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Dedicated to the memory of Academician Mitrofan M. Choban (1942-2021)

Maia type fixed point theorems for some classes of enriched contractive mappings in Banach spaces

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ABSTRACT. We give some extensions of the beautiful 1968 fixed point theorem of Maia [Maia, M. G. Un'osservazione sulle contrazioni metriche. (Italian) *Rend. Sem. Mat. Univ. Padova* **40** (1968), 139–143] to three classes of enriched contractive mappings in Banach spaces: enriched contractions, Kannan enriched contractions and Ćirić-Reich-Rus contractions.

1. INTRODUCTION

The metric fixed point theory has developed by and around Banach's contraction mapping principle, which, in the case of a metric space setting, can be briefly stated as follows.

Theorem 1.1. *Let (X, d) be a complete metric space and $T : X \to X$ a contraction, i.e., a map satisfying*

$$d(Tx, Ty) \leq a\, d(x, y)\,, \quad \text{for all } x, y \in X\,, \tag{1.1}$$

where $0 \leq a < 1$ is constant. Then: $(p1)$ T has a unique fixed point p in X (i.e., $Tp = p$); $(p2)$ The Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = Tx_n\,, \quad n = 0, 1, 2, \dots \tag{1.2}$$

converges to p, for any $x_0 \in X$.

Remark 1.1. A map satisfying $(p1)$ and $(p2)$ in Theorem 1.1 is said to be *a Picard operator*, see [71], [72] for more details.

Theorem 1.1, which has been stated first by Banach [8] in the setting of a complete normed linear space (what we call now a *Banach space*), has been transposed by Cacciop-poli [26] to metric spaces.

Being a simple and versatile tool in establishing existence and uniqueness theorems for solving many kinds of nonlinear problems - especially when the setting is a Banach space - Theorem 1.1 plays a very important role in nonlinear analysis.

This fact motivated researchers to try to extend and generalise Theorem 1.1 in such a way that its area of potential applications should be enlarged as much as possible, see the monographs [11], [70]-[73] for many of such kind of generalizations.

In 1968, by distributing the assumptions on two comparable metrics d and ρ defined on the set X, Maia [44] established a very interesting and beautiful generalization of Theorem 1.1.

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Theorem 1.2 ([44], Teorema 1)**.** *Let X be a set endowed with two metrics d and ρ satisfying*

$$d(x,y) \le \rho(x,y), \text{ for all } x,y \in X. \tag{1.3}$$

Suppose

(i) (X,d) is a complete metric space;

(ii) $T: X \to X$ is continuous with respect to d;

(iii) T is a contraction with respect to ρ, that is,

$$\rho(Tx,Ty) \le a\,\rho(x,y)\,, \quad \text{for all } x,y \in X\,, \tag{1.4}$$

where $0 \le a < 1$ is constant. Then T is a Picard operator.

Remark 1.2. It is easily seen that, if $d \equiv \rho$, then Theorem 1.2 reduces to Theorem 1.1.

Due to the beautiful idea on which the Maia's fixed point theorem is builded, it attracted much interest and still attracts many researchers working in fixed point theory, see Albu [1], Ansari et al. [4], Balazs [5]-[7], Bayen [9], Berinde [10], Berinde and Vetro [22], Bhola and Sharma [23], Bylka [25], Dhage [30]-[33], Dhage and Dhobale [32], Filip [34], [35], Garg [36], Gheorghiu [37], Ilea [38], Iseki [39], Kasahara [42], A. S. Mureşan [45], [46], V. Mureşan [47]-[50], Nădăban et al. [51], Nagare [52], Namdeo and Gupta [53], Pachpatte [54], Păcurar [55], [56], Păcurar and Rus [57], Pande [58], Pathak and Dubey [59], Petracovici [60], Petruşel and Rus [61], Petruşel et al. [62], Popa [63], Precup [64], Ray [65], Rus [68]-[75], Rzepecki [76]-[78], Sharma [79], Shrivastava and Dubey [80], Shukla and Radenović [81], S. P. Singh [82], S. L. Singh [83], M. R. Singh [84], Trif [85], Turinici [86], [87] etc.

On the other hand, in the recent papers [15]-[20], the authors used the technique of enrichment nonlinear mapings in order to generalize, in the setting of a Banach space, some classes of contractive mappings, amongst which we mention the Banach contractions, for which they introduced and studied the corresponding and larger class of *enriched contractions*.

