CARPATHIAN J. MATH.
Volume **38** (2022), No. 1,
Pages 231 - 248

Online version at `https://www.carpathian.cunbm.utcluj.ro/`
Print Edition: ISSN 1584 - 2851; Online Edition: ISSN 1843 - 4401
DOI: https://doi.org/10.37193/CJM.2022.01.19

Dedicated to the memory of Academician Mitrofan M. Choban (1942-2021)

Extension of Haar's theorem

JATURON WATTANAPAN, WATCHAREEPAN ATIPONRAT, SANTI TASENA and TEERAPONG SUKSUMRAN

ABSTRACT. Haar's theorem ensures a unique nontrivial regular Borel measure on a locally compact Hausdorff topological group, up to multiplication by a positive constant. In this article, we extend Haar's theorem to the case of locally compact Hausdorff strongly topological gyrogroups. We simultaneously prove the existence and uniqueness of a Haar measure on a locally compact Hausdorff strongly topological gyrogroup, using the method of Steinlage. We then find a natural relationship between Haar measures on gyrogroups and on their related groups. As an application of this result, we study some properties of a convolution-like operation on the space of Haar integrable functions defined on a locally compact Hausdorff strongly topological gyrogroup.

1. INTRODUCTION

Usually, analyzing the structure of topological groups (especially compact and locally compact groups) involves the presence of invariant measures. In fact, if G is a locally compact Hausdorff topological group, then *Haar's theorem* states that there is, up to multiplication by a positive constant, a unique nontrivial regular measure μ on the Borel subsets of G, the so-called (left) Haar measure, that satisfies the following properties:

(1) The measure μ is invariant under left translation: $\mu(gB) = \mu(B)$ for all $g \in G$ and for all Borel subsets B of G.
(2) The measure μ is finite on all compact sets: $\mu(K) < \infty$ for all compact sets $K \subseteq G$.

Existence, uniqueness (up to scaling), and applications of Haar measures attracted the attention of several mathematicians, including A. Haar, A. Weil, H. Cartan, and R. C. Steinlage, to name a few. The importance of Haar's theorem lies in the fact that it was used to solve Hilbert's fifth problem (on studying Lie groups) for compact groups by John von Neumann. Actually, the notion of a Haar measure has applications in several fields such as analysis, number theory, group theory, representation theory, statistics, probability theory, and ergodic theory. For an introduction to the theory of Haar measures, we refer the reader to [6].

The notion of a gyrogroup was introduced by A. A. Ungar, arising from the study of the parametrization of the Lorentz transformation group [13]. Roughly speaking, a gyrogroup is a nonassociative group-like structure that shares many properties with groups and, in fact, every group may be viewed as a gyrogroup with trivial gyroautomorphisms. In [1], W. Atiponrat introduced the notion of a topological gyrogroup, a gyrogroup with a compatible topology. In [2], M. Bao and F. Lin defined the notion of a strongly topological gyrogroup. It seems that strongly topological gyrogroups suitably generalize topological groups. Several results that are valid for topological groups can be extended to the case of strongly topological gyrogroups in a natural way; see, for instance, [2, 3, 15, 16].

Received: 21.09.2021. In revised form: 04.11.2021. Accepted: 10.11.2021
2010 *Mathematics Subject Classification.* 28C10, 20N05, 54E15, 43A05.
Key words and phrases. *Haar measure, topological gyrogroup, uniformizable space, Haar integral, convolution.*
Corresponding author: T. Suksumran; teerapong.suksumran@cmu.ac.th

This work is a continuation of the study of strongly topological gyrogroups. In fact, we prove the existence and uniqueness of a Haar measure on a locally compact Hausdorff strongly topological gyrogroup, following the method of Steinlage. This result extends Haar's theorem to the case of gyrogroups. As shown in [14], every locally compact Hausdorff topological gyrogroup G can be embedded in a completely regular topological group $\Gamma(G, \mathcal{A}_{\mathrm{gyr}})$. We then find a natural relationship between Haar measures on gyrogroups and on their related groups. We also investigate some properties of a convolution-like operation on the space of Haar integrable functions defined on a locally compact Hausdorff strongly topological gyrogroup. We emphasize that topological gyrogroups enjoy several (but not all) of the properties of topological groups as we will see in Section 3.2.

2. Preliminaries

In this section, we collect basic definitions and relevant results for reference. See, for instance, [5,7,8,13].

2.1. **Strongly topological gyrogroups.** In the case when G is a nonempty set equipped with a binary operation $\oplus$ on G, let $\operatorname{Aut} G$ be the group of automorphisms of $(G, \oplus)$.

Definition 2.1 (Definition 2.7, [13])**.** A nonempty set G, together with a binary operation $\oplus$ on G, is called a *gyrogroup* if it satisfies the following axioms:

(G1) There exists an element $e \in G$ such that $e \oplus a = a$ for all $a \in G$.

(G2) For each $a \in G$, there exists an element $b \in G$ such that $b \oplus a = e$.

(G3) For all $a, b \in G$, there is an automorphism $\mathrm{gyr}[a, b] \in \operatorname{Aut} G$ such that

$$a \oplus (b \oplus c) = (a \oplus b) \oplus \mathrm{gyr}[a, b]c \tag{2.1}$$

for all $c \in G$. (left gyroassociative law)

(G4) For all $a, b \in G$, $\mathrm{gyr}[a \oplus b, b] = \mathrm{gyr}[a, b]$. (left loop property)

We remark that the axioms in Definition 2.1 imply the right counterparts. In fact, any gyrogroup has a unique two-sided identity, denoted by e, and that an element a of the gyrogroup has a unique two-sided inverse, denoted by $\ominus a$. The automorphism $\mathrm{gyr}[a, b]$ is called the *gyroautomorphism* generated by a and b. The *gyrogroup cooperation* (cf. Definition 2.9, [13]) of a gyrogroup G, denoted by $\boxplus$, is defined by

$$a \boxplus b = a \oplus \mathrm{gyr}[a, \ominus b]b, \qquad a, b \in G. \tag{2.2}$$

Let G be a gyrogroup. Define $a \ominus b = a \oplus (\ominus b)$ and $a \boxminus b = a \boxplus (\ominus b)$ for all $a, b \in G$. For each $a \in G$, the *left gyrotranslation* by a, denoted by L_a, is defined by $L_a(g) = a \oplus g$ for all $g \in G$. Similarly, the *right gyrotranslation* by a, denoted by R_a, is defined by $R_a(g) = g \oplus a$ for all $g \in G$. They are indeed bijections from G to itself (cf. Theorem 2.22, [13]). For a subgyrogroup H of a gyrogroup G, set $G/H = \{a \oplus H \mid a \in G\}$, where $a \oplus H$ is the *left coset* defined by $a \oplus H = \{a \oplus h \mid h \in H\}$.

Definition 2.2 (Definition 17, [12])**.** A subgyrogroup H of a gyrogroup G is an *L-subgyrogroup*, denoted by $H \leqslant_L G$, if $\mathrm{gyr}[a, h](H) = H$ for all $a \in G, h \in H$.

We remark that a generic subgyrogroup H of a gyrogroup G do not partition G into left cosets. However, if H is an L-subgyrogroup of G, then G/H forms a partition of G and, in particular, two distinct left cosets of H are disjoint (cf. Theorem 20, [12]).

Theorem 2.1 (see [11,13]). *The following properties hold in any gyrogroup G:*

(1) $\ominus(a \oplus b) = \text{gyr}[a,b](\ominus b \ominus a)$;
(2) $(b \boxminus a) \oplus a = b$; (right cancellation law)
(3) $(\ominus a \oplus b) \oplus \text{gyr}[\ominus a, b](\ominus b \oplus c) = \ominus a \oplus c$;
(4) $\text{gyr}^{-1}[a,b] = \text{gyr}[b,a]$, where $\text{gyr}^{-1}[a,b]$ denotes the inverse of $\text{gyr}[a,b]$ with respect to composition of functions; (inversive symmetry)
(5) $L_a^{-1} = L_{\ominus a}$.

for all $a, b, c \in G$.

Recall that a gyrogroup G endowed with a topology is called a *topological* gyrogroup if (i) the gyroaddition map $(x, y) \mapsto x \oplus y$ is jointly continuous, and (ii) the inversion map $x \mapsto \ominus x$ is continuous (cf. Definition 1, [1]).

Proposition 2.1 (Lemma 4, [1]). *Let G be a topological gyrogroup. Then* $\text{gyr}[a,b]$ *is a homeomorphism of G for all $a, b \in G$.*

Proposition 2.2 (Proposition 7, [1]). *Let G be a topological gyrogroup, and let A be a subgyrogroup of G. If A is open, then it is also closed.*

Proposition 2.3 (Corollary 5, [1]). *Suppose that G is a topological gyrogroup, and let A, B be subsets of G. If A and B are compact, then $A \oplus B$ is compact.*

Definition 2.3 (p. 5116, [2]). A topological gyrogroup G is *strong* if there exists a neighborhood base $\mathcal{U}$ at the identity e of G such that $\text{gyr}[x,y](U) = U$ for all $x, y \in G, U \in \mathcal{U}$. In this case, we say that G is a *strongly topological gyrogroup* with neighborhood base $\mathcal{U}$ at e.

