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Dedicated to the memory of Academician Mitrofan M. Choban (1942-2021)

Criteria of closedness of nilradicals in zero dimensional locally compact rings

MIHAIL URSUL and JOHN LANTA

ABSTRACT. We study in this paper conditions under which nilradicals of totally disconnected locally compact rings are closed. In the paper is given a characterization of locally finite compact rings via identities.

1. INTRODUCTION

Classical radicals are important tools in the study of the structure of rings. Sometimes studying properties of a ring we can reduce its study to the radical and to the factor ring with respect to the radical. For example, Jacobson uses in the proof of the theorem of I. Kaplansky ([10]) that a nil PI-algebra is locally nilpotent the Levitzki nilradical (see [8], Chapter X, Theorem 1, p. 232). Initially the Jacobson radical was used for Banach algebras ([21]), locally compact rings ([11], [12], [13], [14]) and left linearly compact rings (see also [18], [19]). It is well known that the Jacobson radical is closed in a Banach algebra, in a locally compact, in a compact, and in a linearly compact riing. These results are important in the structure theory of the corresponding classes of topological rings. However there are some obstacles in the class of all topological rings because the Jacobson radical of a topological ring is not necessarily closed. Thus the factor ring of a topological ring with respect to the radical is not Hausdorff and we have no information about it. The closedness of radicals was studied since forties of the last century (see [1], [11], [22], [25]). We study in this paper the following question: Let R be a totally disconnected locally compact ring and ρ a nilradical. Under which conditions $\rho(R)$ is closed?
We give a complete answer to this question. Moreover, we will find criteria of closedness of the weakly finite and locally finite radicals of a locally compact totally disconnected ring.

2. NOTIONS AND NOTATION

Rings are assumed associative not necessarily with identity. By $\mathbb{N}$ is denoted the set $\{1, 2, 3, \ldots\}$. A ring R has a *finite charactersitic* if there exists $n \in \mathbb{N}$ such that $nx = 0$ for all $x \in R$.
The additive group of a ring R is denoted R^+. The subring of a ring generated by a subset S is denoted $\langle S\rangle$. *Ideal* means a two-sided ideal. The symbol $\cong$ ($\cong_{top}$) means abstract (topological) isomorphism.

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Corresponding author: Mihail Ursul; mihail.ursul@pnguot.ac.pg, mihail.ursul@gmail.com

The symbol $\overline{A}$ denotes the closure of a subset A of a topological space. Neighborhoods of points are not assumed to be open. Topological rings are assumed to be Hausdorff and associative.

All topogical spaces are assumed to be completely regular. We refer to [5] for all facts from the set-theoretic topology used in this paper.

Recall that a topological ring is called *totally bounded* if its completion is compact, equivalently, it is a subring of a compact ring.

We will recall for the reader's convenience the notion of a radical in the sense of Kurosh in the class of associative rings (see [15], p. 15):

Consider a class of rings having a property S. These rings are called S-rings. An ideal of a ring which also is an S-ring is called S-ideal. If a ring R has an S-ideal J containing all S-ideals, then we will say that J is the S-radical of R. A ring which does not contain nonzero S-ideals is called S-semisimple.

We say that in the class of associative rings is defined the S-radical if:

(1) A homomorphic image of an S-ring is an S-ring.

(2) Every ring has the S-radical.

(3) The factor ring of a ring with respect to the S-radical is S-semisimple.

We say that a radical ρ is a *nilradical* if all ρ-radical rings are nilrings and all nilpotent rings are radical.

Radical ring means always radical in the sense of Jacobson. We will recall briefly the construction of the *lower nilradical*, the *Levitzki nilradical* and the *upper nilradical*. Recall that the *lower nilradical* (*the Baer's radical*) of a ring R is constructed by transfinite induction: Set $\mathfrak{B}(0) = 0$ and $\mathfrak{B}(\alpha + 1)$ is the inverse image of the sum of all nilpotent ideals of the ring $R/\mathfrak{B}(\alpha)$ under the canonical homomorphism of R on $R/\mathfrak{B}(\alpha)$. If α is a limit ordinal, put $\mathfrak{B}(\alpha) = \cup_{\beta<\alpha}\mathfrak{B}(\alpha)$. There exists an ordinal α such that $\mathfrak{B}(\alpha) = \mathfrak{B}(\alpha + 1)$. The ideal $\mathfrak{B}(\alpha)$ is called the *Baer radical* or the *lower nilradical* of R and is denoted $\mathfrak{B}(R)$.