Starting from the above facts, the main aim of this paper is to use the approach based on the technique of enrichment of contractive type mappings in order to establish some Maia fixed point theorems for some important classes of enriched contractions in Banach spaces.

2. Maia type fixed point theorems for enriched contractions

The concept of *enriched contraction* has been introduced and studied in [15] as a natural generalization of the classical concept of Banach contraction.

Definition 2.1 (Definition 2.1, [15])**.** Let $(X, \|\cdot\|)$ be a linear normed space. A mapping $T: X \to X$ is said to be a (b,θ)-*enriched contraction* if there exist $b \in [0,+\infty)$ and $\theta \in [0, b+1)$ such that

$$\|b(x-y) + Tx - Ty\| \le \theta \|x-y\|, \forall x,y \in X. \tag{2.5}$$

Obviously, any Banach contraction satisfies (2.5) with $b=0$. The next theorem is the main result in [15] and represents an effective generalization of Banach's fixed point theorem in the setting of a Banach space.

Theorem 2.3 ([15])**.** *Let $(X, \|\cdot\|)$ be a Banach space and $T: X \to X$ a (b,θ)-enriched contraction. Then*

(i) $Fix\,(T) = \{p\}$, for some $p \in X$;

(ii) There exists $\lambda \in (0,1]$ such that the iterative method $\{x_n\}_{n=0}^{\infty}$, given by

$$x_{n+1} = (1-\lambda)x_n + \lambda T x_n,\; n \ge 0,$$

converges to p, for any $x_0 \in X$*;*

(iii) *The following estimate holds*

$$\|x_{n+i-1} - p\| \le \frac{c^i}{1-c} \cdot \|x_n - x_{n-1}\|, \quad n = 0, 1, 2, \ldots; \; i = 1, 2, \ldots, \tag{2.6}$$

where $c = \dfrac{\theta}{b+1}$.

We now state a Maia type fixed point theorem for enriched contractions defined on a linear vector space which is endowed with a metric d which is subordinated to a norm $\|\cdot\|$.

Theorem 2.4. *Let* X *be a linear vector space endowed with a metric* d *and a norm* $\|\cdot\|$ *satisfying the condition*

$$d(x, y) \le \|x - y\|, \text{ for all } x, y \in X. \tag{2.7}$$

Suppose

(i) (X, d) *is a complete metric space;*

(ii) $T : X \to X$ *is continuous with respect to d;*

(iii) T *is an enriched contraction with respect to* $\|\cdot\|$*, that is, there exist* $b \in [0, +\infty)$ *and* $\theta \in [0, b+1)$ *such that*

$$\|b(x - y) + Tx - Ty\| \le \theta \|x - y\|, \forall x, y \in X. \tag{2.8}$$

Then

(i) $Fix\,(T) = \{p\}$*, for some* $p \in X$*;*

(ii) *There exists* $\lambda \in (0, 1]$ *such that the iterative method* $\{x_n\}_{n=0}^{\infty}$*, given by*

$$x_{n+1} = (1 - \lambda)x_n + \lambda T x_n, \; n \ge 0, \tag{2.9}$$

converges in (X, d) *to p, for any* $x_0 \in X$*;*

(iii) *The estimate*

$$d(x_n, p) \le \frac{c^n}{1-c} \cdot \|x_1 - x_0\|, \; n \ge 1 \tag{2.10}$$

and

$$d(x_n, p) \le \frac{c}{1-c} \cdot \|x_n - x_{n-1}\|, \; n \ge 1, \tag{2.11}$$

hold with $c = \dfrac{\theta}{b+1}$.

Proof. We consider the case $b > 0$ (when $b = 0$, the proof is immediate). In this case, let us denote $\lambda = \dfrac{1}{b+1}$. Obviously, $0 < \lambda < 1$ and the enriched contractive condition (2.8) becomes

$$\left\| \left(\frac{1}{\lambda} - 1 \right)(x - y) + Tx - Ty \right\| \le \theta \|x - y\|, \forall x, y \in X,$$

which can be written in an equivalent form as

$$\|T_\lambda x - T_\lambda y\| \le c \cdot \|x - y\|, \forall x, y \in X, \tag{2.12}$$

where we denoted $c = \lambda\theta$, while T_λ is the averaged mapping defined by

$$T_\lambda x = (1 - \lambda)x + \lambda T x, \forall x \in C. \tag{2.13}$$

Since $\theta \in (0, b+1)$, it follows that $c \in (0, 1)$ and therefore by (4.44) T_λ is a c-contraction.

