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Dedicated to the memory of Academician Mitrofan M. Choban (1942-2021)

On the structure of the Levinson center for monotone non-autonomous dynamical systems with a first integral

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ABSTRACT. In this paper we give a description of the structure of compact global attractor (Levinson center) for monotone Bohr/Levitan almost periodic dynamical system $x' = f(t,x)$ (*) with the strictly monotone first integral. It is shown that Levinson center of equation (*) consists of the Bohr/Levitan almost periodic (respectively, almost automorphic, recurrent or Poisson stable) solutions. We establish the main results in the framework of general non-autonomous (cocycle) dynamical systems. We also give some applications of theses results to different classes of differential/difference equations.

1. INTRODUCTION

The aim of this paper is to study the structure of the compact global attractor (Levinson center) of differential equation

$$u'(t) = f(t,u(t)), \quad (f \in C(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n)) \tag{1.1}$$

with the strictly monotone first integral, when the right hand side is monotone with respect to spacial variable, and Bohr/Levitan almost periodic (respectively, almost automorphic, recurrent or Poisson stable) in $t \in \mathbb{R}$ uniformly with respect to u on every compact subset from $\mathbb{R}^n$. It is proved that

(1) Levinson center of equation (1.1) consists of Bohr/Levitan almost periodic (respectively, almost automorphic, recurrent or Poisson stable) solutions;
(2) each solution of equation (1.1) converges as $t \to \infty$ to some Bohr/Levitan almost periodic (respectively, almost automorphic, recurrent or Poisson stable) solution of equation (1.1) lying in the Levinson center.

Theses results we establish in the framework of general non-autonomous (cocycle) dynamical systems.

This study is a continuation of the author's work [10], which gives a positive answer to the I. U. Bronshtein's conjecture for monotone systems.

I. U. Bronshtein's conjecture [4, ChIV,p.273]. If an equation (1.1) with right hand side (Bohr) almost periodic in t satisfies the conditions of uniform positive stability and positive dissipativity, then it has at least one (Bohr) almost periodic solution.

If $n \leq 3$, then the positive answer to this conjecture follows from the results of V. V. Zhikov [44, ChII] (see also [27, ChVII] and [4, ChIV]).

Even for scalar equations ($n = 1$) as was shown by A. M. Fink and P. O. Frederickson [20] (see also [19, ChXII]), dissipation (without uniform positive stability) does not imply the existence of almost periodic solutions.

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The paper is organized as follows.

In Section 2 we collected some notions and facts from the theory of dynamical systems (both autonomous and non-autonomous) which we use in this paper: cocycles, skew-product dynamical systems, shift (Bebutov's) dynamical systems, Poisson stable (Bohr/Levitan almost periodic, almost automorphic and so on) motions and functions and their comparability by character of recurrence, monotone non-autonomous dynamical systems, global attractors of cocycle etc.

Section 3 is dedicated to the study the structure of global compact attractors of monotone non-autonomous (cocycles) dynamical systems having the strictly monotone first integral. The main result of this paper (Theorem 3.8) gives a description of the structure of Levinson center (compact global attractor) for this type of dynamical systems.

Section 4 is dedicated to the applications of our general results for different classes of evolution equations (linear and nonlinear ordinary differential/difference equations).

2. Preliminaries

In this section we collect some notions and facts from the theory of autonomous and non-autonomous dynamical systems [6] (see also, [9, Ch.IX]) which we will use in the paper.

2.1. **Cocycles.** Let Y be a complete metric space, let $\mathbb{R} := (-\infty, +\infty)$, $\mathbb{Z} := \{0, \pm 1, \pm 2, \ldots\}$, $\mathbb{T} = \mathbb{R}$ or $\mathbb{Z}$, $\mathbb{T}_+ = \{t \in \mathbb{T}|\quad t \geq 0\}$ and $\mathbb{T}_- = \{t \in \mathbb{T}|\quad t \leq 0\}$. Let $(Y, \mathbb{T}, \sigma)$ be an autonomous two-sided dynamical system on Y and E be a real or complex Banach space with the norm $|\cdot|$.

