CARPATHIAN J. MATH.
Volume **38** (2022), No. 1,
Pages 249 - 262

Online version at `https://www.carpathian.cunbm.utcluj.ro/`
Print Edition: ISSN 1584 - 2851; Online Edition: ISSN 1843 - 4401
DOI: https://doi.org/10.37193/CJM.2022.01.20

Dedicated to the memory of Academician Mitrofan M. Choban (1942-2021)

Halpern subgradient extragradient algorithm for solving quasimonotone variational inequality problems

PONGSAKORN YOTKAEW[1], HABIB UR REHMAN [2], BANCHA PANYANAK [3,4] and NUTTAPOL PAKKARANANG[5,†]

ABSTRACT. In this paper, we study the numerical solution of the variational inequalities involving quasi-monotone operators in infinite-dimensional Hilbert spaces. We prove that the iterative sequence generated by the proposed algorithm for the solution of quasimonotone variational inequalities converges strongly to a solution. The main advantage of the proposed iterative schemes is that it uses a monotone and non-monotone step size rule based on operator knowledge rather than its Lipschitz constant or some other line search method.

1. INTRODUCTION

Assume that $\mathcal{E}$ is a real Hilbert space and $\mathcal{K}$ be a nonempty closed convex subset of $\mathcal{E}$. Let $\mathcal{L} : \mathcal{E} \to \mathcal{E}$ be an operator. The problem (VIP) for $\mathcal{L}$ on $\mathcal{K}$ is defined as follows [17, 24]:

$$\text{(VIP)} \qquad \text{Find } x^* \in \mathcal{K} \text{ such that } \big\langle \mathcal{L}(x^*), y - x^* \big\rangle \geq 0, \ \forall\, y \in \mathcal{K}.$$

Our main concern here is to study the iterative methods that are used to approximate the solution of the *variational inequality problem* (shortly, VIP) involving quasimonotone operators in any real Hilbert space. In order to prove the strong convergence, it is considered that the following conditions have been satisfied:

($\mathcal{L}1$) The solution set of a problem (VIP) is denoted by $VI(\mathcal{K}, \mathcal{L})$ and it is nonempty;
($\mathcal{L}2$) An operator $\mathcal{L} : \mathcal{E} \to \mathcal{E}$ is said to be quasimonotone if

$$\text{(QM)} \qquad \big\langle \mathcal{L}(y_1), y_2 - y_1 \big\rangle > 0 \Longrightarrow \big\langle \mathcal{L}(y_2), y_1 - y_2 \big\rangle \leq 0, \ \forall\, y_1, y_2 \in \mathcal{K};$$

($\mathcal{L}3$) An operator $\mathcal{L} : \mathcal{E} \to \mathcal{E}$ is said to be *Lipschitz continuous* if there exists a constant $L > 0$ such that

$$\text{(LC)} \qquad \|\mathcal{L}(y_1) - \mathcal{L}(y_2)\| \leq L\|y_1 - y_2\|, \ \forall\, y_1, y_2 \in \mathcal{K};$$

($\mathcal{L}4$) An operator $\mathcal{L} : \mathcal{E} \to \mathcal{E}$ is said to be *sequentially weakly continuous* if $\{\mathcal{L}(x_n)\}$ converges weakly to $\mathcal{L}(x)$ for each sequence $\{x_n\}$ converges weakly to x.

It is well-established that the problem (VIP) is an important problem in the field of nonlinear analysis. It is an important mathematical model that unifies many crucial concepts in applied mathematics, such as a nonlinear system of equation, optimization conditions for problems with the optimization problems, the complementarity problems, the network equilibrium problems and finance (see for more details [8, 12, 13, 15, 16, 21, 25]). As

Received: 30.03.2021. In revised form: 09.08.2021. Accepted: 16.09.2021
2010 *Mathematics Subject Classification.* 47J25, 47H05, 47H10, 90C2, 47J25.
Key words and phrases. *Subgradient extragradient method, Variational inequality problem, Strong convergence theorems, Quasimonotone mapping, Lipschitz continuity.*
Corresponding author: †Nuttapol Pakkaranang; nuttapol.pak@pcru.ac.th

a consequence, this concept has various applications in the field of engineering, mathematical programming, network economics, transport analysis, game theory and computer science.

The regularized and the projection methods are two prominent and general iterative schemes to approximate a solution to the variational inequalities. It is also noted that the first approach is most commonly used to solve variational inequalities accompanied by the class of monotone operators. The regularized subproblem in this method is strongly monotone and its unique solution exists more conveniently than the initial problem. In this paper, we look at some well-known projection methods that are well-known for their ease of numerical computation. The first well-known projection method is the gradient projection method that is used to solve variational inequalities. Moreover, several other projection methods have been established including the well-known extragradient method [18] the subgradient extragradient method [4, 5] and others in [6, 26, 20, 30, 11] and others in [22, 7, 23, 14, 9, 27, 28, 2, 1, 10]. The above numerical techniques are used to examine the variational inequalities involving monotone, strongly monotone, or inverse monotone. The common feature of these methods is that fixed or variable step size rules are frequently used in constructing approximation solutions and establishing their convergence, depending on the Lipschitz constant of the involved operator. This can limit implementations because these parameters may be undefined or difficult to approximate in some situations.

The aim of this paper is to examine the quasimonotone variational inequalities in infinite-dimensional Hilbert space and to verify that the iterative sequence proposed by the extragradient algorithm for solving quasimonotone variational inequalities converges strongly to a solution. The proposed subgradient extragradient algorithm uses both the monotone and the new non-monotone variable step size rule.

The paper is arranged in the following manner. In Sect. 2, some preliminary results were presented. Sect. 3 gives two new algorithms and their convergence analysis. Finally, Sect. 4 gives some numerical results to explain the practical efficiency of the proposed methods.

2. Preliminaries

For all $x, y \in \mathcal{E}$, we have

$$\|x+y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2.$$

A *metric projection* $P_{\mathcal{K}}(y_1)$ of $y_1 \in \mathcal{E}$ is defined by

$$P_{\mathcal{K}}(y_1) = \arg\min\{\|y_1 - y_2\| : y_2 \in \mathcal{K}\}.$$

Lemma 2.1. [3] *For all $y_1, y_2 \in \mathcal{E}$ and $\ell \in \mathbb{R}$. Then*

(i) $\|\ell y_1 + (1-\ell)y_2\|^2 = \ell\|y_1\|^2 + (1-\ell)\|y_2\|^2 - \ell(1-\ell)\|y_1 - y_2\|^2$.
(ii) $\|y_1 + y_2\|^2 \le \|y_1\|^2 + 2\langle y_2, y_1 + y_2\rangle$.