In view of (2.13), the Krasnoselskij iterative process $\{x_n\}_{n=0}^{\infty}$ defined by (4.38) is exactly the Picard iteration associated to T_λ, that is,

$$x_{n+1} = T_\lambda x_n,\ n \geq 0. \tag{2.14}$$

Take $x = x_n$ and $y = x_{n-1}$ in (4.44) to get

$$\|x_{n+1} - x_n\| \leq c \cdot \|x_n - x_{n-1}\|,\ n \geq 1. \tag{2.15}$$

By (4.47) one obtains routinely the following two estimates

$$\|x_{n+m} - x_n\| \leq c^n \cdot \frac{1-c^m}{1-c} \cdot \|x_1 - x_0\|,\ n \geq 0, m \geq 1. \tag{2.16}$$

and

$$\|x_{n+m} - x_n\| \leq c \cdot \frac{1-c^m}{1-c} \cdot \|x_n - x_{n-1}\|,\ n \geq 1,\ m \geq 1. \tag{2.17}$$

Now, by (4.48) it follows that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence in $(X, \|\cdot\|)$. By the inequality (2.7), we have

$$d(x_{n+m}, x_n) \leq c^n \cdot \frac{1-c^m}{1-c} \cdot \|x_1 - x_0\|,\ n \geq 0, m \geq 1,$$

which shows that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence in (X, d), too.

Hence $\{x_n\}_{n=0}^{\infty}$ is convergent in $(X, \|\cdot\|)$. Let us denote

$$p = \lim_{n\to\infty} x_n. \tag{2.18}$$

By letting $n \to \infty$ in (2.14) and, using the continuity of T_λ with respect to d (which follows by the continuity of T with respect to d), we immediately obtain

$$p = T_\lambda p \Leftrightarrow p \in Fix\,(T_\lambda).$$

Next, we prove that p is the unique fixed point of T_λ. Assume that $q \neq p$ is another fixed point of T_λ. Then, by (4.44)

$$0 < \|p - q\| \leq c \cdot \|p - q\| < \|p - q\|,$$

a contradiction. Hence $Fix\,(T_\lambda) = \{p\}$ and since, by (2.14), $Fix\,(T) = Fix(T_\lambda)$, claim (i) is proven.

Conclusion (ii) follows by (4.50).

To prove (iii), we first observe that by combining (4.48) and (4.49) and (2.7), one obtains

$$d(x_{n+m}, x_n) \leq c^n \cdot \frac{1-c^m}{1-c} \cdot \|x_1 - x_0\|,\ n \geq 0, m \geq 1 \tag{2.19}$$

and

$$d(x_{n+m}, x_n) \leq c \cdot \frac{1-c^m}{1-c} \cdot \|x_n - x_{n-1}\|,\ n \geq 1,\ m \geq 1. \tag{2.20}$$

Now, we let $m \to \infty$ in (3.34) and (3.35) to get the desired estimate (2.10):

$$d(x_n, p) \leq \frac{c^n}{1-c} \cdot \|x_1 - x_0\|,\ n \geq 1$$

and (2.11):

$$d(x_n, p) \leq \frac{c}{1-c} \cdot \|x_n - x_{n-1}\|,\ n \geq 1,$$

respectively, where $c = \dfrac{\theta}{b+1}$. □

Remark 2.3. If $d(x, y) = \|x - y\|$, for all $x, y \in X$, then by Theorem 2.4 we obtain Theorem 2.3.

In this case, the two estimates (2.10) and (2.11) in Theorem 2.4 can be merged to yield the unified estimate (2.6) in Theorem 2.3.

3. Maia type fixed point theorems for enriched Kannan contractions

The concept of *enriched Kannan contraction* has been introduced and studied in [16] as a natural generalization of that of Kannan mappings [40], [41], which are self mapings $T : X \to X$ satisfying the Kannan's contraction condition

$$d(Tx, Ty) \le b\,(d(x, Tx) + d(y, Ty))\,,\ \text{for all } x, y \in X, \tag{3.21}$$

where $b \in [0, 1/2)$ is a constant.