Definition 2.1. (Cocycle on the state space E with the base $(Y, \mathbb{T}, \sigma)$). The triplet $\langle W, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ (or briefly φ) is said to be a cocycle (see, for example, [9] and [32]) on the state space W with the base $(Y, \mathbb{T}, \sigma)$ if the mapping $\varphi : \mathbb{T}_+ \times Y \times W \to W$ satisfies the following conditions:

(1) $\varphi(0, y, u) = u$ for all $u \in W$ and $y \in Y$;
(2) $\varphi(t + \tau, y, u) = \varphi(t, \varphi(\tau, u, y), \sigma(\tau, y))$ for all $t, \tau \in \mathbb{T}_+, u \in W$ and $y \in Y$;
(3) the mapping φ is continuous.

Definition 2.2. (Skew-product dynamical system). Let $\langle W, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ be a cocycle on $W, X := W \times Y$ and π be a mapping from $\mathbb{T}_+ \times X$ to X defined by equality $\pi = (\varphi, \sigma)$, i.e., $\pi(t, (u, y)) = (\varphi(t, \omega, u), \sigma(t, y))$ for all $t \in \mathbb{T}_+$ and $(u, y) \in W \times Y$. The triplet $(X, \mathbb{T}_+, \pi)$ is an autonomous dynamical system and it is called [32] a skew-product dynamical system.

Definition 2.3. (Non-autonomous dynamical system.) Let $\mathbb{T}_1 \subseteq \mathbb{T}_2$ be two sub-semigroup of the group $\mathbb{T}$, $(X, \mathbb{T}_1, \pi)$ and $(Y, \mathbb{T}_2, \sigma)$ be two autonomous dynamical systems and $h : X \to Y$ be a homomorphism from $(X, \mathbb{T}_1, \pi)$ to $(Y, \mathbb{T}_2, \sigma)$ (i.e., $h(\pi(t, x)) = \sigma(t, h(x))$ for all $t \in \mathbb{T}_1$, $x \in X$ and h is continuous), then the triplet $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h\rangle$ is called (see [4] and [9]) a non-autonomous dynamical system.

Example 2.1. (The non-autonomous dynamical system generated by cocycle φ.) Let $\langle W, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ be a cocycle, $(X, \mathbb{T}_+, \pi)$ be a skew-product dynamical system ($X = W \times Y, \pi = (\varphi, \sigma)$) and $h = pr_2 : X \to Y$, then the triplet $\langle (X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma), h\rangle$ is a non-autonomous dynamical system.

2.2. **Bohr/Levitan almost periodic, almost automorphic, recurrent and Poisson stable motions.**

Definition 2.4. A number $\tau \in \mathbb{T}$ is called an $\varepsilon > 0$ shift of x (respectively, almost period of x), if $\rho(\pi(\tau, x), x) < \varepsilon$ (respectively, $\rho(\pi(t + \tau, x), \pi(t, x)) < \varepsilon$ for all $t \in \mathbb{T}$).

Definition 2.5. A point $x \in X$ is called almost recurrent (respectively, Bohr almost periodic), if for any $\varepsilon > 0$ there exists a positive number l such that at any segment of length l there is an ε shift (respectively, almost period) of point $x \in X$.

Definition 2.6. If the point $x \in X$ is almost recurrent and the set $H(x) := \overline{\{\pi(t,x) \mid t \in \mathbb{T}\}}$ is compact, then x is called recurrent.

Denote by $\mathfrak{N}_x := \{\{t_n\} \subset \mathbb{T}\ :\ \text{such that } \{\pi(t_n,x)\} \text{ converges to } x\ \}$.