Recall that a nonempty subset A of a gyrogroup G is *symmetric* if $\ominus A = A$, where $\ominus A = \{\ominus a \mid a \in A\}$. In the case when G is a strongly topological gyrogroup, there exists a neighborhood base $\mathcal{N}$ at e such that for all $U \in \mathcal{N}$,

(1) $\text{gyr}[a,b](U) = U$ for all $a, b \in G$;
(2) U is symmetric.

In fact, if G is a strongly topological gyrogroup with neighborhood base $\mathcal{U}$ at e, then the collection $\mathcal{N} = \{U \cap (\ominus U) \mid U \in \mathcal{U}\}$ is the desired neighborhood base. We call any neighborhood base that also satisfies property (2) a *symmetric neighborhood base* at e.

2.2. **Basic knowledge in topology and measure theory.** Standard terminology in topology and measure theory used throughout this article are defined as usual.

Let X be a nonempty set, and let $A, B \subseteq X \times X$. We will use the symbol A^{-1} to denote the set $\{(x, y) \in X \times X \mid (y, x) \in A\}$ and $A \circ B$ to denote the set $\{(x, z) \in X \times X \mid (x, y) \in A$ and $(y, z) \in B$ for some $y \in X\}$. The *diagonal* of X is defined as $\Delta_X = \{(x, x) \mid x \in X\}$.

Definition 2.4 (p. 45, [7]). A *uniformity* on a set X is a collection $\mathcal{U}$ of subsets of $X \times X$ such that the following properties hold:

(1) If $U \in \mathcal{U}$ and $U \subseteq V$, then $V \in \mathcal{U}$.
(2) If $U \in \mathcal{U}$ and $V \in \mathcal{U}$, then $U \cap V \in \mathcal{U}$.
(3) $\Delta_X \subseteq U$ for all $U \in \mathcal{U}$.
(4) If $U \in \mathcal{U}$, then $U^{-1} \in \mathcal{U}$.
(5) If $U \in \mathcal{U}$, then there is a set $V \in \mathcal{U}$ such that $V \circ V \subseteq U$.

If $\mathcal{U}$ is a uniformity on X, then the pair $(X, \mathcal{U})$ is called a *uniform space*.

Theorem 2.2 (p. 48, [7]). *Let $(X, \mathcal{U})$ be a uniform space. For all $U \in \mathcal{U}, x \in X$, define $U[x] = \{y \mid (x, y) \in U\}$. Then the collection*

$$\mathcal{T}_{\mathcal{U}} = \{O \subseteq X \mid \textit{for each } x \in O, \textit{ there is a set } U \in \mathcal{U} \textit{ such that } U[x] \subseteq O\} \tag{2.3}$$

is a topology on X. The topology $\mathcal{T}_{\mathcal{U}}$ is called the uniform topology on X derived from $\mathcal{U}$.

Definition 2.5 (p. 48, [7])**.** Let X be a topological space. If the topology of X can be derived from a uniformity on X, then X is *uniformizable*.

Let X be a nonempty set. A *filterbase* $\mathcal{B}$ on X is a collection of subsets of X such that (i) $\emptyset \notin \mathcal{B}$, and (ii) if $A, B \in \mathcal{B}$, then there is a set $C \in \mathcal{B}$ satisfying $C \subseteq A \cap B$. A filterbase $\mathcal{B}$ on $X \times X$ is said to be a *base for a uniformity* if the collection

$$\{E \subseteq X \times X \mid B \subseteq E \text{ for some } B \in \mathcal{B}\} \tag{2.4}$$

is a uniformity on X (cf. Theorem 1 on p. 46 of [7]).

Definition 2.6 (p. 225, [7])**.** Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be uniform spaces. A function f from X to Y is *uniformly continuous* on X if for each $V \in \mathcal{V}$, there is a set $U \in \mathcal{U}$ such that $(f(x_1), f(x_2)) \in V$ for all $(x_1, x_2) \in U$.

Definition 2.7 (Definition 4.8, [10])**.** Let X be a topological space, and let $(Y,\mathcal{U})$ be a uniform space. A collection $\mathcal{F}$ of functions from X to Y is *equicontinuous* at a point x of X if for each $U \in \mathcal{U}$, there is an open set O containing x such that $f(O) \subseteq U[f(x)]$ for all $f \in \mathcal{F}$. If $\mathcal{F}$ is equicontinuous at every point of X, we say that $\mathcal{F}$ is equicontinuous.

Definition 2.8 (see [10])**.** Let X be a topological space, and let $\mathcal{G}$ be a group of homeomorphisms of X to X.

(1) $\mathcal{G}$ is *weakly transitive* if for every nonempty open set O, $X = \bigcup_{g\in\mathcal{G}} g(O)$.
(2) $\mathcal{G}$ is *x_0-weakly transitive* if $X \subseteq \bigcup_{g\in\mathcal{G}} g(O)$ for all neighborhoods O of x_0.
(3) $\mathcal{G}$ *separates compact sets* if for each pair of disjoint compact sets B, C in X, there is a nonempty open set O such that $g(O) \cap C = \emptyset$ or $g(O) \cap B = \emptyset$ for all $g \in \mathcal{G}$.
(4) $\mathcal{G}$ *separates compact sets with neighborhoods of x_0* if for each pair of disjoint compact sets B, C in X, there is a neighborhood O of x_0 such that $g(O)\cap C = \emptyset$ or $g(O)\cap B = \emptyset$ for all $g \in \mathcal{G}$.

It is not difficult to see that "being weakly transitive" implies "being x_0-weakly transitive" and that "being able to separate compact sets with neighborhoods of x_0" implies "being able to separate compact sets".

Let X be a locally compact Hausdorff space, and let $\mathcal{G}$ be a group of homeomorphisms of X to X. A *Haar measure* μ on X with respect to $\mathcal{G}$ is a nontrivial, regular, Borel measure μ such that $\mu(g(B)) = \mu(B)$ for all Borel sets B and for all $g \in \mathcal{G}$ (cf. Definition 3.1, [10]).

Theorem 2.3 (Theorem 4.4, [10])**.** *If $\mathcal{G}$ is an x_0-weakly transitive group of homeomorphisms of a nonempty locally compact Hausdorff space X and if $\mathcal{G}$ separates compact sets with neighborhoods of x_0, then there is a Haar measure μ on X with respect to $\mathcal{G}$. Furthermore, $\mu(O) > 0$ whenever O is a neighborhood of x_0.*

Theorem 2.4 (Theorem 5.3, [10])**.** *Let $\mathcal{G}$ be an equicontinuous group of homeomorphisms of a nonempty locally compact Hausdorff space X whose topology is generated by a uniformity $\mathcal{U}$. Then a Haar measure on X is unique if and only if $\mathcal{G}$ is weakly transitive.*

In view of Theorems 2.3 and 2.4, if X is a certain topological group and if $\mathcal{G}$ is chosen to be the group of left translations in X, we recover the classic version of a Haar measure as mentioned in Haar's theorem. As shown in Section 9.4 of [6], the Haar measure μ constructed via Theorems 2.3 and 2.4 satisfies the property that $\mu(K) < \infty$ for all compact sets K in X.

Let X and Y be nonempty sets, and let $U \subseteq X \times Y$. For all $x \in X$, $y \in Y$, define

$$U_x = \{z \in Y \mid (x,z) \in U\} \quad \text{and} \quad U^y = \{z \in X \mid (z,y) \in U\}.$$

Let f be a function with domain $X \times Y$. For all $x \in X$, $y \in Y$, let f_x and f^y be the functions defined by

$$f_x(w) = f(x, w) \quad \text{and} \quad f^y(z) = f(z, y)$$

for all $w \in Y, z \in X$, respectively.

Proposition 2.4 (Proposition 7.6.5, [5]). *Let X and Y be locally compact Hausdorff spaces, let μ and ν be regular Borel measures on X and Y, respectively, and let $\mu \times \nu$ be the regular Borel product of μ and ν. If U is an open subset of $X \times Y$, then*

(1) the functions $x \mapsto \nu(U_x)$ and $y \mapsto \mu(U^y)$ are lower semicontinuous and hence Borel measurable;

(2) $(\mu \times \nu)(U) = \displaystyle\int_X \nu(U_x)\mu(dx) = \int_Y \mu(U^y)\nu(dy)$.

Proposition 2.5 (Exercise 7.6.4, [5]). *Let X and Y be locally compact Hausdorff spaces, let μ and ν be regular Borel measures on X and Y, respectively, and let $\mu \times \nu$ be the regular Borel product of μ and ν. If $f : X \times Y \to [0, \infty]$ is a nonnegative Borel measurable function that vanishes outside a Borel rectangle with σ-finite sides, then*

(1) $x \mapsto \displaystyle\int f(x, y)\nu(dy)$ and $y \mapsto \displaystyle\int f(x, y)\mu(dx)$ are Borel measurable;

(2) $\displaystyle\int f d(\mu \times \nu) = \int \int f(x, y)\nu(dy)\mu(dx) = \int \int f(x, y)\mu(dx)\nu(dy)$.