The *Levitzki's nilradical* $\mathfrak{Le}(R)$ of a ring R is the sum of all locally nilpotent ideals of a ring ([8], page 197). The *upper nilradical(=Köthe's radical)*, $\mathfrak{K}(R)$, of a ring R is the sum of all nilideals of a ring.

It is known that $\mathfrak{B}(R) \subseteq \mathfrak{Le}(R) \subseteq \mathfrak{K}(R)$ for every ring R. However there exist rings R, S such that $\mathfrak{Le}(R) \nsubseteq \mathfrak{B}(R)$ and $\mathfrak{K}(S) \nsubseteq \mathfrak{Le}(S)$.

A topological ring R is called *locally pseudocompact (locally countably compact)* if its underlying topological group is locally pseudocompact (locally countably compact) (see [2]), i.e., if it has a pseudocompact (countably compact) neighborhood of zero.

Recall that an element x of a ring R is called *right (left) quasi–regular* if there exists an element $y \in R$ such that $x + y + xy = 0 \quad (x + y + yx = 0)$. An element is called *quasi-regular* if it is left and right quasi-regular. An element a of a ring is called *nilpotent* if there exists $n \in \mathbb{N}$ such that $a^n = 0$. A ring R is called *nilpotent* if there exists $n \in \mathbb{N}$ such that $x_1x_2\cdots x_n = 0$ for all $x_1, x_2, \ldots, x_n \in R$.

A right ideal I of a ring R is called *right quasi-regular* if every element of I is right quasi-regular.

Recall that a right ideal I of a ring R is called *modular* if there exists $e \in R$ such that $x - ex \in I$ for all $x \in R$.

The Jacobson radical of a ring has different characterizations.

We will recall the following characterization ([8], Chapter I, §6, Theorem 1(2), p. 9): The *Jacobson radical* of a ring R is a quasi-regular ideal which contains every quasi-regular

right ideal.
Every nilring is a radical ring. The Jacobson radical of a ring R is denoted $J(R)$.

Recall that a module is said to be *artinian* if each nonempty set of submodules contains a minimal element. A ring R is called *left (right) artinian* if the left (right) R-module ${}_RR(R_R)$ is artinian. Basic theory of artinian rings can be found in ([8], Chapter 3).

A topological ring R is called a *Q_r-ring* provided the set of all right quasi-regular elements is open and a Q-ring if the set of all quasi-regular elements is open. It is unknown if there exists a Q_r-ring which is not a Q-ring ([11], p.155). It is well-known that every right maximal modular ideal of a Q_r-ring is closed ([23], Theorem 6.2, p.124). Therefore the Jacobson radical of a Q_r-ring is closed.

3. Preliminaries

Theorem 3.1. [2] *The completion of a locally pseudocompact (locally countably compact) topological ring is a locally compact ring.*

Remark 3.1. Let R be a ring whose Jacobson radical $J(R)$ is a nilring of bounded degree, i.e., there exists $n \in \mathbb{N}$ such that $\forall_{x \in J(R)}[x^n = 0]$.
Then the Jacobson radical is closed for every ring topology $\mathcal{T}$. In particular, if R is a ring and $J(R)$ is nilpotent then the Jacobson radical is closed for every ring topology $\mathcal{T}$.

Theorem 3.2. (Levitzki) [6] *Let R be a ring and $0 \neq \rho$ a right ideal which is a nilring of bounded degree. Then R has a nonzero nilpotent ideal.*

Corollary 3.1. *Every nilring R of bounded degree is ρ-radical for every nilradical ρ.*

Indeed, $R/\rho(R)$ has no nonzero nilpotent ideals, hence $R/\rho(R) = 0$, i.e., $R = \rho(R)$.