Definition 2.7. A point $x \in X$ of the dynamical system $(X, \mathbb{T}, \pi)$ is called Levitan almost periodic [27], if there exists a dynamical system $(Y, \mathbb{T}, \sigma)$ and a Bohr almost periodic point $y \in Y$ such that $\mathfrak{N}_y \subseteq \mathfrak{N}_x$.

Definition 2.8. A point $x \in X$ is called stable Lagrange stable, if its trajectory $\Sigma_x := \{\pi(t,x)\ :\ t \in \mathbb{T}\}$ is relatively compact.

Definition 2.9. A point $x \in X$ is called almost automorphic in the dynamical system $(X, \mathbb{T}, \pi)$, if the following conditions hold:

(1) x is Lagrange stable;
(2) the point $x \in X$ is Levitan almost periodic.

Definition 2.10. A point $x_0 \in X$ is called [39, 41]

- pseudo recurrent if for any $\varepsilon > 0$, $t_0 \in \mathbb{T}$ and $p \in \Sigma_{x_0}$ there exist numbers $L = L(\varepsilon, t_0) > 0$ and $\tau = \tau(\varepsilon, t_0, p) \in [t_0, t_0 + L]$ such that $\tau \in \mathfrak{T}(p, \varepsilon))$;
- pseudo periodic (or uniformly Poisson stable) if for any $\varepsilon > 0$, $t_0 \in \mathbb{T}$ there exists a number $\tau = \tau(\varepsilon, t_0) > t_0$ such that $\tau \in \mathfrak{T}(p, \varepsilon))$ for any $p \in \Sigma_{x_0}$;
- Poisson stable in the positive (respectively, negative) direction if for any $\varepsilon > 0$ and $l > 0$ (respectively, $l < 0$) there exists a number $\tau > l$ (respectively, $\tau < l$) such that $\rho(\pi(\tau, x_0), x_0) < \varepsilon$. The point $x_0 \in X$ is called Poisson stable if it is stable (in the sense of Poisson) in the both directions.

Remark 2.1. 1. Every pseudo periodic point is pseudo recurrent.

2. If $x \in X$ is pseudo recurrent, then

- it is Poisson stable;
- every point $p \in H(x)$ is pseudo recurrent;
- there exist pseudo recurrent points for which the set $H(x_0)$ is compact but not minimal [36, ChV];
- there exist pseudo recurrent points which are not almost automorphic (respectively, pseudo periodic) [36, ChV].

3. If x_0 is a Lagrange stable point and $p \in \omega_p$ for any $p \in H(x_0)$, then the point x_0 is pseudo recurrent.

Below we will present some notions and results stated and proved by B. A. Shcherbakov [36]-[39].

Let $(X, \mathbb{T}_1, \pi)$ and $(Y, \mathbb{T}_2, \sigma)$ ($\mathbb{T}_1 \subseteq \mathbb{T}_2$) be two dynamical systems.

Definition 2.11. A point $x \in X$ is said to be comparable with $y \in Y$ by the character of recurrence, if for all $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that every δ–shift of y is an ε–shift for x, i.e., $d(\sigma(\tau, y), y) < \delta$ implies $\rho(\pi(\tau, x), x) < \varepsilon$, where d (respectively, ρ) is the distance on Y (respectively, on X).

Theorem 2.1. *Let x be comparable with $y \in Y$. If the point $y \in Y$ is stationary (respectively, τ–periodic, Levitan almost periodic, almost recurrent, Poisson stable), then the point $x \in X$ is so.*

Definition 2.12. A point $x \in X$ is called *uniformly comparable with $y \in Y$ by character of recurrence*, if for all $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that every δ–shift of $\sigma(t, y)$ is an ε–shift for $\pi(t, x)$ for all $t \in \mathbb{T}_1$, i.e., $d(\sigma(t + \tau, y), \sigma(t, y)) < \delta$ implies $\rho(\pi(t + \tau, x), x) < \varepsilon$ for all $t \in \mathbb{T}_1$ (or equivalently, $d(\sigma(t_1, y), \sigma(t_2, y)) < \delta$ implies $\rho(\pi(t_1, x), \pi(t_2, x)) < \varepsilon$ for all $t_1, t_2 \in \mathbb{T}_1$).