Lemma 2.2. [29] *Let $\{p_n\} \subset [0, +\infty)$ be a sequence such that*

$$p_{n+1} \le (1-q_n)p_n + q_n r_n, \; \forall\, n \in \mathbb{N}.$$

Moreover, two sequences $\{q_n\} \subset (0,1)$ and $\{r_n\} \subset \mathbb{R}$ satisfying the following conditions:

$$\lim_{n\to+\infty} q_n = 0, \; \sum_{n=1}^{+\infty} q_n = +\infty \textit{ and } \limsup_{n\to+\infty} r_n \le 0.$$

Then, $\lim_{n\to+\infty} p_n = 0$.

Lemma 2.3. [19] *Let $\{p_n\}$ be a sequence and there exists a subsequence $\{n_i\}$ of $\{n\}$ such that*

$$p_{n_i} < p_{n_{i+1}}, \forall i \in \mathbb{N}.$$

Then, there exists a nondecreasing sequence $m_k \subset \mathbb{N}$ such that $m_k \to +\infty$ as $k \to +\infty$, and satisfying the following inequality for $k \in \mathbb{N}$:

$$p_{m_k} \le p_{m_{k+1}} \text{ and } p_k \le p_{m_{k+1}}.$$

Indeed, $m_k = \max\{j \le k : p_j \le p_{j+1}\}$.

3. Main results

In this section, we propose an initial method to solve quasimonotone variational inequalities in real Hilbert spaces and prove a strong convergence result for the proposed method. The first method involves the monotonic self-adaptive step rule to make the method independent of the Lipschitz constant. The first method is written as follows:

Algorithm 1 (Monotonic Explicit Halpern-Type Subgradient Extragradient Method)

Step 0: Let $x_1 \in \mathcal{K}$, $\rho_1 > 0$, $\mu \in (0,1)$ and $\{\gamma_n\} \subset (0,1)$ meet the following conditions:

$$\lim_{n\to+\infty} \gamma_n = 0 \text{ and } \sum_{n=1}^{+\infty} \gamma_n = +\infty.$$

Step 1: Compute $y_n = P_{\mathcal{K}}(x_n - \rho_n \mathcal{L}(x_n))$. If $x_n = y_n$, STOP. Otherwise, go to **Step 2**.
Step 2: Compute $z_n = P_{\mathcal{E}_n}(x_n - \rho_n \mathcal{L}(y_n))$ where

$$\mathcal{E}_n = \{z \in \mathcal{E} : \langle x_n - \rho_n \mathcal{L}(x_n) - y_n, z - y_n \rangle \le 0\}.$$

Step 3: Compute $x_{n+1} = \gamma_n x_1 + (1-\gamma_n) z_n$.
Step 4: Compute

$$\rho_{n+1} = \begin{cases} \min\left\{\rho_n, \frac{\mu\|x_n-y_n\|^2+\mu\|z_n-y_n\|^2}{2\left[\left\langle \mathcal{L}(x_n)-\mathcal{L}(y_n), z_n-y_n\right\rangle\right]}\right\} & \text{if } \left\langle \mathcal{L}(x_n)-\mathcal{L}(y_n), z_n-y_n\right\rangle > 0, \\ \rho_n, & \text{otherwise.} \end{cases} \tag{3.1}$$

Set $n := n+1$ and go back to **Step 1**.

Lemma 3.4. *A sequence $\{\rho_n\}$ generated by* (3.1) *is monotonically decreasing and convergent.*

Proof. Due to the Lipschitz continuity of a mapping $\mathcal{L}$ there exists a fixed number $L > 0$. Suppose that $\langle \mathcal{L}(x_n) - \mathcal{L}(y_n), z_n - y_n \rangle > 0$ such that

$$\begin{aligned} \frac{\mu(\|x_n-y_n\|^2+\|z_n-y_n\|^2)}{2\langle \mathcal{L}(x_n)-\mathcal{L}(y_n), z_n-y_n\rangle} &\ge \frac{2\mu\|x_n-y_n\|\|z_n-y_n\|}{2\|\mathcal{L}(x_n)-\mathcal{L}(y_n)\|\|z_n-y_n\|} \\ &\ge \frac{2\mu\|x_n-y_n\|\|z_n-y_n\|}{2L\|x_n-y_n\|\|z_n-y_n\|} \ge \frac{\mu}{L}. \end{aligned} \tag{3.2}$$

We can easily determine that the sequence $\{\rho_n\}$ is bounded and monotonically decreasing. Hence, sequence $\{\rho_n\}$ is convergent to some $\rho > 0$. ☐

Lemma 3.5. *Let $\mathcal{L} : \mathcal{E} \to \mathcal{E}$ be an operator satisfies the conditions* ($\mathcal{L}$1)–($\mathcal{L}$4). *For $x^* \in VI(\mathcal{K}, \mathcal{L}) \neq \emptyset$, we have*

$$\|z_n - x^*\|^2 \le \|x_n - x^*\|^2 - \left(1 - \frac{\mu\rho_n}{\rho_{n+1}}\right)\|x_n - y_n\|^2 - \left(1 - \frac{\mu\rho_n}{\rho_{n+1}}\right)\|z_n - y_n\|^2.$$

Proof. Now consider that

$$\begin{aligned}\|z_n - x^*\|^2 &= \|P_{\mathcal{E}_n}[x_n - \rho_n\mathcal{L}(y_n)] - x^*\|^2\\ &= \|P_{\mathcal{E}_n}[x_n - \rho_n\mathcal{L}(y_n)] + [x_n - \rho_n\mathcal{L}(y_n)] - [x_n - \rho_n\mathcal{L}(y_n)] - x^*\|^2\\ &= \|[x_n - \rho_n\mathcal{L}(y_n)] - x^*\|^2 + \|P_{\mathcal{E}_n}[x_n - \rho_n\mathcal{L}(y_n)] - [x_n - \rho_n\mathcal{L}(y_n)]\|^2\\ &\quad + 2\langle P_{\mathcal{E}_n}[x_n - \rho_n\mathcal{L}(y_n)] - [x_n - \rho_n\mathcal{L}(y_n)], [x_n - \rho_n\mathcal{L}(y_n)] - x^*\rangle. \qquad (3.3)\end{aligned}$$

By using $x^* \in VI(\mathcal{K},\mathcal{L}) \subset \mathcal{K} \subset \mathcal{E}_n$, we get

$$\begin{aligned}&\|P_{\mathcal{E}_n}[x_n - \rho_n\mathcal{L}(y_n)] - [x_n - \rho_n\mathcal{L}(y_n)]\|^2\\ &\quad + \langle P_{\mathcal{E}_n}[x_n - \rho_n\mathcal{L}(y_n)] - [x_n - \rho_n\mathcal{L}(y_n)], [x_n - \rho_n\mathcal{L}(y_n)] - x^*\rangle\\ &= \langle [x_n - \rho_n\mathcal{L}(y_n)] - P_{\mathcal{E}_n}[x_n - \rho_n\mathcal{L}(y_n)], x^* - P_{\mathcal{E}_n}[x_n - \rho_n\mathcal{L}(y_n)]\rangle \le 0. \qquad (3.4)\end{aligned}$$