Proposition 2.6 (Proposition 2.6.8, [5]). *Let $(X, \mathcal{A}, \mu)$ be a measure space, let $(Y, \mathcal{B})$ be a measurable space, and let $f : (X, \mathcal{A}) \to (Y, \mathcal{B})$ be measurable. Let g be an extended real-valued $\mathcal{B}$-measurable function on Y. Then g is μf^{-1}-integrable if and only if $g \circ f$ is μ-integrable. If these functions are integrable, then $\displaystyle\int_Y g d(\mu f^{-1}) = \int_X (g \circ f) d\mu$.*

Proposition 2.7 (Proposition 7.6.7, [5]). *Let X and Y be locally compact Hausdorff spaces, let μ and ν be regular Borel measures on X and Y, respectively, and let $\mu \times \nu$ be the regular Borel product of μ and ν. If f belongs to $\mathcal{L}^1(X \times Y, \mathcal{B}(X \times Y), \mu \times \nu)$ and vanishes outside a rectangle whose sides are Borel sets that are σ-finite under μ and ν, respectively, then*

(1) $f_x \in \mathcal{L}^1(Y, \mathcal{B}(Y), \nu)$ for μ-almost every x and $f^y \in \mathcal{L}^1(X, \mathcal{B}(X), \mu)$ for ν-almost every y;

(2) the functions

$$x \mapsto \begin{cases} \displaystyle\int f_x d\nu & \text{if } f_x \in \mathcal{L}^1(Y, \mathcal{B}(Y), \nu), \\ 0 & \text{otherwise,} \end{cases}$$

and

$$y \mapsto \begin{cases} \displaystyle\int f^y d\mu & \text{if } f^y \in \mathcal{L}^1(X, \mathcal{B}(X), \mu), \\ 0 & \text{otherwise,} \end{cases}$$

belong to $\mathcal{L}^1(X, \mathcal{B}(X), \mu)$ and $\mathcal{L}^1(Y, \mathcal{B}(Y), \nu)$, respectively; and

(3) $\displaystyle\int f d(\mu \times \nu) = \int_X \int_Y f(x, y)\nu(dy)\mu(dx) = \int_Y \int_X f(x, y)\mu(dx)\nu(dy)$.

3. Main results

In Section 3.1, we prove the existence and uniqueness of a Haar measure on a locally compact Hausdorff strongly topological gyrogroup, following the method of Steinlage. We then find a natural relationship between Haar measures on gyrogroups and on their appropriate corresponding groups.

3.1. **Haar measures on locally compact Hausdorff strongly topological gyrogroups.** We begin this section by showing that a concrete example of a strongly topological gyrogroup in which symmetric neighborhood base at the identity does exist.

Example 3.1. Consider the open unit ball in $\mathbb{C}$, $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$, together with Möbius addition $\oplus_M$ given by

$$a \oplus_M b = \frac{a+b}{1+\overline{a}b} \tag{3.5}$$

for all $a, b \in \mathbb{D}$. The pair $(\mathbb{D}, \oplus_M)$ is called the *(complex) Möbius gyrogroup*. When $\mathbb{D}$ is equipped with the subspace topology of $\mathbb{C}$, it becomes a topological gyrogroup (cf. Example 2, [1]). Furthermore, if $a, b \in \mathbb{D}$, then the gyroautomorphism $\mathrm{gyr}[a, b]$ is given by

$$\mathrm{gyr}[a, b]c = \frac{1+a\overline{b}}{1+\overline{a}b}c$$

for all $c \in \mathbb{D}$. Hence, $|\mathrm{gyr}[a, b]c| = |c|$ for all $a, b, c \in \mathbb{D}$. This implies that

$$\mathrm{gyr}[a, b](B(0, r)) = B(0, r)$$

for all $a, b \in \mathbb{D}$ and for all r with $0 < r \leq 1$, where $B(0, r)$ is the open ball in $\mathbb{C}$ of radius r centered at 0. Therefore, $\mathbb{D}$ is a strongly topological gyrogroup with symmetric neighborhood base $\{B(0, r) \mid 0 < r \leq 1\}$ at 0.

Inspired by the definition of a Haar measure on a locally compact Hausdorff topological group, we formulate the following definition for topological gyrogroups.

Definition 3.9. Let G be a locally compact Hausdorff strongly topological gyrogroup. A *(left) Haar measure* μ on G is a nontrivial, regular, Borel measure such that

(i) $\mu(K) < \infty$ for all compact sets K in G;
(ii) $\mu(x \oplus B) = \mu(B)$ for all Borel sets B and for all $x \in G$.

The next theorem is the main result of this section. By following the method of Steinlage in [10], its proof will be divided into several parts; some of them are important in their own right.

Theorem 3.5. *Let G be a locally compact Hausdorff strongly topological gyrogroup. Then there exists a unique Haar measure on G, up to multiplication by a positive constant. Moreover, every nonempty open set has nonzero measure.*

Let G be a strongly topological gyrogroup with symmetric neighborhood base $\mathcal{N}$ at e. For each $U \in \mathcal{N}$, set

$$E_U = \{(x, y) \in G \times G \mid \ominus x \oplus y \in U\}. \tag{3.6}$$

Observe that if $U \in \mathcal{N}$ and if $x \in X$, then

$$E_U[x] = \{y \mid (x, y) \in E_U\} = \{y \mid \ominus x \oplus y \in U\} = x \oplus U.$$

In order to simultaneously prove the existence and uniqueness of a Haar measure on a locally compact Hausdorff strongly topological gyrogroup, we first show that any strongly topological gyrogroup is a uniformizable space in the sense of Definition 2.5.

Theorem 3.6. *Every strongly topological gyrogroup is uniformizable.*

Proof. Let G be a strongly topological gyrogroup with symmetric neighborhood base $\mathcal{N}$ at e, and let $\mathcal{B} = \{E_U \mid U \in \mathcal{N}\}$. We first prove that $\mathcal{B}$ is a base for a uniformity on G. Obviously, $\Delta_G \subseteq E_U$ for all $U \in \mathcal{N}$. If U and V are in $\mathcal{N}$, then there is a set $W \in \mathcal{N}$ such that $W \subseteq U \cap V$. Let $(x, y) \in E_W$. Then $\ominus x \oplus y \in W \subseteq U \cap V$. Thus, $(x, y) \in E_U \cap E_V$. This proves that $E_W \subseteq E_U \cap E_V$. Let $V \in \mathcal{N}$. If $\ominus x \oplus y \in V$, then $\ominus(\ominus x \oplus y) \in V$ because V is symmetric. Since $\ominus(\ominus x \oplus y) = \mathrm{gyr}[\ominus x, y](\ominus y \oplus x)$ and $V \in \mathcal{N}$, it follows that $\ominus y \oplus x \in V$ by Theorem 2.1 (4). This implies that $E_V^{-1} = E_V$. Let $V \in \mathcal{N}$. Then there is a set $W \in \mathcal{N}$ such that $W \oplus W \subseteq V$. Suppose that (x, y) and (y, z) lie in E_W. Then $\ominus x \oplus y$ and $\ominus y \oplus z$ are in W. It follows that $\mathrm{gyr}[\ominus x, y](\ominus y \oplus z) \in W$. Thus, by Theorem 2.1 (3),

$$\ominus x \oplus z = (\ominus x \oplus y) \oplus \mathrm{gyr}[\ominus x, y](\ominus y \oplus z) \in W \oplus W \subseteq V.$$

Therefore, $E_W \circ E_W \subseteq E_V$. According to Theorem 1 on p. 46 of [7], the collection

$$\mathcal{U} = \{E \subseteq G \times G \mid E_U \subseteq E \text{ for some } U \in \mathcal{N}\}$$

is a uniformity on G.

Next, we prove that the uniform topology derived from $\mathcal{U}$ is the same as the original topology of G. Let O be an open set in G, and let $x \in O$. Then there is a set $U \in \mathcal{N}$ such that $x \oplus U \subseteq O$. Thus, $E_U[x] \subseteq O$. Conversely, let O' be an open set with respect to the uniform topology derived from $\mathcal{U}$, and let $x \in O'$. Then there is a set $E \in \mathcal{U}$ such that $E[x] \subseteq O'$. Let $U \in \mathcal{N}$ be such that $E_U \subseteq E$. Note that $x \oplus U = E_U[x] \subseteq E[x] \subseteq O'$. Thus, O' is an open set in G. This shows that the two topologies are the same. Hence, G is uniformizable. □

It turns out that gyroautomorphisms of a strongly topological gyrogroup, which are fundamental functions that encode information about the gyrogroup structure, are uniformly continuous, as shown in the following proposition.