Denote by $\mathfrak{M}_x := \{\{t_n\} \subset \mathbb{T}$: such that $\{\pi(t_n, x)\}$ converges $\}$.

Definition 2.13. A point $x \in X$ is said [5],[8, ChII] to be strongly comparable by character of recurrence with the point $y \in Y$, if $\mathfrak{M}_y \subseteq \mathfrak{M}_x$.

Theorem 2.2. *Let X and Y be two complete metric spaces, the point x be uniformly comparable with $y \in Y$ by the character of recurrence. If the point $y \in Y$ is recurrent (respectively, almost periodic, almost automorphic, uniformly Poisson stable), then so is the point $x \in X$.*

Definition 2.14. Let (X, h, Y) be a fiber space [22], i.e., X and Y be two metric spaces and $h : X \to Y$ be a homomorphism from X into Y. The subset $M \subseteq X$ is said to be conditionally precompact [6],[9, Ch.IX],[11, Ch.III], if the primage $h^{-1}(Y') \bigcap M$ of every precompact subset $Y' \subseteq Y$ is a precompact subset of X. In particularly $M_y = h^{-1}(y) \bigcap M$ is a precompact subset of X_y for every $y \in Y$. The set M is called conditionally compact if it is closed and conditionally precompact.

Remark 2.2. Let W be a compact metric space, $X := W \times Y$ and (X, h, Y), where $h := pr_2 : X \to Y$, then X is conditionally pre-compact (with respect to (X, h, Y)).

Lemma 2.1. [11, ChIII] *Suppose that the following conditions are fulfilled:*

(1) $y \in Y$ is a two-sided Poisson stable point;
(2) $\langle (X, \mathbb{T}, \pi), (Y, \mathbb{T}, \sigma), h \rangle$ is a two-sided non-autonomous dynamical system;
(3) X is a conditionally compact space;
(4)
$$\inf_{t \le 0} \rho(\pi(t, x_1), \pi(t, x_2)) > 0$$
for any $x_1, x_2 \in X_y \quad (x_1 \neq x_2)$.

Then for any pair of points $x_1, x_2 \in X_y$ with $x_1 \neq x_2$ there are the sequences $\{t_k^-\} \in \mathfrak{N}_y^{-\infty}$ and $\{t_k^+\} \in \mathfrak{N}_y^{+\infty}$ such that
$$\lim_{k \to \infty} \pi(t_k^{\pm}, x_i) = x_i \ (i = 1, 2).$$

2.3. **Global Attractors of Cocycles.** Let W (respectively, Y be a complete metric space and $(Y, \mathbb{R}, \sigma)$ be a two-sided dynamical system.

Definition 2.15. The family $\{I_y \mid y \in Y\}$ $(I_y \subset W)$ of nonempty compact subsets W is called (see, for example, [1] and [21]) a compact pullback attractor (respectively, uniform pullback attractor) of a cocycle φ, if the following conditions hold:

(1) the set $I := \bigcup \{I_y \mid y \in Y\}$ is relatively compact;
(2) the family $\{I_y \mid y \in Y\}$ is invariant with respect to the cocycle φ, i.e. $\varphi(t, I_y, y) = I_{\sigma(t,y)}$ for all $t \in \mathbb{T}_+$ and $y \in Y$;
(3) for all $y \in Y$ (respectively, uniformly in $y \in Y$) and $K \in C(W)$
$$\lim_{t \to +\infty} \beta(\varphi(t, K, \sigma(-t, y)), I_y) = 0,$$
where $\beta(A, B) := \sup\{\rho(a, B) : a \in A\}$ is a semi-distance of Hausdorff.