Thus, above expression implies that

$$\begin{aligned}&\langle P_{\mathcal{E}_n}[x_n - \rho_n\mathcal{L}(y_n)] - [x_n - \rho_n\mathcal{L}(y_n)], [x_n - \rho_n\mathcal{L}(y_n)] - x^*\rangle\\ &\le -\|P_{\mathcal{E}_n}[x_n - \rho_n\mathcal{L}(y_n)] - [x_n - \rho_n\mathcal{L}(y_n)]\|^2. \qquad (3.5)\end{aligned}$$

From expressions (3.3) and (3.5), we obtain

$$\begin{aligned}\|z_n - x^*\|^2 &\le \|x_n - \rho_n\mathcal{L}(y_n) - x^*\|^2 - \|P_{\mathcal{E}_n}[x_n - \rho_n\mathcal{L}(y_n)] - [x_n - \rho_n\mathcal{L}(y_n)]\|^2\\ &\le \|x_n - x^*\|^2 - \|x_n - z_n\|^2 + 2\rho_n\langle \mathcal{L}(y_n), x^* - z_n\rangle. \qquad (3.6)\end{aligned}$$

Since x^* is the solution of problem (VIP), we have

$$\langle \mathcal{L}(x^*), y - x^*\rangle \ge 0, \text{ for all } y \in \mathcal{K}.$$

Due to the mapping $\mathcal{L}$ on $\mathcal{K}$, we obtain

$$\langle \mathcal{L}(y), y - x^*\rangle \ge 0, \text{ for all } y \in \mathcal{K}.$$

By substituting $y = y_n \in \mathcal{K}$, we get

$$\langle \mathcal{L}(y_n), y_n - x^*\rangle \ge 0.$$

Thus, we have

$$\langle \mathcal{L}(y_n), x^* - z_n\rangle = \langle \mathcal{L}(y_n), x^* - y_n\rangle + \langle \mathcal{L}(y_n), y_n - z_n\rangle \le \langle \mathcal{L}(y_n), y_n - z_n\rangle. \qquad (3.7)$$

Combining expressions (3.6) and (3.7), we obtain

$$\begin{aligned}\|z_n - x^*\|^2 &\le \|x_n - x^*\|^2 - \|x_n - z_n\|^2 + 2\rho_n\langle \mathcal{L}(y_n), y_n - z_n\rangle\\ &\le \|x_n - x^*\|^2 - \|x_n - y_n + y_n - z_n\|^2 + 2\rho_n\langle \mathcal{L}(y_n), y_n - z_n\rangle\\ &\le \|x_n - x^*\|^2 - \|x_n - y_n\|^2 - \|y_n - z_n\|^2 + 2\langle x_n - \rho_n\mathcal{L}(y_n) - y_n, z_n - y_n\rangle. \qquad (3.8)\end{aligned}$$

Note that $z_n = P_{\mathcal{E}_n}[x_n - \rho_n\mathcal{L}(y_n)]$ and by the definition of ρ_{n+1}, we have

$$\begin{aligned}&2\langle x_n - \rho_n\mathcal{L}(y_n) - y_n, z_n - y_n\rangle\\ &= 2\langle x_n - \rho_n\mathcal{L}(x_n) - y_n, z_n - y_n\rangle + 2\rho_n\langle \mathcal{L}(x_n) - \mathcal{L}(y_n), z_n - y_n\rangle\\ &\le \frac{\rho_n}{\rho_{n+1}}2\rho_{n+1}\langle \mathcal{L}(x_n) - \mathcal{L}(y_n), z_n - y_n\rangle\\ &\le \frac{\mu\rho_n}{\rho_{n+1}}\|x_n - y_n\|^2 + \frac{\mu\rho_n}{\rho_{n+1}}\|z_n - y_n\|^2. \qquad (3.9)\end{aligned}$$

Combining expressions (3.8) and (3.9), we obtain

$$\begin{aligned}
&\|z_n - x^*\|^2 \\
&\leq \|x_n - x^*\|^2 - \|x_n - y_n\|^2 - \|y_n - z_n\|^2 + \frac{\rho_n}{\rho_{n+1}}\left[\mu\|x_n - y_n\|^2 + \mu\|z_n - y_n\|^2\right] \\
&\leq \|x_n - x^*\|^2 - \Big(1 - \frac{\mu\rho_n}{\rho_{n+1}}\Big)\|x_n - y_n\|^2 - \Big(1 - \frac{\mu\rho_n}{\rho_{n+1}}\Big)\|z_n - y_n\|^2. \qquad (3.10)
\end{aligned}$$

□

Lemma 3.6. *Let $\mathcal{L} : \mathcal{E} \to \mathcal{E}$ be an operator satisfying the conditions* ($\mathcal{L}$1)–($\mathcal{L}$4). *If there exists a subsequence $\{x_{n_k}\}$ weakly convergent to $\hat{x}$ and $\lim_{k\to\infty} \|x_{n_k} - y_{n_k}\| = 0$. Then, $\hat{x}$ is the solution of the problem* (VIP).

Proof. Since $\{x_{n_k}\}$ weakly convergent to $\hat{x}$ and due to $\lim_{k\to\infty} \|x_{n_k} - y_{n_k}\| = 0$, sequence $\{y_{n_k}\}$ also weakly convergent to $\hat{x}$. Next, we need to prove that $\hat{x} \in VI(\mathcal{K}, \mathcal{L})$. Indeed, we have

$$y_{n_k} = P_{\mathcal{K}}[x_{n_k} - \rho_{n_k}\mathcal{L}(x_{n_k})]$$

that is equivalent to

$$\langle x_{n_k} - \rho_{n_k}\mathcal{L}(x_{n_k}) - y_{n_k}, y - y_{n_k}\rangle \leq 0, \ \forall\, y \in \mathcal{K}. \qquad (3.11)$$

The inequality mentioned above implies that

$$\langle x_{n_k} - y_{n_k}, y - y_{n_k}\rangle \leq \rho_{n_k}\langle \mathcal{L}(x_{n_k}), y - y_{n_k}\rangle, \ \forall\, y \in \mathcal{K}. \qquad (3.12)$$