Proposition 3.8. *Every gyroautomorphism of a strongly topological gyrogroup is uniformly continuous.*

Proof. Let G be a strongly topological gyrogroup with symmetric neighborhood base $\mathcal{N}$ at e. Let $a, b \in G$, and let $U \in \mathcal{N}$. Suppose that $(x, y) \in E_U$. Then

$$\ominus \mathrm{gyr}[a, b]x \oplus \mathrm{gyr}[a, b]y = \mathrm{gyr}[a, b](\ominus x \oplus y) \in \mathrm{gyr}[a, b](U) = U.$$

Thus, $(\mathrm{gyr}[a, b]x, \mathrm{gyr}[a, b]y) \in E_U$, which completes the proof. □

For any Hausdorff strongly topological gyrogroup G, let $\mathrm{H}(G)$ be the group of all homeomorphisms from G to G, and let $\mathcal{T}$ be the smallest subgroup of $\mathrm{H}(G)$ that contains all left gyrotranslations of G. According to Part (5) of Theorem 2.1, $L_a^{-1} = L_{\ominus a}$ for all $a \in G$. It follows that $\mathcal{T}$ consists precisely of all finite compositions of left gyrotranslations. Lemma 3.1 shows that the image of any neighborhood of the identity under a transformation in $\mathcal{T}$ coincides with the image under a left gyrotranslation. This lemma will be used to prove Lemma 3.2, which shows that $\mathcal{T}$ is weakly transitive, separates compact sets with neighborhoods of e, and is equicontinuous. Therefore, Theorems 2.3 and 2.4 apply.

Lemma 3.1. *Let G be a strongly topological gyrogroup with neighborhood base $\mathcal{N}$ at e. For all $g \in \mathcal{T}$, there is an element $x \in G$ such that $g(U) = L_x(U)$ for all $U \in \mathcal{N}$.*

Proof. Let $x, y \in G$. Note that

$$(L_x \circ L_y)(U) = (x \oplus y) \oplus \mathrm{gyr}[x, y](U) = (x \oplus y) \oplus U = L_{x \oplus y}(U)$$

for all $U \in \mathcal{N}$. Hence, the lemma follows from mathematical induction. □

Remark 3.1. From the proof of Lemma 3.1, we obtain that for all $x_1, x_2, \ldots, x_n \in G$ and for all $U \in \mathcal{N}$,

$$x_n \oplus (x_{n-1} \oplus (\cdots (x_2 \oplus (x_1 \oplus U)))) = (x_n \oplus (x_{n-1} \oplus (\cdots \oplus (x_2 \oplus x_1)))) \oplus U.$$

Lemma 3.2. *Let G be a Hausdorff strongly topological gyrogroup with symmetric neighborhood base $\mathcal{N}$ at e. Then $\mathcal{T}$ is weakly transitive, separates compact sets with neighborhoods of e, and is equicontinuous. In particular, $\mathcal{T}$ is e-weakly transitive and separates compact sets.*

Proof. First, we prove that $\mathcal{T}$ is weakly transitive. Let O be a nonempty open set, let $x \in O$, and let $y \in G$. Then $L_{y \boxminus x} \in \mathcal{T}$ and, by the right cancellation law (cf. Theorem 2.1 (2)), $L_{y\boxminus x}(x) = y$. Hence, $G \subseteq \bigcup_{g \in \mathcal{T}} g(O)$ and so equality holds.

Next, we prove that $\mathcal{T}$ separates compact sets with neighborhoods of e. Let B, C be disjoint compact sets in G. Suppose that for each $U \in \mathcal{N}$, there is an element $x_U \in G$ such that $L_{x_U}(U) \cap B \neq \emptyset$ and $L_{x_U}(U) \cap C \neq \emptyset$. Then for each $U \in \mathcal{N}$, there are elements $b_U, c_U \in U$ such that $L_{x_U}(b_U) \in B$ and $L_{x_U}(c_U) \in C$. Recall that the collection $\mathcal{N}$ with the reverse inclusion—that is, for all sets $U, V \in \mathcal{N}$, $U \leq V$ if and only if $V \subseteq U$—is a directed set. It follows that $\{x_U \oplus b_U\}_{U\in\mathcal{N}}$ is a net in B. Since B is compact, $\{x_U \oplus b_U\}_{U\in\mathcal{N}}$ has a convergent subnet, say $\{x_U \oplus b_U\}_{U\in\mathcal{B}}$, where $\mathcal{B}$ is cofinal in $\mathcal{N}$. Assume that $\{x_U \oplus b_U\}_{U\in\mathcal{B}}$ converges to $b \in B$. Notice that $\{x_U \oplus c_U\}_{U\in\mathcal{B}}$ is a net in C. Since C is compact, $\{x_U \oplus c_U\}_{U\in\mathcal{B}}$ has a convergent subnet, say $\{x_U \oplus c_U\}_{U\in\mathcal{C}}$, where $\mathcal{C}$ is cofinal in $\mathcal{B}$. Assume that $\{x_U \oplus c_U\}_{U\in\mathcal{C}}$ converges to $c \in C$. Note that $\{x_U \oplus b_U\}_{U\in\mathcal{C}}$ is a subnet of $\{x_U \oplus b_U\}_{U\in\mathcal{B}}$ that converges to b. Since $b_U \in U$, $\mathrm{gyr}[x_U, b_U]b_U \in U$. It follows that the net $\{\mathrm{gyr}[x_U, b_U]b_U\}_{U\in\mathcal{C}}$ converges to e. By the continuity of $\ominus$, the net $\{\mathrm{gyr}[x_U, b_U](\ominus b_U)\}_{U\in\mathcal{C}} = \{\ominus \mathrm{gyr}[x_U, b_U]b_U\}_{U\in\mathcal{C}}$ converges to e. Since the operation $\oplus$ is continuous, the net

$$\{x_U\}_{U\in\mathcal{C}} = \{x_U \oplus (b_U \ominus b_U)\}_{U\in\mathcal{C}} = \{(x_U \oplus b_U) \oplus \mathrm{gyr}[x_U, b_U](\ominus b_U)\}_{U\in\mathcal{C}}$$

converges to $b \oplus e = b$, noting that the second equality follows from the left gyroassociative law. Similarly, one can show that the net $\{x_U\}_{U\in\mathcal{C}}$ converges to c. Since G is Hausdorff, $b = c$. This contradicts the fact that $B \cap C = \emptyset$. Therefore, there is a set $U \in \mathcal{N}$ such that $L_x(U) \cap B = \emptyset$ or $L_x(U) \cap C = \emptyset$ for all $x \in G$. This combined with Lemma 3.1 shows that $\mathcal{T}$ separates compact sets with neighborhoods of e.

Finally, we prove that $\mathcal{T}$ is equicontinuous. Let $x \in G$, let $U \in \mathcal{N}$, and let $f \in \mathcal{T}$. Then $x \oplus U$ is an open set containing x. We can assume that $f = L_{x_1} \circ L_{x_2} \circ \cdots \circ L_{x_n}$ for some $x_1, x_2, \ldots, x_n \in G$. As noted in Remark 3.1,

$$f(x \oplus U) = (f \circ L_x)(U) = L_{f(x)}(U) = f(x) \oplus U = E_U[f(x)],$$

which completes the proof. $\square$

We are now in a position to prove Theorem 3.5.

Proof of Theorem 3.5. By Theorems 2.3, 2.4, 3.6, and Lemma 3.2, there exists a unique Haar measure μ on G with respect to $\mathcal{T}$ such that (i) every neighborhood of e has nonzero measure, and (ii) $\mu(K) < \infty$ for all compact sets K in G. Hence, μ is indeed a Haar measure in the sense of Definition 3.9. Let O be a nonempty open set in G. Suppose that $x \in O$. Then $\ominus x \oplus O$ is a neighborhood of e and so $\mu(O) = \mu(\ominus x \oplus O) > 0$. $\square$

As indicated in Example 1.1 of [14], the aforementioned Möbius gyrogroup can be embedded in the general linear group $\mathrm{GL}_2(\mathbb{C})$ via the topological embedding $a \mapsto \begin{bmatrix} 1 & a \\ \bar{a} & 1 \end{bmatrix}$ for all $a \in \mathbb{D}$. This suggests studying connections between Haar measures on groups and on gyrogroups. We emphasize that an explicit formula for the Haar measure on the open disk $\mathbb{D}$ is known. In fact, if dA represents the area measure on $\mathbb{D}$, then the function

$$\mu(B) = \int_{\mathbb{D}} \frac{1_B}{(1-|z|^2)^2} dA(z) \tag{3.7}$$

for all Borel sets B of $\mathbb{D}$ defines a Haar measure on $\mathbb{D}$ that is invariant under Möbius addition $\oplus_M$; see, for instance, [9] or Section 4.1 of [17].

Let G be a locally compact Hausdorff topological gyrogroup. As shown in [14], the group $\mathcal{A}_{\mathrm{gyr}}$ generated by all the gyroautomorphisms of G,

$$\mathcal{A}_{\mathrm{gyr}} = \{\mathrm{gyr}[a_1, b_1] \circ \mathrm{gyr}[a_2, b_2] \circ \cdots \circ \mathrm{gyr}[a_n, b_n] \mid n \in \mathbb{N}, a_i, b_i \in G\}, \tag{3.8}$$

forms a Hausdorff topological group with respect to the subspace topology induced by the g-topology of the homeomorphism group of G. Recall that the space

$$\Gamma(G, \mathcal{A}_{\mathrm{gyr}}) = \{(a, \tau) \mid a \in G, \tau \in \mathcal{A}_{\mathrm{gyr}}\}, \tag{3.9}$$

endowed with the product topology, forms a completely regular topological group with group law

$$(a, \tau)(b, \sigma) = (a \oplus \tau(b),\ \mathrm{gyr}[a, \tau(b)] \circ \tau \circ \sigma) \tag{3.10}$$

for all $a, b \in G$ and for all $\tau, \sigma \in \mathcal{A}_{\mathrm{gyr}}$. Furthermore, according to Theorem 3.9 of [14], G is embedded in $\Gamma(G, \mathcal{A}_{\mathrm{gyr}})$ via the topological embedding $a \mapsto (a, i)$, where i is the identity function of G. By Theorem 3.5, if G is strong, then G possesses a unique Haar measure. However, the existence of a Haar measure on $\Gamma(G, \mathcal{A}_{\mathrm{gyr}})$ is in question because the local compactness of this group is not known. Assuming the local compactness of $\mathcal{A}_{\mathrm{gyr}}$, we obtain a strong connection between Haar measures on G and $\Gamma(G, \mathcal{A}_{\mathrm{gyr}})$, as shown in the following theorem and corollary.