Remark 2.3. 1. Let $\{I_y \mid y \in Y\}$ be a family of compact subsets from W such that $I = \bigcup \{I_y \mid y \in Y\}$ is precompact, then the set $J = \bigcup \{J_y \mid y \in Y\} \subset X = W \times Y$, where $J_y = I_y \times \{y\}$, is conditionally compact with respect to (X, h, Y) $(h = pr_2)$.

2. Let $\mathbf{I} = \{I_y|\ y \in Y\}$ be the pullback attractor (respectively, compact global attractor) of cocycle φ, then the set $J = \bigcup\{J_y|\ y \in Y\}$ ($J_y = I_y \times \{y\}$) is conditionally compact with respect to (X, h, Y) ($h = pr_2$).

Proof. Let Y' be a compact subset of Y, then $h^{-1}(Y') \bigcap J = \bigcup\{J_y|\ y \in Y'\} \subseteq I \times Y'$ and, consequently, it is precompact.

The second statement follows from the first one. □

Definition 2.16. A cocycle φ over $(Y, \mathbb{T}, \sigma)$ with the fiber W is said to be compactly dissipative, if there exits a nonempty compact $K \subseteq W$ such that

$$\lim_{t\to+\infty} \sup\{\beta(U(t, y)M, K) \mid y \in Y\} = 0 \tag{2.2}$$

for any $M \in C(W)$, where $U(t, y) := \varphi(t, \cdot, y)$.

Definition 2.17. A family $\{I_y \mid y \in Y\}(I_y \subset W)$ of nonempty compact subsets is called a compact (forward) global attractor of the cocycle φ, if the following conditions are fulfilled:

(1) the set $I := \bigcup\{I_y \mid y \in Y\}$ is relatively compact;
(2) the family $\{I_y \mid y \in Y\}$ is invariant with respect to the cocycle φ;
(3) the equality
$$\lim_{t\to+\infty} \sup_{y\in Y} \beta(\varphi(t, K, y), I) = 0$$
holds for every $K \in C(W)$.

Let $M \subseteq W$ and

$$\omega_y(M) := \bigcap_{t\geq 0} \overline{\bigcup_{\tau\geq t} \varphi(\tau, M, \sigma(-\tau, y))}$$

for any $y \in Y$.

Theorem 2.3. [9, ChII] *Let $\langle W, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ be compactly dissipative and K be the nonempty compact subset of W appearing in the equality (2.2), then:*

1. *$I_y = \omega_y(K) \neq \emptyset$, is compact, $I_y \subseteq K$ and*
$$\lim_{t\to+\infty} \beta(U(t, \sigma(-t, y))K, I_y) = 0$$
for every $y \in Y$;
2. *$U(t, y)I_y = I_{\sigma(t,y)}$ for all $y \in Y$ and $t \in \mathbb{T}_+$;*
3. $$\lim_{t\to+\infty} \beta(U(t, \sigma(-t, y))M, I_y) = 0$$
for all $M \in C(W)$ and $y \in Y$;
4. *the set I is relatively compact, where $I := \cup\{I_y \mid y \in Y\}$.*
5. *if Y is compact, then*
$$\lim_{t\to+\infty} \sup\{\beta(U(t, \sigma(-t, y))M, I)|\ y \in Y\} = 0$$
for any $M \in C(W)$.

Definition 2.18. Let $\langle W, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ be compactly dissipative, K be the nonempty compact subset of W appearing in the equality (2.2) and $I_y := \omega_y(K)$ for any $y \in Y$. The family of compact subsets $\{I_y|\ y \in Y\}$ is said to be a Levinson center (compact global attractor) of non-autonomous (cocycle) dynamical system $\langle W, \varphi, (Y, \mathbb{R}, \sigma)\rangle$.

Remark 2.4. According to Theorem 3.6 [10] by definition 2.18 is defined correctly the notion of Levinson center (compact global attractor) for non-autonomous (cocycle) dynamical system $\langle W, \varphi, (Y, \mathbb{T}, \sigma)\rangle$.