Thus, we obtain

$$\frac{1}{\rho_{n_k}}\langle x_{n_k} - y_{n_k}, y - y_{n_k}\rangle + \langle \mathcal{L}(x_{n_k}), y_{n_k} - x_{n_k}\rangle \leq \langle \mathcal{L}(x_{n_k}), y - x_{n_k}\rangle, \ \forall\, y \in \mathcal{K}. \qquad (3.13)$$

Since $\min\left\{\frac{\mu}{L}, \rho_1\right\} \leq \rho \leq \rho_1$ and $\{x_{n_k}\}$ is a bounded sequence. By the use of $\lim_{k\to\infty} \|x_{n_k} - y_{n_k}\| = 0$ and $k \to \infty$ in expression (3.13), we obtain

$$\liminf_{k\to\infty}\langle \mathcal{L}(x_{n_k}), y - x_{n_k}\rangle \geq 0, \ \forall\, y \in \mathcal{K}. \qquad (3.14)$$

Moreover, we have

$$\begin{aligned}
&\langle \mathcal{L}(y_{n_k}), y - y_{n_k}\rangle \\
&= \langle \mathcal{L}(y_{n_k}) - \mathcal{L}(x_{n_k}), y - x_{n_k}\rangle + \langle \mathcal{L}(x_{n_k}), y - x_{n_k}\rangle + \langle \mathcal{L}(y_{n_k}), x_{n_k} - y_{n_k}\rangle. \qquad (3.15)
\end{aligned}$$

Since $\lim_{k\to\infty} \|x_{n_k} - y_{n_k}\| = 0$ and $\mathcal{L}$ is L-Lipschitz continuity on $\mathcal{E}$ implies that

$$\lim_{k\to\infty} \|\mathcal{L}(x_{n_k}) - \mathcal{L}(y_{n_k})\| = 0, \qquad (3.16)$$

which together with expressions (3.15) and (3.16), we obtain

$$\liminf_{k\to\infty}\langle \mathcal{L}(y_{n_k}), y - y_{n_k}\rangle \geq 0, \ \forall\, y \in \mathcal{K}. \qquad (3.17)$$

To prove further, let us take a positive sequence $\{\epsilon_k\}$ that is convergent to zero and decreasing. For each $\{\epsilon_k\}$ we denote by m_k the smallest positive integer such that

$$\langle \mathcal{L}(x_{n_i}), y - x_{n_i}\rangle + \epsilon_k > 0, \ \forall\, i \geq m_k \qquad (3.18)$$

where the existence of m_k follows from expression (3.17). Since $\{\epsilon_k\}$ is decreasing and it is easy to see that the sequence m_k is increasing.
Case I: If there is a subsequence $\{x_{n_{m_{k_j}}}\}$ of $\{x_{n_{m_k}}\}$ such that $\mathcal{L}(x_{n_{m_{k_j}}}) = 0$ $(\forall j)$. Let $j \to \infty$, we obtain

$$\langle \mathcal{L}(\hat{x}), y - \hat{x}\rangle = \lim_{j\to\infty}\langle \mathcal{L}(x_{n_{m_{k_j}}}), y - \hat{x}\rangle = 0. \qquad (3.19)$$

Thus, $\hat{x} \in \mathcal{K}$ and imply that $\hat{x} \in VI(\mathcal{K}, \mathcal{L})$.

Case II: If there exists $N_0 \in \mathbb{N}$ such that for all $n_{m_k} \geq N_0$, $\mathcal{L}(x_{n_{m_k}}) \neq 0$. Consider that

$$\Upsilon_{n_{m_k}} = \frac{\mathcal{L}(x_{n_{m_k}})}{\|\mathcal{L}(x_{n_{m_k}})\|^2}, \ \forall n_{m_k} \geq N_0. \tag{3.20}$$

Due to the above definition, we obtain

$$\langle \mathcal{L}(x_{n_{m_k}}), \Upsilon_{n_{m_k}} \rangle = 1, \ \forall n_{m_k} \geq N_0. \tag{3.21}$$

Moreover, expressions (3.18) and (3.21) for all $n_{m_k} \geq N_0$, we have

$$\langle \mathcal{L}(x_{n_{m_k}}), \ y + \epsilon_k \Upsilon_{n_{m_k}} - x_{n_{m_k}} \rangle > 0. \tag{3.22}$$

Since $\mathcal{L}$ is quasimonotone, then

$$\langle \mathcal{L}(y + \epsilon_k \Upsilon_{n_{m_k}}), \ y + \epsilon_k \Upsilon_{n_{m_k}} - x_{n_{m_k}} \rangle > 0. \tag{3.23}$$

For all $n_{m_k} \geq N_0$, we have

$$\langle \mathcal{L}(y), y - x_{n_{m_k}} \rangle \geq \langle \mathcal{L}(y) - \mathcal{L}(y + \epsilon_k \Upsilon_{n_{m_k}}), \ y + \epsilon_k \Upsilon_{n_{m_k}} - x_{n_{m_k}} \rangle - \epsilon_k \langle \mathcal{L}(y), \Upsilon_{n_{m_k}} \rangle. \tag{3.24}$$

Due to $\{x_{n_k}\}$ weakly converges to $\hat{x} \in \mathcal{K}$ through $\mathcal{L}$ is sequentially weakly continuous on the set $\mathcal{K}$, we get $\{\mathcal{L}(x_{n_k})\}$ weakly converges to $\mathcal{L}(\hat{x})$. Suppose that $\mathcal{L}(\hat{x}) \neq 0$, we have

$$\|\mathcal{L}(\hat{x})\| \leq \liminf_{k \to \infty} \|\mathcal{L}(x_{n_k})\|. \tag{3.25}$$

Since $\{x_{n_{m_k}}\} \subset \{x_{n_k}\}$ and $\lim_{k \to \infty} \epsilon_k = 0$, we have

$$0 \leq \lim_{k \to \infty} \|\epsilon_k \Upsilon_{n_{m_k}}\| = \lim_{k \to \infty} \frac{\epsilon_k}{\|\mathcal{L}(x_{n_{m_k}})\|} \leq \frac{0}{\|\mathcal{L}(\hat{x})\|} = 0. \tag{3.26}$$

Next, consider $k \to \infty$ in (3.24), we obtain

$$\langle \mathcal{L}(y), y - \hat{x} \rangle \geq 0, \ \forall y \in \mathcal{K}. \tag{3.27}$$

Let $x \in \mathcal{K}$ be arbitrary element and for $0 < \lambda \leq 1$, let

$$\hat{x}_\lambda = \lambda x + (1 - \lambda)\hat{x}. \tag{3.28}$$

Then $\hat{x}_\lambda \in \mathcal{K}$ and from (3.27) we have

$$\lambda \big\langle \mathcal{T}(\hat{x}_\lambda), x - \hat{x} \big\rangle \geq 0. \tag{3.29}$$