Theorem 3.7. *Let G be a locally compact Hausdorff strongly topological gyrogroup with a Haar measure μ. If $\mathcal{A}_{\mathrm{gyr}}$ is locally compact, then there exists a unique Haar measure ν on $\mathcal{A}_{\mathrm{gyr}}$. Consequently, the regular Borel product $\mu \times \nu$ is the unique Haar measure on $\Gamma(G, \mathcal{A}_{\mathrm{gyr}})$.*

Proof. The existence and uniqueness of a Haar measure on a locally compact Hausdorff topological group is well known. Note that $\mu \times \nu$ is a nontrivial regular Borel measure on $\Gamma(G, \mathcal{A}_{\mathrm{gyr}})$. Let K be a compact subset of $\Gamma(G, \mathcal{A}_{\mathrm{gyr}})$. Then

$$(\mu \times \nu)(K) \le (\mu \times \nu)(p_1(K) \times p_2(K)) = \mu(p_1(K))\nu(p_2(K)) < \infty.$$

Next, let U and V be open sets in G and $\mathcal{A}_{\mathrm{gyr}}$, respectively, and let $(a, \tau) \in \Gamma(G, \mathcal{A}_{\mathrm{gyr}})$. Note that

$$\begin{aligned}(a, \tau)(U \times V) &= \{(a, \tau)(b, \sigma) \mid b \in U, \sigma \in V\}\\ &= \{(a \oplus \tau(b), \mathrm{gyr}[a, \tau(b)] \circ \tau \circ \sigma) \mid b \in U, \sigma \in V\}.\end{aligned}$$

Then, for each $(x, \alpha) \in \Gamma(G, \mathcal{A}_{\mathrm{gyr}})$, $(x, \alpha) \in (a, \tau)(U \times V)$ if and only if $x = a \oplus \tau(b), \alpha = \mathrm{gyr}[a, \tau(b)] \circ \tau \circ \sigma$ for some $(b, \sigma) \in U \times V$ if and only if $x \in a \oplus \tau(U), \alpha \in (\mathrm{gyr}[a, \ominus a \oplus x] \circ \tau)V$. It follows that

$$((a, \tau)(U \times V))_x = \begin{cases} (\mathrm{gyr}[a, \ominus a \oplus x] \circ \tau)V & \text{if } x \in a \oplus \tau(U);\\ \emptyset & \text{otherwise.}\end{cases}$$

This shows that

$$\nu(((a, \tau)(U \times V))_x) = \begin{cases} \nu(V) & \text{if } x \in a \oplus \tau(U);\\ 0 & \text{otherwise.}\end{cases}$$

By Proposition 2.4,

$$\begin{aligned}(\mu\times\nu)((a,\tau)(U\times V)) &= \int \nu(((a,\tau)(U\times V))_x)\mu(dx)\\ &= \int \nu(V)1_{a\oplus\tau(U)}\mu(dx)\\ &= \nu(V)\mu(a\oplus\tau(U))\\ &= (\mu\times\nu)(U\times V).\end{aligned}$$

Claim. *Let $W_1, W_2, \dots, W_n$ be Borel sets in G, and let $Z_1, Z_2, \dots, Z_n$ be Borel sets in $\mathcal{A}_{\mathrm{gyr}}$. Then, for each $\epsilon > 0$, there are open sets $U_1, U_2, \dots, U_k$ of G and open sets $V_1, V_2, \dots, V_k$ of $\mathcal{A}_{\mathrm{gyr}}$ such that*

$$\bigcup_{i=1}^{n}(W_i\times Z_i)\subseteq\bigcup_{i=1}^{k}(U_i\times V_i),$$

and

$$\sum_{i=1}^{k}(\mu\times\nu)(U_i\times V_i) < (\mu\times\nu)\left(\bigcup_{i=1}^{n}(W_i\times Z_i)\right)+\epsilon.$$

Proof of the Claim. Let $Q(n)$ be the statement "If $W_1, W_2, \dots, W_n$ are Borel sets in G, and if $Z_1, Z_2, \dots, Z_n$ are Borel sets in $\mathcal{A}_{\mathrm{gyr}}$, then there are Borel sets $A_1, A_2, \dots, A_k$ in G and $B_1, B_2, \dots, B_k$ in $\mathcal{A}_{\mathrm{gyr}}$ such that $\bigcup_{i=1}^{n}(W_i\times Z_i)=\bigcup_{i=1}^{k}(A_i\times B_i)$ and $(A_i\times B_i)\cap(A_j\times B_j)=\emptyset$ whenever $i\neq j$." Note that $Q(1)$ holds trivially. Assume that $Q(N)$ holds. Recall that for all sets A, B, C, and D, $(A\times B)\setminus(C\times D)=((A\cap C)\times(B\setminus D))\cup((A\setminus C)\times B)$. Let $W_1, W_2, \dots, W_{N+1}$ be Borel sets in G, and let $Z_1, Z_2, \dots, Z_{N+1}$ be Borel sets in $\mathcal{A}_{\mathrm{gyr}}$. Then,

$$\begin{aligned}\bigcup_{i=1}^{N+1}(W_i\times Z_i) &= \left(\bigcup_{i=1}^{N}(W_i\times Z_i)\right)\cup(W_{N+1}\times Z_{N+1})\\ &= \left(\bigcup_{i=1}^{N}(W_i\times Z_i)\right)\cup\left((W_{N+1}\times Z_{N+1})\setminus\left(\bigcup_{i=1}^{N}(W_i\times Z_i)\right)\right)\\ &= \left(\bigcup_{i=1}^{N}(W_i\times Z_i)\right)\\ &\quad\cup((\cdots((W_{N+1}\times Z_{N+1})\setminus(W_1\times Z_1))\setminus\cdots)\setminus(W_N\times Z_N)).\end{aligned}$$

Therefore, $Q(N+1)$ holds.

Let W and Z be Borel sets in G and $\mathcal{A}_{\mathrm{gyr}}$, respectively, and let $\epsilon > 0$. Then, there is a number $\delta > 1$ such that $\delta\mu(W)\nu(Z) < \mu(W)\nu(Z)+\epsilon$. By regularity of μ and ν, there are open sets U and V with $W\subseteq U$ and $Z\subseteq V$ such that $\mu(U) < \sqrt{\delta}\mu(W)$ and $\nu(V) < \sqrt{\delta}\nu(Z)$. Then,

$$\begin{aligned}(\mu\times\nu)(U\times V) &= \mu(U)\nu(V)\\ &< (\sqrt{\delta}\mu(W))(\sqrt{\delta}\nu(Z))\\ &= \delta\mu(W)\nu(Z)\\ &< \mu(W)\nu(Z)+\epsilon.\end{aligned}$$

Let $W_1, W_2, \dots, W_n$ be Borel sets in G, let $Z_1, Z_2, \dots, Z_n$ be Borel sets in $\mathcal{A}_{\rm gyr}$, and let $\epsilon > 0$. Then, there are Borel sets $A_1, A_2, \dots, A_k$ in G and $B_1, B_2, \dots, B_k$ in $\mathcal{A}_{\rm gyr}$ such that

$$\bigcup_{i=1}^{n}(W_i \times Z_i) = \bigcup_{i=1}^{k}(A_i \times B_i),$$

and $(A_i \times B_i) \cap (A_j \times B_j) = \emptyset$ whenever $i \neq j$. For each i with $1 \leq i \leq k$, there are open sets U_i and V_i with $A_i \subseteq U_i$ and $B_i \subseteq V_i$ such that

$$(\mu \times \nu)(U_i \times V_i) < (\mu \times \nu)(A_i \times B_i) + \frac{\epsilon}{k}.$$

It follows that

$$\begin{aligned}
(\mu \times \nu)\left(\bigcup_{i=1}^{k}(U_i \times V_i)\right) &\leq \sum_{i=1}^{k}(\mu \times \nu)(U_i \times V_i) \\
&< \sum_{i=1}^{k}(\mu \times \nu)(A_i \times B_i) + \epsilon \\
&= (\mu \times \nu)\left(\bigcup_{i=1}^{k}(A_i \times B_i)\right) + \epsilon \\
&= (\mu \times \nu)\left(\bigcup_{i=1}^{n}(W_i \times Z_i)\right) + \epsilon.
\end{aligned}$$