Definition 3.2 ([16], Definition 2.1). Let $(X, \|\cdot\|)$ be a normed linear space. A mapping $T : X \to X$ is said to be a (k, a)*-enriched Kannan mapping* if there exist $a \in [0, 1/2)$ and $k \in [0, \infty)$ such that

$$\|k(x - y) + Tx - Ty\| \le a\,(\|x - Tx\| + \|y - Ty\|)\,,\ \text{for all } x, y \in X. \tag{3.22}$$

Obviously, any Kannan mapping satisfies (3.22) with $k = 0$.

The next theorem, the main result in [16], is a genuine generalization of the Kannan fixed point theorem in the setting of Banach spaces, see Example 2.1 in [16].

Theorem 3.5 ([16]). *Let $(X, \|\cdot\|)$ be a Banach space and $T : X \to X$ a (k, a)-enriched Kannan mapping. Then*

(i) $Fix\,(T) = \{p\}$, for some $p \in X$;

(ii) There exists $\lambda \in (0, 1]$ such that the iterative method $\{x_n\}_{n=0}^{\infty}$, given by

$$x_{n+1} = (1 - \lambda)x_n + \lambda T x_n,\ n \ge 0, \tag{3.23}$$

converges to p, for any $x_0 \in X$;

(iii) The following estimate holds

$$\|x_{n+i-1} - p\| \le \frac{\delta^i}{1 - \delta} \cdot \|x_n - x_{n-1}\|\,, \quad n = 0, 1, 2, \ldots;\ i = 1, 2, \ldots$$

where $\delta = \dfrac{a}{1 - a}$.

Our aim in this section is to extend 3.5 and thus obtain a Maia type fixed point theorem for enriched Kannan contractions in Banach spaces.

Theorem 3.6. *Let X be a linear vector space endowed with a metric d and a norm $\|\cdot\|$ satisfying the condition*

$$d(x, y) \le \|x - y\|,\ \text{ for all } x, y \in X. \tag{3.24}$$

Suppose

(i) (X, d) is a complete metric space; (ii) $T : X \to X$ is continuous with respect to d;

(iii) T is an enriched Kannan contraction with respect to $\|\cdot\|$, that is, there exist $a \in [0, 1/2)$ and $k \in [0, \infty)$ such that (3.22) *holds.*

Then

(i) $Fix\,(T) = \{p\}$, for some $p \in X$;

(ii) There exists $\lambda \in (0, 1]$ such that the iterative method $\{x_n\}_{n=0}^{\infty}$, given by

$$x_{n+1} = (1 - \lambda)x_n + \lambda T x_n,\ n \ge 0, \tag{3.25}$$

converges in (X, d) to p, for any $x_0 \in X$;

(*iii*) *The following estimates hold*

$$d(x_n, p) \le \frac{\delta^n}{1-\delta} \cdot \|x_1 - x_0\|,\ n \ge 1 \tag{3.26}$$

and

$$d(x_n, p) \le \frac{\delta}{1-\delta} \cdot \|x_n - x_{n-1}\|,\ n \ge 1, \tag{3.27}$$

where $\delta = \dfrac{a}{1-a}$.

Proof. We analyze the case when $k > 0$ (the case $k = 0$ is immediate). Consider the averaged mapping T_λ for $\lambda = \dfrac{1}{k+1}$, as $0 < \lambda < 1$. In this case we have that $k = 1/\lambda - 1$ and thus the contractive condition (3.22) becomes

$$\left\| \left(\frac{1}{\lambda} - 1\right)(x-y) + Tx - Ty \right\| \le a\left(\|x - Tx\| + \|y - Ty\|\right), \text{ for all } x, y \in X,$$

which can be written in an equivalent form as

$$\|T_\lambda x - T_\lambda y\| \le a\left(\|x - T_\lambda x\| + \|y - T_\lambda y\|\right), \text{ for all } x, y \in X. \tag{3.28}$$

The above inequality shows that T_λ is a Kannan mapping.