Thus, we have

$$\big\langle \mathcal{T}(\hat{x}_\lambda), x - \hat{x} \big\rangle \geq 0. \tag{3.30}$$

Let $\lambda \to 0$. Then $\hat{x}_\lambda \to \hat{x}$ along a line segment. By the continuity of an operator, $\mathcal{T}(\hat{x}_\lambda)$ converges to $\mathcal{T}(\hat{x})$ as $\lambda \to 0$. It follows from (3.30) that

$$\big\langle \mathcal{T}(\hat{x}), x - \hat{x} \big\rangle \geq 0. \tag{3.31}$$

Therefore, $\hat{x}$ is a solution of the problem (VIP). □

Theorem 3.1. *Let $\mathcal{L} : \mathcal{E} \to \mathcal{E}$ be an operator satisfies the conditions ($\mathcal{L}$1)–($\mathcal{L}$4). Then, the sequence $\{x_n\}$ generated by the Algorithm 1 converges strongly to a solution $x^* \in VI(\mathcal{K}, \mathcal{L})$.*

Proof. From Lemma 3.5, we have

$$\|z_n - x^*\|^2 \leq \|x_n - x^*\|^2, \ \forall n \geq n_1. \tag{3.32}$$

Since $\rho_n \to \rho$, thus there exists a fixed number $\epsilon \in (0, 1 - \mu)$ such that

$$\lim_{n \to \infty} \left(1 - \frac{\mu \rho_n}{\rho_{n+1}}\right) = 1 - \mu > \epsilon > 0.$$

Thus, there exists a finite number $n_1 \in \mathbb{N}$ such that

$$\Big(1-\frac{\mu\rho_n}{\rho_{n+1}}\Big) > \epsilon > 0,\ \forall\, n \geq n_1. \tag{3.33}$$

By the use of definition of $\{x_{n+1}\}$ we obtain

$$\begin{aligned}\big\|x_{n+1}-x^*\big\| &= \big\|\alpha_n x_1 + (1-\alpha_n)z_n - x^*\big\| \\ &= \big\|\alpha_n[x_1-x^*] + (1-\alpha_n)[z_n-x^*]\big\| \\ &\leq \alpha_n\big\|x_1-x^*\big\| + (1-\alpha_n)\big\|z_n-x^*\big\|.\end{aligned} \tag{3.34}$$

Combining expressions (3.33) and (3.34), we obtain

$$\begin{aligned}\big\|x_{n+1}-x^*\big\| &\leq \alpha_n\big\|x_1-x^*\big\| + (1-\alpha_n)\big\|x_n-x^*\big\|. \\ &\leq \max\Big\{\big\|x_1-x^*\big\|, \big\|x_n-x^*\big\|\Big\} \\ &\leq \max\Big\{\big\|x_1-x^*\big\|, \big\|x_{n_1}-x^*\big\|\Big\}.\end{aligned} \tag{3.35}$$

Thus, we conclude that $\{x_n\}$ is a bounded sequence. By using Lemma 2.1 (i), we have

$$\begin{aligned}\big\|x_{n+1}-x^*\big\|^2 &= \big\|\alpha_n x_1 + (1-\alpha_n)z_n - x^*\big\|^2 \\ &= \big\|\alpha_n[x_1-x^*] + (1-\alpha_n)[z_n-x^*]\big\|^2 \\ &= \alpha_n\|x_1-x^*\|^2 + (1-\alpha_n)\|z_n-x^*\|^2 - \alpha_n(1-\alpha_n)\|x_1-z_n\|^2 \\ &\leq \alpha_n\|x_1-x^*\|^2 + (1-\alpha_n)\Big[\|x_n-x^*\|^2 - \Big(1-\frac{\mu\rho_n}{\rho_{n+1}}\Big)\|x_n-y_n\|^2 \\ &\quad - \Big(1-\frac{\mu\rho_n}{\rho_{n+1}}\Big)\|z_n-y_n\|^2\Big] - \alpha_n(1-\alpha_n)\|x_1-z_n\|^2 \\ &\leq \alpha_n\|x_1-x^*\|^2 + \|x_n-x^*\|^2 \\ &\quad - (1-\alpha_n)\Big(1-\frac{\mu\rho_n}{\rho_{n+1}}\Big)\|x_n-y_n\|^2 - (1-\alpha_n)\Big(1-\frac{\mu\rho_n}{\rho_{n+1}}\Big)\|z_n-y_n\|^2.\end{aligned} \tag{3.36}$$

The above relation implies that

$$\begin{aligned}&(1-\alpha_n)\Big(1-\frac{\mu\rho_n}{\rho_{n+1}}\Big)\|x_n-y_n\|^2 + (1-\alpha_n)\Big(1-\frac{\mu\rho_n}{\rho_{n+1}}\Big)\|z_n-y_n\|^2 \\ &\leq \alpha_n\|x_1-x^*\|^2 + \|x_n-x^*\|^2 - \|x_{n+1}-x^*\|^2.\end{aligned} \tag{3.37}$$

From Lemma 2.1, we have

$$\begin{aligned}\big\|x_{n+1}-x^*\big\|^2 &= \big\|\alpha_n x_1 + (1-\alpha_n)z_n - x^*\big\|^2 \\ &= \big\|\alpha_n[x_1-x^*] + (1-\alpha_n)[z_n-x^*]\big\|^2 \\ &\leq (1-\alpha_n)^2\big\|z_n-x^*\big\|^2 + 2\alpha_n\langle x_1-x^*, (1-\alpha_n)[z_n-x^*] + \alpha_n[x_1-x^*]\rangle \\ &= (1-\alpha_n)^2\big\|z_n-x^*\big\|^2 + 2\alpha_n\langle x_1-x^*, x_{n+1}-x^*\rangle \\ &\leq (1-\alpha_n)\big\|x_n-x^*\big\|^2 + 2\alpha_n\langle x_1-x^*, x_{n+1}-x^*\rangle.\end{aligned} \tag{3.38}$$

Case 1: Assume that there exists a fixed number $n_2 \in \mathbb{N}$ such that

$$\|x_{n+1}-x^*\| \leq \|x_n-x^*\|,\ \forall\, n \geq n_2. \tag{3.39}$$

Thus, the above expression implies that $\lim_{n\to\infty}\|x_n - x^*\|$ exists and let $\lim_{n\to\infty}\|x_n - x^*\| = l$. From expression (3.37), we have

$$(1-\alpha_n)\Big(1-\frac{\mu\rho_n}{\rho_{n+1}}\Big)\|x_n - y_n\|^2 + (1-\alpha_n)\Big(1-\frac{\mu\rho_n}{\rho_{n+1}}\Big)\|z_n - y_n\|^2 \\ \le \alpha_n\|x_1 - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2. \tag{3.40}$$