Corollary 2.1. *Let Y be compact, $\langle W, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ be compactly dissipative, $\boldsymbol{I} : \{I_y |\ y \in Y\}$ be its Levinson center and $I = \cup\{I_y \mid y \in Y\}$, then*

$$\lim_{t\to+\infty} \sup\{\beta(U(t, \sigma(-t, y))M, I)|\ y \in Y\} = 0$$

for all $M \in C(W)$.

Proof. This statement follows from Theorem 2.3 (item 5.), because

$$\sup\{\beta(U(t, y))M, I)|\ y \in Y\} = \sup\{\beta(U(t, \sigma(-t, y))M, I)|\ y \in Y\}$$

for all $M \in C(W)$. □

Definition 2.19. A cocycle φ is said to be positively uniformly Lyapunov stable if for any $\varepsilon > 0$ and nonempty compact subset $K \subseteq W$ there exists a positive number $\delta = \delta(\varepsilon, K)$ such that $\rho(u_1, u_2) < \delta$ $(u_1, u_2 \in K)$ implies

$$\rho(\varphi(t, u_1, y), \varphi(t, u_2, y)) < \varepsilon$$

for any $t \geq 0$ and $y \in Y$.

3. Levinson center for monotone non-autonomous dynamical systems with the strictly monotone first integral

Assume that E is an ordered Banach space [23, Ch.II]. A subset U of E is called lower-bounded (respectively, upper-bounded) if there exists an element $a \in E$ such that $a \leq U$ (respectively, $a \geq U$). Such an a is said to be a lower bound (respectively, upper bound) for U. A lower bound α is said to be the *greatest lower bound* (g.l.b.) or *infimum*, if any other lower bound a satisfies $a \leq \alpha$. Similarly, we can define the *least upper bound* (l.u.b.) or *supremum*.

Let (X, h, Y) be a Banach vector bundle with fiber E (see, for example, [26, Ch.I]). A bundle (X, h, Y) [22] is said to be *ordered* if each fiber $X_y := h^{-1}(y)$ $(y \in Y)$ is ordered. Note that only points on the same fiber may be order related: if $x_1 \leq x_2$ or $x_1 < x_2$, then it implies $h(x_1) = h(x_2)$. We assume that the order relation and the topology on X are compatible in the sense that $x \leq \tilde{x}$ if $x_n \leq \tilde{x}_n$ for all n and $x_n \to x$, $\tilde{x}_n \to \tilde{x}$ as $n \to \infty$.

Definition 3.20. For given bundle (X, h, Y), a non-autonomous dynamical system $\langle(X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma), h\rangle$ defined on it is said to be *monotone* if $x_1 \leq x_2$ implies $\pi(t, x_1) \leq \pi(t, x_2)$ for any $t > 0$.

For given non-autonomous dynamical system $\langle(X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma), h\rangle$, let $\mathcal{S} \subseteq X$ be a nonempty closed ordered and positively invariant subset possessing the following properties:

1. $h(\mathcal{S}) = Y$;
2. $\mathcal{S}$ is positively invariant with respect to π, i.e. $\langle(\mathcal{S}, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma), h\rangle$ is a non-autonomous dynamical system.

Below we will use the following assumptions:

- (C1) For every conditionally compact subset K of $\mathcal{S}$ and $y \in Y$ the set $K_y := h^{-1}(y) \bigcap K$ has both infimum $\alpha_y(K)$ and supremum $\beta_y(K)$.
- (C2) For every $x \in \mathcal{S}$, the semi-trajectory Σ_x^+ is conditionally precompact and its ω-limit set ω_x is positively uniformly stable.
- (C3) The non-autonomous dynamical system

$$\langle(\mathcal{S}, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma), h\rangle$$

is monotone.