Theorem 3.3. [24] *Any compact nilring is a nilring of bounded degree.*

Corollary 3.2. *A compact nilring is ρ-radical for every nilradical ρ.*

Theorem 3.4. ([12], Lemma 4) *A locally compact totally disconnected ring R has a fundamental system of neighborhoods of zero consisting of compact open subrings.*

Theorem 3.5. ([8], Chapter 3, Theorem 1, p. 38) *The Jacobson radical of a left artinian ring is nilpotent.*

4. Conditions under which nilradicals of zero dimensional locally compact rings are closed

The study of locally compact rings started after publication of classical results of L. S. Pontryagin about classification of connected locally compact division rings and H. J. Kowalsky about nondiscrete totally disconnected locally compact division rings.

The papers of Kaplansky [11], [12], [13], [14] contain basic results about locally compact rings. Kaplansky has proved [12] that the Jacobson radical of a locally compact ring is closed and is the intersection of all closed left (or right) maximal modular ideals.

An important tool in the study of locally compact rings plays Lemma 4 from [12] (see Theorem 3.4). It should be mentioned that this theorem has a counterpart in the theory of locally compact groups, namely, the van Dantzig theorem stating that every locally compact totally disconnected group has a fundamental system of neighborhhods of identity consisting of compact subgroups ([7], Theorem (7.8), p. 78).

Remark 4.2. For any radical ρ the class of ρ-radical rings is closed under extensions.

Indeed, let A be a ring and B an ideal of A such that A/B and B are ρ-radical. Then $B \subseteq \rho(A)$. The induced homomorphism $\lambda : A/B \to A/\rho(A)$ is surjective. Since A/B is ρ-radical and $A/\rho(A)$ is ρ-semisimple, $A/\rho(A) = 0$, i.e., $A = \rho(A)$.

Lemma 4.1. *Let R be a totally disconnected locally compact ring having a compact open nilsubring V and ρ be a nilradical. If I is a dense ρ-radical ideal, then R is a ρ-radical ring.*

Proof. By Theorem 3.3, V is a nilring of bounded degree. As follows from Corollary 3.2, V is ρ-radical. Since I is dense, we have $R = V + I$. The factor ring R/I is isomorphic to the ring $V/(V \cap I)$, hence it is ρ-radical. Since the class of ρ-radical rings is closed under extensions , R is a ρ-radical ring. $\square$

Theorem 4.6. *Let R be a totally disconnected locally compact ring and ρ be a nilradical. Then $\rho(R)$ is closed iff there exists an open compact subring V of R such that $V \cap \rho(R)$ is a nilring of bounded degree.*

Proof. " $\Rightarrow$ " : Let V be any compact open subring. Then $V \cap \rho(R)$ is a compact nilring. By Theorem 3.3 it is a nilring of bounded degree.

" $\Leftarrow$ " : Let V be a compact open subring of R such that $V \cap \rho(R)$ is a nilring of bounded degree. The closure $\overline{V \cap \rho(R)}$ is an open compact nilsubring of bounded degree of the closure $\overline{\rho(R)}$ of $\rho(R)$. By Lemma 4.1 $\overline{\rho(R)}$ is ρ-radical, hence $\overline{\rho(R)} \subseteq \rho(R)$ and so $\overline{\rho(R)} = \rho(R)$.

$\square$

5. The weakly finite and locally finite radicals in locally compact rings

Definition 5.1. [4] A ring is called weakly locally finite if each its element is contained in a finite subring.

Definition 5.2. A ring is called locally finite if every finite subset is contained in a finite subring.

Proposition 5.1. *Every commutative weakly locally finite ring is locally finite.*

We leave the proof to the reader.

Lemma 5.2. *If a ring R has a weakly locally finite ideal I such that R/I is weakly locally finite, then R is a weakly locally finite ring.*

Proof. If $x \in R$, then $(\langle x\rangle + I)/I$ is finite. Since $(\langle x\rangle + I)/I \cong \langle x\rangle/(\langle x\rangle \cap I)$

and by Proposition 5.1 $\langle x\rangle \cap I$ is locally finite, $\langle x\rangle$ is finite. $\square$

Corollary 5.3. *If A is a subring and B a weakly locally finite ideal of a ring R, then the subring $A + B$ is weakly locally finite.*
In particular, the sum of two weakly locally finite ideals of a ring is a weakly locally finite ideal.