The existence of $\lim_{n\to\infty}\|x_n - x^*\| = l$, and $\alpha_n \to 0$ we can deduce that

$$\lim_{n\to\infty}\|x_n - y_n\| = \lim_{n\to\infty}\|z_n - y_n\| = 0. \tag{3.41}$$

It follows that

$$\lim_{n\to\infty}\|x_n - z_n\| \le \lim_{n\to\infty}\|x_n - y_n\| + \lim_{n\to\infty}\|y_n - z_n\| = 0. \tag{3.42}$$

Furthermore, we obtain

$$\begin{aligned}\big\|x_{n+1} - x_n\big\| &= \big\|\alpha_n x_1 + (1-\alpha_n)z_n - x_n\big\| \\ &= \big\|\alpha_n[x_1 - x_n] + (1-\alpha_n)[z_n - x_n]\big\| \\ &\le \alpha_n\big\|x_1 - x_n\big\| + (1-\alpha_n)\big\|z_n - x_n\big\|.\end{aligned} \tag{3.43}$$

It follows that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{3.44}$$

Since $\{x_n\}$ is bounded sequence and there exists a subsequence $\{x_{n_k}\}$ that converges weakly to some $\hat{x} \in \mathcal{E}$. By using Lemma 3.6, we have

$$\begin{aligned}&\limsup_{n\to\infty}\langle x_1 - x^*, x_n - x^*\rangle \\ &= \limsup_{k\to\infty}\langle x_1 - x^*, x_{n_k} - x^*\rangle = \langle x_1 - x^*, \hat{x} - x^*\rangle \le 0.\end{aligned} \tag{3.45}$$

Since $x^* = P_{VI(\mathcal{K},\mathcal{L})}(x_1)$. Thus, we have

$$\langle x_1 - x^*, y - x^*\rangle \le 0, \ \forall\, y \in VI(\mathcal{K},\mathcal{L}). \tag{3.46}$$

Combining expressions (3.45) and (3.46), we obtain

$$\begin{aligned}&\limsup_{n\to\infty}\langle x_1 - x^*, x_{n+1} - x^*\rangle \\ &\le \limsup_{n\to\infty}\langle x_1 - x^*, x_{n+1} - x_n\rangle + \limsup_{n\to\infty}\langle x_1 - x^*, x_n - x^*\rangle \le 0.\end{aligned} \tag{3.47}$$

Case 2: Assume that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\|x_{n_i} - x^*\| \le \|x_{n_{i+1}} - x^*\|, \ \forall\, i \in \mathbb{N}.$$

Thus, by Lemma 2.3 there exists a sequence $\{m_k\} \subset \mathbb{N}$ as $\{m_k\} \to \infty$, such that

$$\|x_{m_k} - x^*\| \le \|x_{m_{k+1}} - x^*\| \quad \text{and} \quad \|x_k - x^*\| \le \|x_{m_{k+1}} - x^*\|, \ \text{for all } k \in \mathbb{N}. \tag{3.48}$$

Similar to Case 1 and expression (3.37) provides that

$$(1-\alpha_{m_k})\Big(1-\frac{\mu\rho_{m_k}}{\rho_{m_k+1}}\Big)\|x_{m_k} - y_{m_k}\|^2 + (1-\alpha_{m_k})\Big(1-\frac{\mu\rho_{m_k}}{\rho_{m_k+1}}\Big)\|z_{m_k} - y_{m_k}\|^2 \\ \le \alpha_{m_k}\|x_1 - x^*\|^2 + \|x_{m_k} - x^*\|^2 - \|x_{m_k+1} - x^*\|^2. \tag{3.49}$$

Due to $\alpha_{m_k} \to 0$, we deduce the following results:

$$\lim_{n\to\infty}\|x_{m_k} - y_{m_k}\| = \lim_{n\to\infty}\|z_{m_k} - y_{m_k}\| = 0. \tag{3.50}$$

Next, we can obtain

$$\begin{aligned} \|x_{m_k+1} - x_{m_k}\| &= \|\alpha_{m_k} x_1 + (1-\alpha_{m_k}) z_{m_k} - x_{m_k}\| \\ &= \|\alpha_{m_k}[x_1 - x_{m_k}] + (1-\alpha_{m_k})[z_{m_k} - x_{m_k}]\| \\ &\leq \alpha_{m_k}\|x_1 - x_{m_k}\| + (1-\alpha_{m_k})\|z_{m_k} - x_{m_k}\| \longrightarrow 0. \end{aligned} \tag{3.51}$$

We use the same argument as in Case 1, which is as follows:

$$\limsup_{k\to\infty} \langle x_1 - x^*, x_{m_k+1} - x^* \rangle \leq 0. \tag{3.52}$$

Now, using expressions (3.38), we have

$$\begin{aligned} \|x_{m_k+1} - x^*\|^2 &\leq (1-\alpha_{m_k})\|x_{m_k} - x^*\|^2 + 2\alpha_{m_k}\langle x_1 - x^*, x_{m_k+1} - x^* \rangle \\ &\leq (1-\alpha_{m_k})\|x_{m_k+1} - x^*\|^2 + 2\alpha_{m_k}\langle x_1 - x^*, x_{m_k+1} - x^* \rangle \end{aligned} \tag{3.53}$$

It continues from that

$$\|x_{m_k+1} - x^*\|^2 \leq 2\langle x_1 - x^*, x_{m_k+1} - x^* \rangle. \tag{3.54}$$

Thus, expressions (3.47) and (3.54) implies that

$$\|x_{m_k+1} - x^*\|^2 \to 0, \text{ as } k \to \infty. \tag{3.55}$$

It implies that

$$\lim_{n\to\infty} \|x_k - x^*\|^2 \leq \lim_{n\to\infty} \|x_{m_k+1} - x^*\|^2 \leq 0. \tag{3.56}$$

Consequently, $x_n \to x^*$. This completes the proof of the theorem. □

Now, we propose a second variant of the first method to solve quasimonotone variational inequalities in real Hilbert spaces and prove a strong convergence result for the proposed method. The second method involves a non-monotonic self adaptive step rule to make the method independent of the Lipschitz constant. The second method is written as follows:

Algorithm 2 (Non-Monotonic Explicit Halpern-Type Subgradient Extragradient Method)

Step 0: Let $x_1 \in \mathcal{K}$, $\rho_1 > 0$, $\mu \in (0,1)$ and sequence $\{\varphi_n\}$ satisfying $\sum_{n=1}^{+\infty} \varphi_n < +\infty$. Moreover, $\{\gamma_n\} \subset (0,1)$ satisfying the following conditions:

$$\lim_{n\to+\infty} \gamma_n = 0 \text{ and } \sum_{n=1}^{+\infty} \gamma_n = +\infty.$$