Let $K \subseteq \Gamma(G, \mathcal{A}_{\rm gyr})$ be a compact set, let O be an open set containing K, and let $\epsilon > 0$. Since K is compact, there are open sets $U_1, U_2, \dots, U_n$ of G and open sets $V_1, V_2, \dots, V_n$ of $\mathcal{A}_{\rm gyr}$ such that $K \subseteq \bigcup_{i=1}^{n}(U_i \times V_i)$. By the claim, there are open sets $A_1, A_2, \dots, A_k$ of G and open sets $B_1, B_2, \dots, B_k$ of $\mathcal{A}_{\rm gyr}$ such that

$$\bigcup_{i=1}^{n}(U_i \times V_i) \subseteq \bigcup_{i=1}^{k}(A_i \times B_i),$$

and

$$\sum_{i=1}^{k}(\mu \times \nu)(A_i \times B_i) < (\mu \times \nu)(\bigcup_{i=1}^{n}(U_i \times V_i)) + \epsilon \leq (\mu \times \nu)(O) + \epsilon.$$

Thus,

$$\begin{aligned}
(\mu \times \nu)((a,\tau)K) &\leq (\mu \times \nu)\left(\bigcup_{i=1}^{n}((a,\tau)(U_i \times V_i))\right) \\
&\leq \sum_{i=1}^{n}(\mu \times \nu)((a,\tau)(U_i \times V_i)) \\
&= \sum_{i=1}^{n}(\mu \times \nu)(U_i \times V_i) \\
&< (\mu \times \nu)(O) + \epsilon.
\end{aligned}$$

This shows that

$$\begin{aligned}
(\mu \times \nu)((a,\tau)K) &\leq \inf\{(\mu \times \nu)(O) \mid O \text{ is an open set containing } K\} + \epsilon \\
&= (\mu \times \nu)(K) + \epsilon.
\end{aligned}$$

Similarly, one can show that $(\mu \times \nu)(K) \leq (\mu \times \nu)((a,\tau)K) + \epsilon$. This proves that

$$(\mu \times \nu)((a,\tau)K) = (\mu \times \nu)(K).$$

Let O be an open set in $\Gamma(G, \mathcal{A}_{\mathrm{gyr}})$, and let $(a,\tau) \in \Gamma(G, \mathcal{A}_{\mathrm{gyr}})$. Set

$$A = \{(\mu \times \nu)(K) \mid K \text{ is a compact subset of } O\}$$

and

$$B = \{(\mu \times \nu)(K) \mid K \text{ is a compact subset of } (a,\tau)O\}.$$

Then, for $r \geq 0$,

$$\begin{aligned} r \in A \quad &\iff \quad r = (\mu \times \nu)(K) \text{ for some compact subset } K \text{ of } O \\ &\iff \quad r = (\mu \times \nu)((a,\tau)K) \text{ for some compact subset } K \text{ of } O \\ &\iff \quad r = (\mu \times \nu)(K) \text{ for some compact subset } K \text{ of } (a,\tau)O \\ &\iff \quad r \in B. \end{aligned}$$

Thus, $(\mu\times\nu)(O) = \sup A = \sup B = (\mu\times\nu)((a,\tau)O)$. By the same argument, $(\mu\times\nu)(W) = (\mu\times\nu)((a,\tau)W)$ for all Borel sets W. Therefore, $\mu\times\nu$ is a Haar measure on $\Gamma(G, \mathcal{A}_{\mathrm{gyr}})$. □

Corollary 3.1. *Let G be a locally compact Hausdorff strongly topological gyrogroup and suppose that $\mathcal{A}_{\mathrm{gyr}}$ is locally compact. Let ν and θ be the Haar measures on $\mathcal{A}_{\mathrm{gyr}}$ and $\Gamma(G, \mathcal{A}_{\mathrm{gyr}})$, respectively. Let $K \subseteq \mathcal{A}_{\mathrm{gyr}}$ be a Borel set such that $0 < \nu(K) < \infty$. Then the function μ_K, defined by $\mu_K(A) = \theta(A \times K)$ for all Borel sets A in G, defines a Haar measure on G.*

Proof. Let μ be a fixed Haar measure on G. By Theorem 3.7, there is a number $r > 0$ such that $\theta = r(\mu \times \nu)$. Therefore, $\mu_K(A) = \theta(A \times K) = r(\mu \times \nu)(A \times K) = r\mu(A)\nu(K) = (r\nu(K))\mu(A)$ for all Borel sets A in G. □

Hoping that the assumption of $\mathcal{A}_{\mathrm{gyr}}$ being locally compact may be dropped, we pose the following question.

Question 1. Let G be a locally compact Hausdorff topological gyrogroup. Is the group $\mathcal{A}_{\mathrm{gyr}}$ described by (3.8) locally compact?

We close this section with the remark that Haar measures on finite groups and on finite gyrogroups are the same. In fact, recall that if G is a finite discrete gyrogroup, then every subset of G is open and compact. Therefore, the counting measure μ on G is a regular Borel measure such that every compact set has finite measure. Moreover, μ is invariant under the left gyrotranslations because they are bijective. Hence, the Haar measure on a finite discrete gyrogroup is the counting measure.

3.2. **Convolution and its basic properties.** In this section, we will give an application of Theorem 3.5. We begin this section by proving that there exists a nice subgyrogroup of a locally compact Hausdorff strongly topological gyrogroup. This subgyrogroup will prove useful when we extend the notion of convolution, defined for locally compact Hausdorff topological groups, to the case of gyrogroups.

Lemma 3.3. *If G is a gyrogroup, then the collection*

$$\mathbb{I} = \{\emptyset \neq U \subseteq G \mid \mathrm{gyr}[a,b](U) = U \text{ for all } a,b \in G\}$$

forms a monoid under the operation $\oplus$ defined by

$$U \oplus V = \{u \oplus v \mid u \in U, v \in V\}$$

for all $U, V \in \mathbb{I}$ with identity $\{e\}$. Moreover, if G is a topological gyrogroup, then

(1) $U \in \mathbb{I}$ implies $\overline{U} \in \mathbb{I}$;

(2) $U, V \in \mathbb{I}$ implies $\ominus(U \oplus V) = \ominus V \ominus U$.

Proof. Part (1) of the lemma follows from Proposition 2.1 and the fact that $\mathrm{gyr}[a,b]$ preserves the gyrogroup operation for all $a,b \in G$. Let $U,V \in \mathbb{I}$, let $u \in U$, and let $v \in V$. Then, $\ominus(u \oplus v) = \mathrm{gyr}[u,v](\ominus v \ominus u) = \ominus \mathrm{gyr}[u,v]v \ominus \mathrm{gyr}[u,v]u \in \ominus V \ominus U$. Furthermore, $\ominus v \ominus u = \mathrm{gyr}[v,u](\ominus(u \oplus v)) = \ominus(\mathrm{gyr}[u,v]u \oplus \mathrm{gyr}[u,v]v) \in \ominus(U \oplus V)$. This proves Part (2) of the lemma. □

Proposition 3.9. *Let G be a locally compact Hausdorff strongly topological gyrogroup. Then there is an open, closed, σ-compact subgyrogroup H of G that is invariant under all the gyroautomorphisms. In particular, H is an L-subgyrogroup.*

Proof. Since G is a locally compact Hausdorff strongly topological gyrogroup, there is a neighborhood U of e such that $\mathrm{gyr}[a,b](U) = U$ for all $a,b \in G$, $\ominus U = U$, and $\overline{U}$ is compact. Define $U^1 = U$ and $U^n = U \oplus U^{n-1}$ for $n > 1$. By Lemma 3.3, $U^n \in \mathbb{I}$ for all $n \in \mathbb{N}$ and $U^n \oplus U^m = U^{n+m}$ for all $n,m \in \mathbb{N}$. Note that $\ominus U = U$ and that if $\ominus U^n = U^n$, then $\ominus U^{n+1} = \ominus(U \oplus U^n) = \ominus U^n \ominus U = U^n \oplus U = U^{n+1}$ by Lemma 3.3 (2). By mathematical induction, $\ominus U^n = U^n$ for all $n \in \mathbb{N}$. Set $H = \bigcup_{n\in\mathbb{N}} U^n$. Note that

$$\mathrm{gyr}[a,b](H) = \mathrm{gyr}[a,b]\left(\bigcup_{n\in\mathbb{N}} U^n\right) = \bigcup_{n\in\mathbb{N}} \mathrm{gyr}[a,b](U^n) = \bigcup_{n\in\mathbb{N}} U^n = H$$

for all $a,b \in G$. This implies that H is an L-subgyrogroup of G. Clearly, H is open and hence is closed by Proposition 2.2. Since $\overline{U}$ is compact, $\overline{U}^n$ is compact for all $n \in \mathbb{N}$ by Proposition 2.3. Furthermore, $\overline{U}^n \subseteq H$ for all $n \in \mathbb{N}$ because $\overline{U} \subseteq H$. It follows that $H = \bigcup_{n\in\mathbb{N}} U^n \subseteq \bigcup_{n\in\mathbb{N}} \overline{U}^n \subseteq H$. This shows that H is σ-compact. □

From now on, let G be a locally compact Hausdorff strongly topological gyrogroup, and let μ be a fixed Haar measure on G. Set

$$\mathcal{L}^1(G) = \{f : G \to \mathbb{C} \mid f \text{ is a Haar integrable function}\}, \tag{3.11}$$

$$\mathcal{N}^1(G) = \{f \in \mathcal{L}^1(G) \mid f = 0 \text{ almost everywhere}\}, \tag{3.12}$$

and let $L^1(G)$ be the quotient vector space given by

$$L^1(G) = \mathcal{L}^1(G)/\mathcal{N}^1(G). \tag{3.13}$$

In other words, $f \in \mathcal{L}^1(G)$ if and only if f is μ-measurable and

$$\|f\|_1 = \int_G |f| d\mu < \infty. \tag{3.14}$$

To define a convolution-like operation on $\mathcal{L}^1(G)$, we first prove the following ad hoc lemmas.