From Lemma 5.2 and Corollary 5.3 follows:

Theorem 5.7. *For every ring R there exists a weakly locally finite ideal $\mathfrak{W}(R)$ such that $R/\mathfrak{W}(R)$ does not contain nonzero weakly locally finite ideals.*

Theorem 5.8. *For every ring R there exists a locally finite ideal $\mathfrak{L}(R)$ such that $R/\mathfrak{L}(R)$ does not contain nonzero locally finite one-sided ideals.*

See, for instance, ([23], Theorem 18.21, p. 220).

Theorem 5.9. *(compare with* [23]*, Lemma 27.35) For a compact ring R the following conditions are equivalent:*

(a) R is weakly locally finite;

(b) R has a finite characteristic and there exist $n, m \in \mathbb{N}, n > m$ such that

$$\forall_{x\in R}[x^n - x^m = 0];$$

(c) R is locally fiinite.

Proof. "$(a) \Rightarrow (b)$" : We notice that the additive group R^+ is a torsion group. Since R^+ is compact there exists $k \in \mathbb{N}$ such that $\forall_{x\in\mathbb{N}}[kx = 0]$, i.e. R has a finite characteristic.

If $x \in J(R)$, then the subring $\langle x\rangle \subseteq J(R)$ is finite, hence left artinian. By Theorem 3.5 $\langle x\rangle$ is a nilring. By ([11], Theorem 4), $J(R)$ is closed, hence compact. It follows from Theorem 3.3 that $J(R)$ is a nilring of bounded degree.

Obviously, $R/J(R)$ is a weakly locally finite ring. As follows from Kaplansky's Theorem, ([11], Theorem 16), $R/J(R) \cong_{top} \prod_{l=1}^{\infty} M(l, F_l)^{\mathfrak{m}_l}$ for some numbers l, where $M(l, F_l)$ is the ring of $l \times l$ matrices over a finite field F_l and $\mathfrak{m}_l$ are cardinal numbers.

Claim 1. The number of nonisomorphic fields F_l is finite.

Indeed, otherwise the ring $R/J(R)$ will contain an invertible element of infinite order, a contradiction with the periodicity of $R/J(R)$.

Claim 2. The sizes of matrices are bounded in totality.

Indeed, otherwise we can find an infinite sequence $R_1, R_2, \cdots$ of factors of $R/J(R)$, containing nilpotent elements $x_1, x_2, \ldots$ whose indexes of nilpotency are not bounded in totality. It is clear that $R/J(R)$ will contain an infinite one-generated subring, a contradiction.

It follows that there exist a finite number $F_1, F_2, \cdots, F_n$ of finite fields, natural numbers $m_1, \ldots, m_n$ and cardinal numbers $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ such that
$R/J(R) \cong_{top} \prod_{i=1}^{n} M(m_i, F_i)^{\mathfrak{m}_i}$. This implies that there exists a finite ring A and a cardinal number τ such that $R/J(R)$ is embedded in A^τ. It follows there exist $n, m \in \mathbb{N}, n > m$ such that $\forall_{x\in R/J(R)}[x^n - x^m = 0]$.

Since $J(R)$ is a nilring of bounded degree there exists a natural number s such that $\forall_{x\in R}[(x^n - x^m)^s = 0]$.

We claim that the cardinalities of one-generated subrings of R are bounded in totality. For, $x^{ns} \in \langle x^{ns-1}, x^{ns-2}, \ldots, x^{ms}\rangle^+$. It follows that $\langle x\rangle \subseteq \langle x^{ns-1}, x^{ns-2}, \ldots, x\rangle^+$, hence $|\langle x\rangle| \leq k^{ns-1}$.

This implies that R belongs to a variety of rings generated by some finite ring B. For, there exists a finite number $B_1, \cdots, B_q$ of nonisomorphic finite rings of characteristic k and of cardinality $\leq k^{ns-1}$. Put $B = B_1 \times \cdots \times B_q$. Then every one-generated subring of R is a homomorphic image of B.

Since B satisfies the identity $x^a = x^b$, where $a, b \in \mathbb{N}, a > b, kx = 0$, the ring R satisfies the same identity.

"$(b) \Rightarrow (c)$" : Every one-generated subring of R is finite. Clearly, the cardinalities of one-generated subrings are bounded in totality. It follows that R belongs to a variety generated by a finite ring. Since any variety generated by a finite ring is locally finite ([20], p. 359), R is a locally finite ring.