Step 1: Compute $y_n = P_{\mathcal{K}}(x_n - \rho_n \mathcal{L}(x_n))$. If $x_n = y_n$, STOP. Otherwise, go to **Step 2**.
Step 2: Compute $z_n = P_{\mathcal{E}_n}(x_n - \rho_n \mathcal{L}(y_n))$ where

$$\mathcal{E}_n = \{z \in \mathcal{E} : \langle x_n - \rho_n \mathcal{L}(x_n) - y_n, z - y_n \rangle \leq 0\}.$$

Step 3: Compute $x_{n+1} = \gamma_n x_1 + (1-\gamma_n) z_n$.
Step 4: Compute

$$\rho_{n+1} = \begin{cases} \min\left\{\rho_n + \varphi_n, \ \frac{\mu\|x_n - y_n\|^2 + \mu\|z_n - y_n\|^2}{2\left[\left\langle \mathcal{L}(x_n) - \mathcal{L}(y_n), z_n - y_n \right\rangle\right]}\right\} & \text{if } \left\langle \mathcal{L}(x_n) - \mathcal{L}(y_n), z_n - y_n \right\rangle > 0, \\ \rho_n + \varphi_n, & \text{otherwise.} \end{cases} \tag{3.57}$$

Set $n := n+1$ and go back to **Step 1**.

Lemma 3.7. *A sequence $\{\rho_n\}$ generated by* (3.57) *is convergent to ρ and satisfying the following inequality*

$$\min\left\{\frac{\mu}{L},\rho_1\right\}\leq\rho_n\leq\rho_1+P\quad\textit{where}\quad P=\sum_{n=1}^{+\infty}\varphi_n.$$

Proof. Due to the Lipschitz continuity of a mapping $\mathcal{L}$ there exists a fixed number $L>0$. Consider that $\langle\mathcal{L}(x_n)-\mathcal{L}(y_n),z_n-y_n\rangle>0$, implies that

$$\begin{aligned}\frac{\mu(\|x_n-y_n\|^2+\|z_n-y_n\|^2)}{2\langle\mathcal{L}(x_n)-\mathcal{L}(y_n),z_n-y_n\rangle}&\geq\frac{2\mu\|x_n-y_n\|\|z_n-y_n\|}{2\|\mathcal{L}(x_n)-\mathcal{L}(y_n)\|\|z_n-y_n\|}\\&\geq\frac{2\mu\|x_n-y_n\|\|z_n-y_n\|}{2L\|x_n-y_n\|\|z_n-y_n\|}\geq\frac{\mu}{L}.\end{aligned}\tag{3.58}$$

By using mathematical induction on the definition of ρ_{n+1}, we have

$$\min\left\{\frac{\mu}{L},\rho_1\right\}\leq\rho_n\leq\rho_1+P.$$

Let $[\rho_{n+1}-\rho_n]^+=\max\{0,\rho_{n+1}-\rho_n\}$ and $[\rho_{n+1}-\rho_n]^-=\max\{0,-(\rho_{n+1}-\rho_n)\}$. From the definition of $\{\rho_n\}$, we have

$$\sum_{n=1}^{+\infty}(\rho_{n+1}-\rho_n)^+=\sum_{n=1}^{+\infty}\max\{0,\rho_{n+1}-\rho_n\}\leq P<+\infty.\tag{3.59}$$

That is, the series $\sum_{n=1}^{+\infty}(\rho_{n+1}-\rho_n)^+$ is convergent. Next, we need to prove the convergence of $\sum_{n=1}^{+\infty}(\rho_{n+1}-\rho_n)^-$. Let $\sum_{n=1}^{+\infty}(\rho_{n+1}-\rho_n)^-=+\infty$. Due to the reason that $\rho_{n+1}-\rho_n=(\rho_{n+1}-\rho_n)^+-(\rho_{n+1}-\rho_n)^-$. Thus, we have

$$\rho_{k+1}-\rho_1=\sum_{n=0}^{k}(\rho_{n+1}-\rho_n)=\sum_{n=0}^{k}(\rho_{n+1}-\rho_n)^+-\sum_{n=0}^{k}(\rho_{n+1}-\rho_n)^-.\tag{3.60}$$

By allowing $k\to+\infty$ in (3.60), we have $\rho_k\to-\infty$ as $k\to\infty$. This is a contradiction. Due to the convergence of the series $\sum_{n=0}^{k}(\rho_{n+1}-\rho_n)^+$ and $\sum_{n=0}^{k}(\rho_{n+1}-\rho_n)^-$ taking $k\to+\infty$ in (3.60), we obtain $\lim_{n\to\infty}\rho_n=\rho$. This completes the proof. □

Theorem 3.2. *Let a mapping $\mathcal{L}:\mathcal{E}\to\mathcal{E}$ satisfies the condition* ($\mathcal{L}$1)–($\mathcal{L}$4). *Then, the sequence $\{x_n\}$ generated by the Algorithm 2 converges strongly to a solution of $VI(\mathcal{K},\mathcal{L})$.*

Proof. The proof is the same as of Theorem 3.1. □

4. Numerical Illustrations

The computational results of the proposed schemes are described in this section and study how variations in control parameters affect the numerical effectiveness of the proposed algorithms. All computations are done in MATLAB R2018b and run on HP i 5 Core(TM)i5-6200 8.00 GB (7.78 GB usable) RAM laptop.

Example 4.1. *Let $\mathcal{E}=l_2$ be a real Hilbert space with sequences of real numbers satisfying the following condition*

$$\|x_1\|^2+\|x_2\|^2+\cdots+\|x_n\|^2+\cdots<+\infty.\tag{4.61}$$

Assume that operator $\mathcal{L} : \mathcal{K} \to \mathcal{K}$ is defined by

$$G(x) = (5 - \|x\|)x, \ \forall\, x \in \mathcal{E}$$

where $\mathcal{K} = \{x \in \mathcal{E} : \|x\| \le 3\}$. It is easy to see that $\mathcal{L}$ is weakly sequentially continuous on $\mathcal{E}$ and $VI(\mathcal{K}, \mathcal{L}) = \{0\}$. For any $x, y \in \mathcal{E}$, we have

$$\begin{aligned} \big\|\mathcal{L}(x) - \mathcal{L}(y)\big\| &= \big\|(5 - \|x\|)x - (5 - \|y\|)y\big\| \\ &= \big\|5(x-y) - \|x\|(x-y) - (\|x\| - \|y\|)y\big\| \\ &\le 5\|x-y\| + \|x\|\|x-y\| + \big|\|x\| - \|y\|\big|\|y\| \\ &\le 5\|x-y\| + 3\|x-y\| + 3\|x-y\| \\ &\le 11\|x-y\|. \end{aligned} \tag{4.62}$$