According to (2.13), the iterative process $\{x_n\}_{n=0}^\infty$ defined by (3.25) is the Picard iteration associated to T_λ, that is,

$$x_{n+1} = T_\lambda x_n,\ n \ge 0. \tag{3.29}$$

Take $x = x_n$ and $y = x_{n-1}$ in (4.44) to get

$$\|x_{n+1} - x_n\| \le a\left(\|x_n - x_{n+1}\| + \|x_n - x_{n-1}\|\right),$$

which yields

$$\|x_{n+1} - x_n\| \le \frac{a}{1-a}\|x_n - x_{n-1}\|,\ n \ge 1.$$

Since $0 < a < \dfrac{1}{2}$, by denoting $\delta = \dfrac{a}{1-a}$, we have $0 < \delta < 1$ and therefore the sequence $\{x_n\}_{n=0}^\infty$ satisfies

$$\|x_{n+1} - x_n\| \le \delta\|x_n - x_{n-1}\|,\ n \ge 1. \tag{3.30}$$

By (4.47) one obtains routinely the following two estimates

$$\|x_{n+m} - x_n\| \le \delta^n \cdot \frac{1-\delta^m}{1-\delta} \cdot \|x_1 - x_0\|,\ n \ge 0, m \ge 1 \tag{3.31}$$

and

$$\|x_{n+m} - x_n\| \le \delta \cdot \frac{1-\delta^m}{1-\delta} \cdot \|x_n - x_{n-1}\|,\ n \ge 1,\ m \ge 1. \tag{3.32}$$

Now, by (4.48) and the inequality (3.24), we have

$$d(x_{n+m}, x_n) \le \delta^n \cdot \frac{1-\delta^m}{1-\delta} \cdot \|x_1 - x_0\|,\ n \ge 0, m \ge 1,$$

which shows that $\{x_n\}_{n=0}^\infty$ is a Cauchy sequence in the complete metric space (X, d), hence it is convergent.

Let us denote

$$p = \lim_{n\to\infty} x_n. \tag{3.33}$$

By the continuity of T with respect to d it follows that T_λ is also continuous with respect to d and therefore by passing to the limit in (3.29) we obtain $p \in Fix\,(T_\lambda)$.

To prove that p is the unique fixed point of T_λ, assume that $q \neq p$ is another fixed point of T_λ. Then, by (4.44) we get

$$0 < \|p - q\| \leq a \cdot 0,$$

a contradiction. Hence $Fix\,(T_\lambda) = \{p\}$ and since $Fix\,(T) = Fix(T_\lambda)$, (i) is proven.

Conclusion (ii) now follows by (4.50).

To prove (iii), we first observe that by combining (4.48) and (4.49) with (2.7), one obtains

$$d(x_{n+m}, x_n) \leq \delta^n \cdot \frac{1-\delta^m}{1-\delta} \cdot \|x_1 - x_0\|,\ n \geq 0, m \geq 1 \tag{3.34}$$

and

$$d(x_{n+m}, x_n) \leq \delta \cdot \frac{1-\delta^m}{1-\delta} \cdot \|x_n - x_{n-1}\|,\ n \geq 1,\ m \geq 1, \tag{3.35}$$

respectively. Now, we let $m \to \infty$ in (3.34) and (3.35) to get the desired estimates (3.26) and (3.27). □

Remark 3.4. If $d(x, y) = \|x - y\|$, for all $x, y \in X$, then by Theorem 3.6 we obtain Theorem 3.5.

In this case, the two estimates (3.26) and (3.27) in Theorem 3.6 can be merged to yield the unified estimate in Theorem 3.5.

4. MAIA FIXED POINT THEOREM FOR ENRICHED ĆIRIĆ-REICH-RUS CONTRACTIONS

Let (X, d) be a metric space. In 1971, Ćirić [29], Reich [66] and Rus [67] have established independently a very nice fixed point theorem for mappings $T : X \to X$ satisfying the following condition:

$$d(Tx, Ty) \leq ad(x, y) + b\,(d(x, Tx) + d(y, Ty))\,,\ \text{for all } x, y \in X, \tag{4.36}$$

where $a, b \geq 0$ and $a + 2b < 1$.

We remark that if $b = 0$, condition (4.36) reduces to Banach's contraction condition (1.1) while, for $a = 0$ condition (4.36) reduces to Kannan's contraction condition (3.21).

Therefore, the fixed point results established in [29], [66] and [67], under slightly different forms, are genuine generalizations of the Banach's contraction principle [8], [26] and of Kannan's fixed point theorem [40], see also [41], as shown by examples in [21].