Lemma 3.4. *If $f \in \mathcal{L}^1(G)$, then there is a sequence $\{K_n\}$ of compact subsets of G such that $f(x) = 0$ for all $x \notin \bigcup_{n\in\mathbb{N}} K_n$.*

Proof. It follows from Corollary 2.3.11 of [5] that $A = \{x \in G \mid f(x) \neq 0\}$ is σ-finite. Then there exists a sequence $\{A_n\}$ of Borel sets such that $A = \bigcup_{n\in\mathbb{N}} A_n$ and $\mu(A_n) < \infty$ for all $n \in \mathbb{N}$. Since μ is regular, there exists a sequence $\{U_n\}$ of open sets such that $A \subseteq \bigcup_{n\in\mathbb{N}} U_n$ and $\mu(U_n) < \infty$ for all $n \in \mathbb{N}$. Let H be an open, σ-compact L-subgyrogroup of G (such H exists by Proposition 3.9). Let $k \in \mathbb{N}$ be fixed. For each $N \in \mathbb{N}$, set $M_N = \left\{X \in G/H \mid \mu(U_k \cap X) > \dfrac{1}{N}\right\}$. Then M_N is a finite set since otherwise there is a sequence $\{X_n\}$ in M_N such that $X_n \neq X_m$ if $n \neq m$, which implies that $\mu(U_k) \geq \sum_{n=1}^{\infty} \mu(U_k \cap X_n) \geq \sum_{n=1}^{\infty} \dfrac{1}{N} = \infty$, a contradiction. Let $x \in U_k$. Then, $x \in X$ for some coset $X \in G/H$. It follows that $U_k \cap X$ is a nonempty open set in G and so $\mu(U_k \cap X) > 0$. This means that $X \in \bigcup_{n\in\mathbb{N}} M_n$. Therefore, $U_k = \bigcup \left\{U_k \cap X \mid X \in \bigcup_{n\in\mathbb{N}} M_n\right\}$. This proves that U_n is a subset of a countable union of σ-compact sets (that is, left cosets of H) for all $n \in \mathbb{N}$ and so is $\bigcup_{n\in\mathbb{N}} U_n$. Therefore, $\bigcup_{n\in\mathbb{N}} U_n$ is contained in a countable union of compact sets. □

Lemma 3.5. *Define $F : G\times G \to G\times G$ by $F(x, y) = (x, \ominus x \oplus y)$. Then F is a homeomorphism and $(\mu \times \mu)(A) = (\mu \times \mu)(F^{-1}(A))$ for each Borel set A of $G \times G$, where $\mu \times \mu$ is the regular Borel product of μ.*

Proof. Note that $F^{-1}(x, y) = (x, x \oplus y)$ for all $x, y \in G$. Clearly, F is a homeomorphism. Hence, the measure $(\mu \times \mu)F^{-1}$ is regular. Let U be an open set in $G \times G$, and let $x \in G$. Then

$$\begin{aligned} z \in (F^{-1}(U))_x &\iff (x, z) \in F^{-1}(U) \\ &\iff (x, \ominus x \oplus z) \in U \\ &\iff \ominus x \oplus z \in U_x \\ &\iff z \in x \oplus U_x \end{aligned}$$

for all $z \in G$. Thus, $(F^{-1}(U))_x = x \oplus U_x$. By Proposition 2.4,

$$\begin{aligned} (\mu \times \mu)(U) &= \int \mu(U_x) d\mu \\ &= \int \mu(x \oplus U_x) d\mu \\ &= \int \mu((F^{-1}(U))_x) d\mu \\ &= (\mu \times \mu)(F^{-1}(U)). \end{aligned}$$

Since $\mu \times \mu$ is regular, it follows that $(\mu \times \mu)(A) = (\mu \times \mu)(F^{-1}(A))$ for any Borel set A of $G \times G$. □

Let $f, g \in \mathcal{L}^1(G)$. Define a function $f * g : G \to \mathbb{C}$ by

$$(3.15)\quad f * g(x) = \begin{cases} \displaystyle\int f(y)g(\ominus y \oplus x) d\mu(y) & \text{if } y \mapsto f(y)g(\ominus y \oplus x) \text{ is integrable;} \\ 0 & \text{otherwise.} \end{cases}$$

The function $f * g$ is called the *convolution* of f and g.

Theorem 3.8. *Let $f, g \in \mathcal{L}^1(G)$.*

(1) The function $y \mapsto f(y)g(\ominus y \oplus x)$ is in $\mathcal{L}^1(G)$ almost every x in G.
(2) The convolution $f * g$ is in $\mathcal{L}^1(G)$ and $\|f * g\|_1 \leq \|f\|_1\|g\|_1$.

Proof. Let $f, g \in \mathcal{L}^1(G)$. By Lemma 3.4, there are sequences $\{A_n\}$ and $\{B_n\}$ of compact sets in G such that f and g vanish outside $\bigcup_{n\in\mathbb{N}} A_n$ and $\bigcup_{n\in\mathbb{N}} B_n$, respectively. Define a function $M : G \times G \to \mathbb{C}$ by $M(x, y) = f(x)g(y)$ for all $x, y \in G$. Note that M is a measurable function that vanishes outside $\bigcup_{n,m\in\mathbb{N}} (A_n \times B_m) = \bigcup_{n\in\mathbb{N}} A_n \times \bigcup_{n\in\mathbb{N}} B_n$. By Proposition 2.5,

$$\int |M| d(\mu \times \mu) = \int |f(x)| \left(\int |g(y)| d\mu(y) \right) d\mu(x) = \|f\|_1 \|g\|_1 < \infty.$$

Therefore, M is integrable. By Proposition 2.6, the function $|M \circ F| = |M| \circ F$ is integrable. Observe that $M \circ F$ vanishes outside

$$F^{-1}\left(\bigcup_{n,m\in\mathbb{N}} (A_n \times B_m) \right) = \bigcup_{n,m\in\mathbb{N}} F^{-1}(A_n \times B_m),$$

which is a σ-compact set. In fact, $M \circ F$ vanishes outside

$$p_1\left(\bigcup_{n,m\in\mathbb{N}} F^{-1}(A_n \times B_m) \right) \times p_2\left(\bigcup_{n,m\in\mathbb{N}} F^{-1}(A_n \times B_m) \right),$$

which is a rectangle with σ-finite sides. Here, p_1 and p_2 are the projections to the first and second coordinate, respectively. By Proposition 2.7, the function $y \mapsto f(y)g(\ominus y \oplus x)$ is in $\mathcal{L}^1(G)$ almost every x in G and $f * g \in \mathcal{L}^1(G)$. Direct computation shows that

$$\begin{aligned}
\|f * g\|_1 &= \int |(f * g)(x)| \mu(dx) \\
&\leq \int \left| \int f(y)g(\ominus y \oplus x)\mu(dy) \right| \mu(dx) \\
&\leq \int \int |f(y)g(\ominus y \oplus x)| \mu(dy)\mu(dx) \\
&= \int \int |f(y)g(\ominus y \oplus x)| \mu(dx)\mu(dy) \\
&= \int |f(y)| \left(\int |g(\ominus y \oplus x)| \mu(dx) \right) \mu(dy) \\
&= \int |f(y)| \left(\int |g(x)| \mu(dx) \right) \mu(dy) \\
&= \|f\|_1 \|g\|_1,
\end{aligned}$$

noting that the first and third equalities follow from Propositions 2.7 and 2.6. □

The next proposition lists some basic properties of the convolution on $\mathcal{L}^1(G)$.

Proposition 3.10. *Let $f, g, h \in \mathcal{L}^1(G)$, and let $a \in \mathbb{C}$. Then*

(1) $f * (g + h) = f * g + f * h$ almost everywhere;
(2) $(g + h) * f = g * f + h * f$ almost everywhere;
(3) $a(f * g) = (af) * g = f * (ag)$; and
(4) if $f_1, f_2, g_1, g_2 \in \mathcal{L}^1(G)$ with $f_1 = f_2$ almost everywhere and $g_1 = g_2$ almost everywhere, then $f_1 * g_1 = f_2 * g_2$ almost everywhere.