Hence $\mathcal{L}$ is L-Lipschitz continuous with $L = 11$. For any $x, y \in \mathcal{E}$ and let $\langle \mathcal{L}(x), y - x\rangle > 0$ such that

$$(5 - \|x\|)\langle x, y - x\rangle > 0.$$

Since $\|x\| \le 3$ implies that

$$\langle x, y - x\rangle > 0.$$

Thus, we have

$$\begin{aligned} \langle \mathcal{L}(y), y - x\rangle &= (5 - \|y\|)\langle y, y - x\rangle \\ &\ge (5 - \|y\|)\langle y, y - x\rangle - (5 - \|y\|)\langle x, y - x\rangle \\ &\ge 2\|x-y\|^2 \ge 0. \end{aligned} \tag{4.63}$$

Thus, we shown that $\mathcal{L}$ is quasimonotone on $\mathcal{K}$. Let $x = (\frac{5}{2}, 0, 0, \cdots, 0, \cdots)$ and $y = (3, 0, 0, \cdots, 0, \cdots)$, we have

$$\langle \mathcal{L}(x) - \mathcal{L}(y), x - y\rangle = (2.5 - 3)^3 < 0.$$

A projection on the set C is computed explicitly as follows:

$$P_C(x) = \begin{cases} x \quad \textit{if} \quad \|x\| \le 3, \\ \frac{3x}{\|x\|}, \quad \textit{otherwise.} \end{cases}$$

Numerical results are shown in Figures 1 and 2 and Table 1. The iterative control parameters are taken in the following manner: (i) ***Algorithm 1*** *:* $\rho_1 = 0.20; \mu = 0.55; \gamma_n = \frac{1}{(n+2)}; D_n = \|x_n - y_n\|$; (ii) ***Algorithm 2*** *:* $\rho_1 = 0.20; \mu = 0.55; \gamma_n = \frac{1}{(n+2)}; \varphi_n = \frac{100}{(n+1)}; D_n = \|x_n - y_n\|$.

TABLE 1. Numerical results values for Example 4.1.

	Number of Iterations		Execution Time in Seconds	
x_1	***Algorithm 1***	***Algorithm 2***	***Algorithm 1***	***Algorithm 2***
$(1, 1, \cdots, 1_{5000}, 0, 0, \cdots)$	*120*	*75*	*7.47764030000000*	*4.79321210000000*
$(1, 2, \cdots, 5000, 0, 0, \cdots)$	*220*	*72*	*14.3395908000000*	*6.83192160000000*

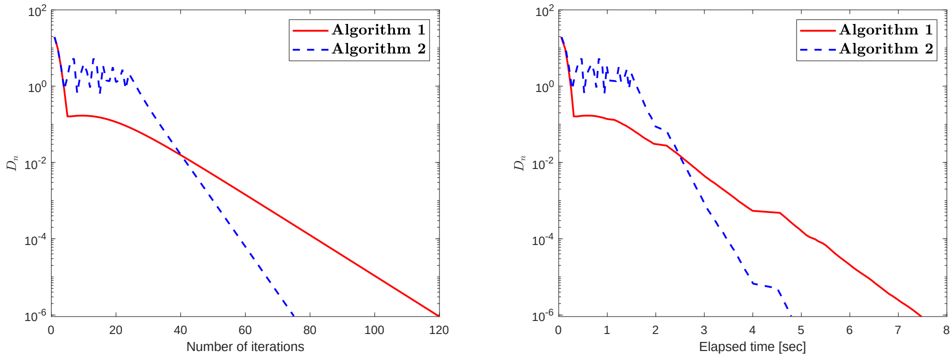

FIGURE 1. Numerical illustration of Algorithm 1 and Algorithm 2 by using $x_1 = (1, 1, \cdots, 1_{5000}, 0, 0, \cdots)$.

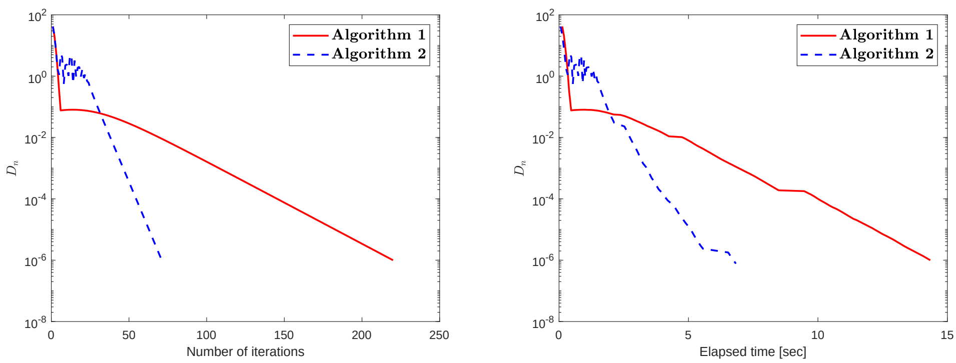

FIGURE 2. Numerical illustration of Algorithm 1 and Algorithm 2 by using $x_1 = (1, 2, \cdots, 5000, 0, 0, \cdots)$.

CONCLUSION

The main idea of this paper is to study quasimonotone variational inequality problems in infinite-dimensional Hilbert spaces and to prove that the iterative sequence generated by the Halpern subgradient extragradient algorithm is convergent strongly to a solution.
Acknowledgement. The first author was supported by National Research Council of Thailand and Khon Kaen University 2017 (Grant on. 600052). The third author was support by Chiang Mai University. The fourth author would like to thank Phetchabun Rajabhat University.

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[1] DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
KHON KAEN UNIVERSITY
KHON KAEN 40002, THAILAND
Email address: pongyo@kku.ac.th

[2]DEPARTMENT OF MATHEMATICS
KING MONGKUT'S UNIVERSITY OF TECHNOLOGY THONBURI (KMUTT)
BANGKOK 10140, THAILAND
Email address: habib.rehman@mail.kmutt.ac.th

[3]RESEARCH CENTER IN MATHEMATICS AND APPLIED MATHEMATICS
DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE, CHIANG MAI UNIVERSITY
CHIANG MAI 50200, THAILAND

[4]DATA SCIENCE RESEARCH CENTER
DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
CHIANG MAI UNIVERSITY
CHIANG MAI 50200, THAILAND
Email address: bancha.p@cmu.ac.th

[5]DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE AND TECHNOLOGY
PHETCHABUN RAJABHAT UNIVERSITY
PHETCHABUN 67000, THAILAND.
Email address: nuttapol.pak@pcru.ac.th