Our aim in this section is to unify and extend Theorems 2.4 and 3.6 and thus obtain a Maia type fixed point theorem for enriched Ćirić-Reich-Rus contractions in Banach spaces.

To this end we need the following concept introduced in [21].

Definition 4.3 ([21], Definition 2.3)**.** Let $(X, \|\cdot\|)$ be a linear normed space. A mapping $T : X \to X$ is said to be a (k, a, b)*-enriched Ćirić-Reich-Rus contraction* if there exist $a, b \geq 0$ satisfying $a + 2b < 1$ and $k \in [0, \infty)$ such that

$$\|k(x - y) + Tx - Ty\| \leq a\|x - y\| + b\,(\|x - Tx\| + \|y - Ty\|)\,,\ \text{for all } x, y \in X. \tag{4.37}$$

Obviously, any Ćirić-Reich-Rus contraction satisfies (4.37) with $k = 0$.

Also, if $b = 0$, then from (4.37) we obtain the contraction condition (2.5) satisfied by an enriched contraction, while, if $a = 0$, from (4.37) we obtain the enriched Kannan contraction condition (3.22). Amongst the main results in [21] we recall the next theorem.

Theorem 4.7 ([21], Theorem 2.3)**.** *Let $(X, \|\cdot\|)$ be a Banach space and $T : X \to X$ a (k, a, b)-enriched Ćirić-Reich-Rus contraction. Then*

(i) $Fix\,(T) = \{p\}$, *for some* $p \in X$;

(ii) There exists $\lambda \in (0,1]$ such that the iterative method $\{x_n\}_{n=0}^{\infty}$, given by

$$x_{n+1} = (1-\lambda)x_n + \lambda T x_n,\; n \geq 0, \tag{4.38}$$

converges to p, for any $x_0 \in X$;

(iii) The following estimate holds

$$\|x_{n+i-1} - p\| \leq \frac{\delta^i}{1-\delta} \cdot \|x_n - x_{n-1}\|\,, \quad n = 0,1,2,\dots;\; i = 1,2,\dots \tag{4.39}$$

where $\delta = \dfrac{a+b}{1-b}$.

The aim of this section is to extend Theorem 4.7 and thus obtain a Maia type fixed point theorem for enriched Ćirić-Reich-Rus contractions in Banach spaces..

Theorem 4.8. *Let X be a linear vector space endowed with a metric d and a norm $\|\cdot\|$ satisfying the condition*

$$d(x,y) \leq \|x-y\|, \text{ for all } x,y \in X. \tag{4.40}$$

Suppose

(i) (X,d) is a complete metric space; (ii) $T : X \to X$ is continuous with respect to d;

(iii) T is an enriched Ćirić-Reich-Rus contraction with respect to $\|\cdot\|$, that is, there exist $a, b \geq 0$ satisfying $a + 2b < 1$ and $k \in [0,\infty)$ such that (4.37) *holds.*

Then

(i) $Fix\,(T) = \{p\}$, for some $p \in X$;

(ii) There exists $\lambda \in (0,1]$ such that the iterative method $\{x_n\}_{n=0}^{\infty}$, given by

$$x_{n+1} = (1-\lambda)x_n + \lambda T x_n,\; n \geq 0, \tag{4.41}$$

converges in (X,d) to p, for any $x_0 \in X$;

(iii) The following estimates hold

$$d(x_n,p) \leq \frac{\delta^n}{1-\delta} \cdot \|x_1 - x_0\|,\; n \geq 1 \tag{4.42}$$

and

$$d(x_n,p) \leq \frac{\delta}{1-\delta} \cdot \|x_n - x_{n-1}\|,\; n \geq 1, \tag{4.43}$$

where $\delta = \dfrac{a+b}{1-a}$.

Proof. First we work in the case when $k > 0$ (the case $k = 0$ is similar) and consider the averaged mapping T_λ defined by (2.13) for $\lambda = \dfrac{1}{k+1} < 1$.