"$(c) \Rightarrow (a)$" : Obvious. □

Remark 5.3. The implication $(b) \Rightarrow (c)$ has been proved also in [17], Proposition 2.1.

Corollary 5.4. *The completion of a totally bounded ring R is a locally finite ring iff the following two conditions are satisfied:*

$i)$ R has a finite characteristic;

$ii)$ there exist $n, m \in \mathbb{N}, n > m$ such that

$$\forall_{x \in R}[x^n - x^m = 0].$$

Remark 5.4. Compact locally finite rings were studied also in [4].

Theorem 5.10. *The weakly locally finite radical $\mathfrak{W}(R)$ (the locally finite radical $\mathfrak{L}(R)$) of a totally disconnected locally compact ring is closed iff there exists an open compact subring V such that $V \cap \mathfrak{W}(R)$ has a finite characteristic and there exist $n, m \in \mathbb{N}, n > m$ such that $\forall_{x \in V \cap \mathfrak{W}(R)}[x^n - x^m = 0]$.*

Proof. The proof is given only for the weakly finite radical because the proof for the locally finite radical is analogous.

" $\Rightarrow$": Let U be an open compact subring of R. Then the subring of $V = U \cap \mathfrak{W}(R)$ will be a compact weakly locally finite subring of R. Applying Theorem 5.9, we finish the proof.

" $\Leftarrow$": Let V be a compact open subring such that such that $V \cap \mathfrak{W}(R)$ has a finite characteristic and there exist $n, m \in \mathbb{N}, n > m$ such that $\forall_{x \in V \cap \mathfrak{W}(R)}[x^n - x^m = 0]$.

The subring $\overline{V \cap \mathfrak{W}(R)} = W$ is compact, open in $\overline{\mathfrak{W}(R)}$, has a finite characteristic and there exist $n, m \in \mathbb{N}, n > m$ such that $\forall_{x \in W}[x^n - x^m = 0]$.

It follows from Theorem 5.9 that W is locally finite. We have $\overline{\mathfrak{W}(R)} = W + \mathfrak{W}(R)$. By Corollary 5.3, $\overline{\mathfrak{W}(R)}$ is weakly locally finite, hence $\overline{\mathfrak{W}(R)} \subseteq \mathfrak{W}(R)$ and so $\overline{\mathfrak{W}(R)} = \mathfrak{W}(R)$. □

Theorem 5.11. *Let R be a locally compact totally disconnected ring. Then every maximal modular left ideal is closed iff R is a Q-ring.*

Proof. " $\Leftarrow$ " : Is true for every topological Q-ring ([23], Chapter 2, §6, Theorem 6.2, p. 124).

" $\Rightarrow$ " : Assume on the contrary that every maximal modular left ideal is closed however R is not a Q-ring. Let V be a compact open ring. Then $V \neq J(V)$ and by ([11], Theorem 16), $V/J(V)$ is a ring with identity. Since idempotents of the factor ring $V/J(V)$ can be lifted modulo $J(V)$ (see, for instance, [11], Lemma 12), there exists an idempotent $e \in V$ such that $e + J(V)$ is the identity of $V/J(V)$.

Let $V/J(V) = \prod_{\alpha \in \Omega} P_\alpha$ be the product of finite simple rings and e'_α be the identity of P_α. Let $f : V \to V/J(V)$ be the canonical homomorphism. There exists a family $\{e_\alpha\}_{\alpha \in \Omega}$ of orthogonal idempotents of V such that $e = \sum_{\alpha \in \Omega} e_\alpha$ and $f(e_\alpha) = e_\alpha$ for each $\alpha \in \Omega$. We notice that $Re = \overline{\sum_{\alpha \in \Omega} Re_\alpha}$.
Since R is not a Q-ring, the set Ω is infinite. Let $_RR$ be R considered as a left R-module. Consider the Peirce decomposition $_RR = Re \oplus R(1 - e)$ which is a topological direct sum of R-submodules. Let $\mathfrak{M} = \{L | \oplus_{\alpha \in \Omega} Re_\alpha \subseteq L \subseteq Re\}$ where L is a submodule of $_RR$ and $L \neq Re$. By Zorn's lemma there exists a maximal module H in $\mathfrak{M}$. Then $H \oplus Re$ will be a maximal modular left ideal of R. Since $\oplus_{\alpha \in \Omega} Re_\alpha \subseteq H \neq Re$ and $\oplus_{\alpha \in \Omega} Re_\alpha$ is dense in Re, H is not closed. Then $H \oplus R(1 - e)$ will is a nonclosed maximal modular left ideal, a contradiction. □

