S^2(BK^3-AS^3)+(B^2K^4-ABKS^3+2S^5)((S-K)x+i(K+S)y),\ i^2=-1$.

In the case (3.16): $l_1=4K^2LS(K^3L+S^4)+(BK-FS)(K^3L-S^4)((K^2L+S^3)x+i(K^2L-S^3)y),\ i^2=-1$.

Under the conditions (3.17) the equalities $CF-DG=AD^3-BC^3=AF^3-BG^3=A^4L^3-B^4K^3=A^2N^3-B^2M^3=A^2R^3-B^2S^3=C^4L-D^4K=C^2N-D^2M=C^2R-D^2S=F^4K-G^4L=F^2M-G^2N=F^2S-G^2R=KN^2-LM^2=KR^2-LS^2=MR-NS=P-Q=0$ hold. Therefore, the system {(1.1),(3.17)} has an axis of symmetry and the origin is a center ([7]).

In the case (3.18) the system (1.1) has the integrating factor of the Darboux form:

$$\mu(x,y)=l_1l_2^{\alpha_2}l_3^{\alpha_3}l_4^{\alpha_4},$$

where

$$\begin{array}{l} l_1=2F-(S^2(Kx+Sx-iKy+iSy))/K^2,\quad l_2=Exp[x+Sx/K-iy+iSy/K],\\ l_3=Exp[16S^2x/K^2-(F(K-S)(Kx+Sx-iKy+iSy)^2)/K^3],\\ l_4=Exp[6S(Kx-Sx-iKy-iSy)(Kx+Sx-iKy+iSy)/K^3\\ \qquad +F(Kx+Sx-iKy+iSy)^3/K^3],\ i^2=-1\\ \alpha_2=FK(K+S)/(2S(K-S)),\ \alpha_3=FK^3/(8S^2(S-K)),\ \alpha_4=-K/24. \end{array}$$

□

To prove the Theorem 1.1, we compute the first three Lyapunov quantities L_1,L_2,L_3 for each sets of conditions (2.13) and (2.14). In the expressions for L_j, we will neglect the non-zero factors.

In the case (2.13) the first Lyapunov quantity is $L_1=4KS^3+C(BK^3-AS^3)$ and $L_1=0$ gives $C=-4KS^3)/(BK^3-AS^3)$. Substituting the expression of C in L_2 and L_3 we obtain $L_2=g_1g_2$ and $L_3=g_1g_3$, where $g_1=B^2K^6+4K^2S^5-A^2S^6$, $g_2=B^2K^6-16K^2S^5-A^2S^6$ and $g_3=(61AB^3K^7+339B^2K^6S^2-122A^2B^2K^4S^3+61A^3BKS^6+1512K^2S^7-339A^2S^8)\cdot g_2+4320K^4S^{12}$.

If $g_1=0$, then Lemma 3.4, (3.15). Taking into account that $KS\neq0$, the system of equalities $\{g_2=0,\ g_3=0\}$ do not have solutions.

In the case (2.14) the first Lyapunov quantity is $L_1=f_1f_2$, where $f_1=(BK^6L^2-2FK^5L^2S+BS^8)(BK-FS)+2LS(K^3L+S^4)^2$ and $f_2=BFK+2KL-F^2S$. If $f_1=0$, then Lemma 3.4, (3.16). Assume that $f_1\neq0$ and let $f_2=0$. Then we find that $B=(F^2S-2KL)/(FK)$ and $L_2=(K^3L-S^4)(K^3L+S^4)$. If $K^3L-S^4=0$, then Lemma 3.4, (3.17), and if $K^3L+S^4=0$, then Lemma 3.4, (3.18). Theorem 1.1 is proved.

From the proof of Theorem 1.1 it results the following statement:

Theorem 3.2. *The cubic system* (1.1) *with the line at infinity of multiplicity four has a center at the origin if and only if the coefficients of* (1.1) *verify one of the sets of conditions* (3.15)–(3.18).

Example 3.1. *Consider the cubic system*

$$\dot{x}=y,\ \ \dot{y}=(-4x+3x^2+4xy+2x^2y-5y^2)/4.$$

For this system the line at infinity has multiplicity four: $E_\infty = -Z^3(2x^2y(3x^2+5y^2)+(9x^4+4x^3y+4xy^3-25y^4)Z-8(3x+2y)(x^2+y^2)Z^2+16(x^2+y^2)Z^3)/16$. *The first two Lyapunov quantities vanish (*$L_1 = L_2 = 0$*) and the third one is* $L_3 = 75/1024 \neq 0$. *Therefore, the origin is a focus of multiplicity three.*

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