Proof. It follows from Theorem 3.8 that $f(g \circ R_x \circ \ominus)$ and $f(h \circ R_x \circ \ominus)$ are both integrable for almost every x in G. Let $x \in G$ be such that both $f(g \circ R_x \circ \ominus)$ and $f(h \circ R_x \circ \ominus)$ are integrable. Then, $f((g+h) \circ R_x \circ \ominus) = f(g \circ R_x \circ \ominus) + f(h \circ R_x \circ \ominus)$ is integrable. Thus,

$$\begin{aligned}(f * (g+h))(x) &= \int f(y)(g+h)(\ominus y \oplus x)d\mu(y) \\ &= \int f(y)g(\ominus y \oplus x) + f(y)h(\ominus y \oplus x)d\mu(y) \\ &= \int f(y)g(\ominus y \oplus x)d\mu(y) + \int f(y)h(\ominus y \oplus x)d\mu(y) \\ &= (f * g + f * h)(x).\end{aligned}$$

Therefore, $f*(g+h) = f*g+f*h$ almost everywhere. Parts (2) and (3) can be proved in a similar fashion. To prove Part (4), let $f_1, f_2, g_1, g_2 \in \mathcal{L}^1(G)$ with $f_1 = f_2$ almost everywhere and $g_1 = g_2$ almost everywhere. Then,

$$\begin{aligned}\|f_1 * g_1 - f_2 * g_2\|_1 &= \|f_1 * g_1 - f_1 * g_2 + f_1 * g_2 - f_2 * g_2\|_1 \\ &\leq \|f_1 * (g_1 - g_2)\|_1 + \|(f_1 - f_2) * g_2\|_1 \\ &\leq \|f_1\|_1 \|g_1 - g_2\|_1 + \|f_1 - f_2\|_1 \|g_2\|_1 \\ &= 0,\end{aligned}$$

which completes the proof. □

We emphasize that the identity $f * (g * h) = (f * g) * h$, where $f, g, h \in \mathcal{L}^1(G)$, which is true for the case of groups, is missing. In fact, the following example shows that this is not the case, in general. Thus, topological groups and topological gyrogroups are somewhat different.

Example 3.2. Let $G = \{0, 1, 2, 3, 4, 5, 6, 7\}$ be the gyrogroup given in Example 3.2 of [4]. Its gyroaddition and gyration tables are given by Table 1. If G is endowed with the discrete topology, then G becomes a locally compact Hausdorff strongly topological gyrogroup. Note that the counting measure μ on G, defined by $\mu(A) = |A|$ for all $A \subseteq G$, is a Haar measure on G. Let f, g, h be the functions from G to $\mathbb{R}$ defined by two-row notation as

$$\begin{aligned}f &= \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{pmatrix}, \\ g &= \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{pmatrix}, \\ h &= \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{pmatrix}.\end{aligned}$$

Then,

$$((f * g) * h)(x) = \sum_{y \in G} \sum_{z \in G} f(z)g(\ominus z \oplus y)h(\ominus y \oplus x)\mu(\{z\})\mu(\{y\}),$$

and

$$(f * (g * h))(x) = \sum_{y \in G} \sum_{z \in G} f(y)g(z)h(\ominus z \oplus (\ominus y \oplus x))\mu(\{z\})\mu(\{y\})$$

for all $x \in G$. We have by inspection that $((f * g) * h)(1) = 5224$, whereas $(f * (g * h))(1) = 5044$. This shows that $f * (g * h) \neq (f * g) * h$ in general. In fact,

$$\mu(\{x \in G \mid (f * (g * h))(x) \neq ((f * g) * h)(x)\}) \geq 1.$$

$\oplus$	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	0	3	2	5	4	7	6
2	2	3	0	1	6	7	4	5
3	3	5	6	0	7	1	2	4
4	4	2	1	7	0	6	5	3
5	5	4	7	6	1	0	3	2
6	6	7	4	5	2	3	0	1
7	7	6	5	4	3	2	1	0

gyr	0	1	2	3	4	5	6	7
0	I	I	I	I	I	I	I	I
1	I	I	A	A	A	A	I	I
2	I	A	I	A	A	I	A	I
3	I	A	A	I	I	A	A	I
4	I	A	A	I	I	A	A	I
5	I	A	I	A	A	I	A	I
6	I	I	A	A	A	A	I	I
7	I	I	I	I	I	I	I	I

TABLE 1. The gyroaddition table (left) and the gyration table (right) for the gyrogroup G in Example 3.2. The gyroautomorphism A is given in cycle notation by $A = (1\ 6)(2\ 5)$.

The seminorm $\|\cdot\|_1$ on $\mathcal{L}^1(G)$ induces a norm on $L^1(G)$ by defining

$$\|f + \mathcal{N}^1(G)\|_1 = \|f\|_1 \tag{3.16}$$

for all $f \in \mathcal{L}^1(G)$. Note that (3.16) is well defined because if $f, g \in \mathcal{L}^1(G)$ with $f = g$ almost everywhere, then $\|f\|_1 = \|g\|_1$. Therefore, $L^1(G)$ forms a normed space. Furthermore, the convolution $*$ on $\mathcal{L}^1(G)$ induces a convolution-like operation on $L^1(G)$ by defining

$$(f + \mathcal{N}^1(G)) * (g + \mathcal{N}^1(G)) = (f * g) + \mathcal{N}^1(G) \tag{3.17}$$

for all $f, g \in \mathcal{L}^1(G)$. Note that (3.17) is well defined by Proposition 3.10 (4). From this point of view, $L^1(G)$, where G is a locally compact Hausdorff strongly topological gyrogroup, is a nonassociative normed-algebra. This makes sense since gyrogroup operations are nonassociative, in general.

Acknowledgment. The authors are grateful to the referees for their useful comments. The first author was supported by the Institute for Promotion of Teaching Science and Technology (IPST), Thailand via Development and Promotion of Science and Technology Talents Project (DPST). This research was supported by Chiang Mai University.

REFERENCES

[1] Atiponrat, W. Topological gyrogroups: Generalization of topological groups. *Topology Appl.* **224** (2017), 73–82.
[2] Bao, M.; Lin, F. Feathered gyrogroups and gyrogroups with countable pseudocharacter. *Filomat* **33** (2019), no. 16, 5113–5124.
[3] Bao, M.; Lin, Y.; Lin, F. Strongly topological gyrogroups with remainders close to metrizable. *Bull. Iran. Math. Soc.* (2021), in press.
[4] Bussaban, L.; Kaewkhao, A.; Suantai, S. Cayley graphs of gyrogroups. *Quasigroups Related Systems.* **27** (2019), no. 1, 25–32.
[5] Cohn, D. L. *Measure theory*, 2nd ed., Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks], Birkhäuser, New York, 2013.
[6] Diestel, J.; Spalsbury, A. *The joys of Haar measure*, Graduate Studies in Mathematics, **vol. 150**, American Mathematical Society, Providence, RI, 2014.
[7] Gaal, S. A. *Point set topology*, Pure and Applied Mathematics, **Vol. XVI**, Academic Press, New York-London, 1964.
[8] Hewitt, E.; Ross, K. A. *Abstract harmonic analysis.* **Vol. I**, 2nd ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 115, Springer-Verlag, Berlin-New York, 1979, Structure of topological groups, integration theory, group representations.
[9] Karapetyants, A.; Samko, S.; Zhu, K. A class of Hausdorff-Berezin operators on the unit disc. *Complex Anal. Oper. Theory* **13** (2019), no. 8, 3853–3870.
[10] Steinlage, R. C. On Haar measure in locally compact T_2 spaces. *Amer. J. Math.* **97** (1975), 291–307.

[11] Suksumran, T. *The algebra of gyrogroups: Cayley's theorem, Lagrange's theorem, and isomorphism theorems*. Essays in mathematics and its applications (Th. M. Rassias and P. M. Pardalos, eds.), Springer, Cham, 2016, pp. 369–437.
[12] Suksumran, T.; Wiboonton, K. Isomorphism theorems for gyrogroups and L-subgyrogroups. *J. Geom. Symmetry Phys.* **37** (2015), 67–83.
[13] Ungar, A. A. *Analytic hyperbolic geometry and Albert Einstein's special Theory of relativity*, World Scientific, Hackensack, NJ, 2008.
[14] Wattanapan, J.; Atiponrat, W.; Suksumran, T. Embedding of locally compact Hausdorff topological gyrogroups in topological groups. *Topology Appl.* **273** (2020), Article 107102 (12 pages).
[15] Wattanapan, J.; Atiponrat, W.; Suksumran, T. Embedding of strongly topological gyrogroups in path-connected and locally path-connected gyrogroups. *Symmetry* **12** (2020), no. 11, Article 1817, 23 pp.
[16] Zhang, J.; Liang, Q.; Wang, H. Uniformities on strongly topological gyrogroups. *Topology Appl.* **302** (2021), Article 107776, 10 pp.
[17] Zhu, K. *Operator theory in function spaces*, 2nd ed., Mathematical Surveys and Monographs, **vol. 138**, American Mathematical Society, Providence, RI, 2007.

DOCTORAL PROGRAM IN MATHEMATICS
GRADUATE SCHOOL, CHIANG MAI UNIVERSITY
CHIANG MAI 50200, THAILAND
Email address: jaturon_watt@cmu.ac.th

DEPARTMENT OF MATHEMATICS
RESEARCH GROUP IN MATHEMATICS AND APPLIED MATHEMATICS
FACULTY OF SCIENCE, CHIANG MAI UNIVERSITY
CHIANG MAI 50200, THAILAND
Email address: watchareepan.a@cmu.ac.th
Email address: santi.tasena@cmu.ac.th
Email address: teerapong.suksumran@cmu.ac.th