Example of a compact ring with nonclosed Baer radical

Let p be a prime number and $R_n = \mathbb{Z}/p^n\mathbb{Z}(n \in \mathbb{N})$ and $R = \prod_{n\in\mathbb{N}}(\mathbb{Z}/p^n\mathbb{Z})$ the product of discrete rings $\mathbb{Z}/p^n\mathbb{Z}$. Then R is a commutative compact zero dimensional ring, hence all nilradicals of R coincide.

Its Jacobson radical $J(R)$ is $\prod_{n\in\mathbb{N}}(p\mathbb{Z}/p^n\mathbb{Z}) = pR$. The element $x = (p + p^n\mathbb{Z})_{n\in\mathbb{N}}$ is not nilpotent and is a limit of nilpotent elements. Therefore the nilradical $N(R)$ is not closed.

Recall that a topological space is said to be *pseudocompact* if each continuous real-valued function on it is bounded. A topological space is called *countably compact* if each its countable open cover contains a finite subcover.

Lemma 5.3. *Any countably compact ring R without nonzero idempotents is radical.*

Proof. We will reduce the proof to the case when R has a fundamental system of neighborhoods of zero consisting of ideals.

Since every countably compact space is pseudocompact, by ([3], Theorem 1.1), the completion $\widehat{R}$ is compact. Let $(\widehat{R})_0$ be its component of zero of $\widehat{R}$ and $\varphi : \widehat{R} \to \widehat{R}/(\widehat{R})_0$ be the canonical homomorphism. Then ([11], Theorem 8) $\widehat{R} \cdot (\widehat{R})_0 = (\widehat{R})_0 \cdot \widehat{R} = 0$.

The ring $\varphi(R) = (R + (\widehat{R})_0)/(\widehat{R})_0$ is countably compact and has a fundamental system of neighborhoods of zero consisting of ideals. It is isomorphic as an abstract ring to $R/(R \cap (\widehat{R})_0)$.

We claim that $(R + (\widehat{R})_0)/(\widehat{R})_0$ has no nonzero idempotents. Indeed, on the contrary $R/(R \cap (\widehat{R})_0)$ will have a nonzero idempotent. Since idempotents can be lifted modulo ideals with trivial multiplication (see [16], §3.6, Proposition 1) R will contain a nonzero idempotent, a contradiction.

We reduced the proof to the case when R has a fundamental system of neighborhoods of zero consisting of ideals.

If $x \in R$, then $\langle x\rangle$ is metrizable, hence $\overline{\langle x\rangle}$ is countably compact and metrizable, hence compact. Since $\overline{\langle x\rangle}$ has no nonzero idempotents it is radical ([23], Theorem 5.29, p. 123). The element x was arbitrary, hence R is radical. □

Lemma 5.4. *If R is a countably compact ring having a dense ideal A without nonzero idempotents, then R is a radical ring.*

Proof. It is known that the Jacobson radical of a countably compact ring is closed ([23], Corollary 13.8, page 182). By Lemma 5.3 a countably compact ring without nonzero idempotents is radical in the sense of Jacobson. If $a \in A$, then Ra is countably compact and $Ra \subseteq A$, hence $Ra \subseteq J(R)$ and so $RA \subseteq J(R)$. Since $J(R)$ is closed, by continuity of ring operations $R^2 \subseteq \overline{RA} \subseteq J(R)$, hence $R = J(R)$. □

Theorem 5.12. *The Jacobson radical of a topological ring R having a fundamental system of neighborhoods of zero consisting of open countably compact subrings is closed.*

Proof. Fix an open countably compact subring V of R. Assume that $\overline{J(R)} \neq J(R)$. Then we may assume that $J(R)$ is dense in R. The ideal $V \cap J(R)$ of V is dense and has no nonzero idempotents. By Lemma 5.4 V is radical in the sense of Jacobson, hence R is a $Q-$ring. It follows that $\overline{J(R)} = J(R)$, a contradiction.