In this case we have that $k = 1/\lambda - 1$ and thus the contractive condition (4.37) becomes

$$\left\|\left(\frac{1}{\lambda} - 1\right)(x-y) + Tx - Ty\right\| \leq a\|x-y\| + b\left(\|x-Tx\| + \|y-Ty\|\right), \text{ for all } x,y \in X,$$

which can be written equivalently as

$$\|T_\lambda x - T_\lambda y\| \leq a\lambda\|x-y\| + b\left(\|x - T_\lambda x\| + \|y - T_\lambda y\|\right), \text{ for all } x,y \in X,$$

and, because $a\lambda \leq a$, this implies that

$$\|T_\lambda x - T_\lambda y\| \leq a\|x-y\| + b\left(\|x - T_\lambda x\| + \|y - T_\lambda y\|\right), \text{ for all } x,y \in X, \tag{4.44}$$

which means that T_λ is a Ćirić-Reich-Rus contraction mapping.

By using triangle inequality in (4.44), we obtain that T_λ satisfies

$$\|T_\lambda x - T_\lambda y\| \leq \delta \cdot \|x-y\| + 2\delta \cdot \|y - T_\lambda x\|, \text{ for all } x,y \in X, \tag{4.45}$$

where $\delta = \dfrac{a+b}{1-b} < 1$.

Consider the iterative process $\{x_n\}_{n=0}^{\infty}$ defined by (4.38), which is in fact the Picard iteration associated to T_λ, that is,

$$x_{n+1} = T_\lambda x_n,\ n \geq 0. \tag{4.46}$$

and take $x = x_n$ and $y = x_{n-1}$ in (4.45) to get

$$\|x_{n+1} - x_n\| \leq \delta \|x_n - x_{n-1}\|,\ n \geq 1. \tag{4.47}$$

By (4.47) one obtains routinely the following two estimates

$$\|x_{n+m} - x_n\| \leq \delta^n \cdot \frac{1-\delta^m}{1-\delta} \cdot \|x_1 - x_0\|,\ n \geq 0, m \geq 1 \tag{4.48}$$

and

$$\|x_{n+m} - x_n\| \leq \delta \cdot \frac{1-\delta^m}{1-\delta} \cdot \|x_n - x_{n-1}\|,\ n \geq 1,\ m \geq 1. \tag{4.49}$$

Now, by (4.48) and the subordination inequality (4.40), we have

$$d(x_{n+m}, x_n) \leq \delta^n \cdot \frac{1-\delta^m}{1-\delta} \cdot \|x_1 - x_0\|,\ n \geq 0, m \geq 1,$$

which shows that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence in the complete metric space (X, d), hence it is convergent. Let us denote

$$p = \lim_{n\to\infty} x_n. \tag{4.50}$$

To prove that p is a fixed point of T_λ, observe that by the continuity of T with respect to d it follows that T_λ is continuous with respect to d, too and therefore by passing to the limit in (4.46) we obtain $p \in Fix\,(T_\lambda)$.

Assume, that $q \neq p$ is another fixed point of T_λ. Then, by (4.44) with $x = p$ and $y = q$ it follows

$$0 < \|p - q\| \leq a\|p - q\| < \|p - q\|,$$

a contradiction. Hence $Fix\,(T_\lambda) = \{p\}$ and since $Fix\,(T) = Fix(T_\lambda)$, (i) is proven.

Conclusion (ii) follows by (4.50).

The rest of the proof is similar to that of Theorem 3.6. □

Remark 4.5. In the particular case $a = 0$, by Theorem 4.8 we obtain Theorem 3.6, while, for $b = 0$, by Theorem 4.8 we obtain Theorem 2.4.

If $d(x, y) = \|x - y\|$, for all $x, y \in X$ and $a = 0$, then by Theorem 4.8 we get Theorem 3.5, while, for $a = 0$, we get Theorem 2.3.

In both these cases, the two estimates (4.42) and (4.43) in Theorem 4.8 can be merged to yield the unified estimate in Theorems 3.5 and 2.3.

5. Conclusions

1. Using the technique of enriching contractive type mappings T by means of the averaged operator T_λ, we established some Maia fixed point theorems for three important classes of enriched contractive mappings in Banach spaces.

2. The obtained fixed point theorems are important generalizations of the corresponding results for enriched contractions, enriched Kannan mappings and Ćirić-Reich-Rus contractions in Banach spaces, respectively.

3. Similar Maia fixed point theorems could be obtained by applying the technique of enriching nonlinear operators for the classes of contractive mappings studied in [1], [4], [5]-[7], [9], [10], [22], [23], [25], [30]-[39], [42], [45]-[87] etc.

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