□

Open question 1. For which locally compact rings its von Neumann radical (= the largest regular ideal) is closed?

Open question 2. For which locally compact rings with identity the Brown-McCoy radical (i.e. the intersection of all maximal two-sided ideals) is closed?

Open question 3. For which radicals ρ, $\rho(R)$ is closed for every compact ring R?

The following radicals possess this property: a) the Jacobson radical; b) von Neumann radical.

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References

[1] Andrunakievich, V. A. The radical of generalized Q-rings. *Izvestia Akademii Nauk SSSR. Seria Matemat-iceskaia* **18** (1954), 419–426.
[2] Comfort, W. W.; Trigos-Arrieta, F. Javier Locally pseudocompact topological groups. *Topology and its Applications* **62** (1995), 263–280.
[3] Comfort, W. W.; Ross, K. A. Pseudocompactness and uniform continuity in topological groups. *Pac. J. Math.* **16** (1966), no. 3, 483-496.
[4] Dobrowolski, J.; Krupinski, K. Locally finite profinite rings. *J. Algebra* **401** (2014), 161–178.
[5] Engelking, R. *General Topology Revised and completed edition*, Heldermann Verlag Berlin, 1989.
[6] Herstein, I *Topics in ring theory*, The University of Chicago Press, 1969.
[7] Hewitt, E; Ross, K. A. *Abstract Harmonic Analysis I Structure of Topological Groups Integration Theory Group Representations Second Edition*, Springer Verlag Grundlehren der Mathematischen Wissenschaften 115 A Series of Comprehensive Series in Mathematics 1979.
[8] Jacobson, N. Structure of Rings. Structure of rings. Revised edition American Mathematical Society Colloquium Publications, **Vol. 37** *American Mathematical Society, Providence, R.I.* 1964
[9] Jacobson, N. Structure of rings. *American Mathematical Society* 1968.
[10] Kaplansky, I. Rings with a polynomial identity. *Bull. Amer. Math. Soc.* **54** (1948), no. 6, 575-580.
[11] Kaplansky, I. Topological rings. *Amer. J. Math.* **69**(1947), 153–183.
[12] Kaplansky, I. Locally compact rings I. *Amer. J. Math.* **70** (1948), 447–459.
[13] Kaplansky, I. Locally compact rings II. *Amer. J. Math.* **73**(1951), 20–24.
[14] Kaplansky, I. Locally compact rings III. *Amer. J. Math.* **74** (1952), 929–935.
[15] Kurosh, A. G. Radicals of rings and algebras. *Mat. Sb. (N.S.)* **33** (1953), no. 75, 13–26.
[16] Lambek, J. *Lectures on rings and modules*, Blaisdell Publishing Company, 1966.
[17] L'vov, I. V. Varieties of associative rings. II. *Algebra i Logika* **12** (1973), no. 3, 269–287.
[18] Leptin, H. Linear kompakte moduln und ringe. *Math. Z.* **63** (1955), 241–267.
[19] Leptin, H. Linear kompakte moduln und ringe. *Math. Z.* **66** (1957), 289–327.
[20] Mal'cev, A. I. *Algebraic systems*. "Nauka", Moscow, 1970.
[21] Rickart, C. E. *General Theory of Banach Algebras*. Van Nostrand Reinhold Company, 1974.
[22] Ursul, M. Tripe, A.; Kuna, R. Closedness of radicals in topological rings. *Serdica Math. J.* **45** (2019), no. 4, 317-324.
[23] Ursul, M. Topological Rings Satisfying Compactness Conditions. Mathematics and its Applications. *Kluwer Academic Publishers* (vol. 549), Dordrecht, 2002.
[24] Ursul, M. Compact nilrings. (Russian) *Mat. Zametki* **36** (1984), no. 6, 839–845
[25] Yood, B. Incomplete normed algebras. *Bull. Amer. Math. Soc.* **79** (1972), no. 1, 50–52.

Department of Mathematics and Computer Science
PNG University of Technology
Lae, Morobe Province Papua New Guinea
Email address: mihail.ursul@pnguot.ac.pg
Email address: john.lanta@pnguot.ac.pg