Let $\mathbb{R}^d_+ := \{x \in \mathbb{R}^d :$ such that $x_i \geq 0$ $(x := (x_1, \dots, x_n))$ for any $i = 1, 2, \dots, d\}$ be the cone of nonnegative vectors of $\mathbb{R}^d$. By $\mathbb{R}^d_+$ on the space $\mathbb{R}^d$ is defined a partial order. Namely: $u \leq v$ if $v - u \in \mathbb{R}^d_+$. Let $K \subset \mathbb{R}^d$ be a compact subset of $\mathbb{R}^d$, and for each $1 \leq i \leq d$, define $\alpha_i(K) := \min\{x_i |\ x = (x_1, \dots, x_d) \in K\}$ and $\beta_i(K) := \max\{x_i |\ x = (x_1, \dots, x_d) \in K\}$. Then $\alpha(K) := (\alpha_1(K), \dots, \alpha_d(K))$ and $\beta(K) := (\beta_1(K), \dots, \beta_d(K))$ are the greatest lower bound (*infimum*) and least upper bound (*supremum*) of with respect to the order on $\mathbb{R}^d$, respectively.

Definition 3.21. Let $\langle E, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ be a cocycle and $\langle (X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma), h\rangle$ be a non-autonomous dynamical system associated by cocycle φ (i.e., $X := E \times Y$, $\pi = (\varphi, \sigma)$ and $h := pr_2 : X \to Y$). The cocycle φ is said to be monotone if $u_1 \leq u_2$ implies $\varphi(t, u_1, y) \leq \varphi(t, u_2, y)$ for any $t > 0$ and $y \in Y$.

Recall that a forward orbit $\{\pi(t, x_0) |\ t \geq 0\}$ of non-autonomous dynamical systems $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{T}, \sigma), h\rangle$ is said to be uniformly stable if for any $\varepsilon > 0$, there is a $\delta = \delta(\varepsilon) > 0$ such that $\rho(\pi(t_0, x_0), \pi(t_0, x_0)) < \delta$ implies $d(\pi(t, x_0), \pi(t, x_0)) < \varepsilon$ for every $t \geq t_0$.

Lemma 3.2. [14] *Assume that $(C1)$–$(C3)$ hold, $x_0 \in X$ such that ω_{x_0} is positively uniformly stable. Let $K := \omega_{x_0}$ be fixed and $y_0 := h(x_0)$. Then if $q \in \omega_q \subseteq \omega_{y_0}$, $\alpha_q := \alpha_q(K)$, $K^1 := \omega_{\alpha_q}$, then the set $K^1_q := \omega_{\alpha_q} \bigcap X_q$ (respectively, $\omega_{\beta_q} \bigcap X_q$) consists a single point γ_q (respectively, δ_q), i.e., $K^1_q = \{\gamma_q\}$ (respectively, $\{\delta_q\}$).*

Definition 3.22. A point $x_0 \in X$ is said to be uniformly Poisson stable [2] (or pseudo periodic [3, ChII,p.32]) if for arbitrary $\varepsilon > 0$ and $l > 0$ there exists a number $\tau > l$ such that $\rho(\pi(t + \tau, x), \pi(t, x)) < \varepsilon$ for any $t \in \mathbb{T}$.

Definition 3.23. Let $(X, \mathbb{T}, \pi)$ be a two-sided dynamical system. A point $x \in X$ is said to be strongly Poisson stable in the positive (respectively, in the negative) direction if $p \in \omega_p$ (respectively, $p \in \alpha_p$) for any $p \in H(x)$. The point $x \in X$ is said to be strongly Poisson stable if it is strongly Poisson stable in the both directions.

Remark 3.5. Every pseudo recurrent point is strongly Poisson stable. The inverse statement, generally speaking, is not true.

Theorem 3.4. [14] *Assume that $(C1)$–$(C3)$ hold, $x_0 \in X$ and $y_0 := h(x_0) \in Y$ is strongly Poisson stable. Then the following statements hold:*

(1) the point γ_{y_0} (respectively, δ_{y_0}) is strongly comparable by character of recurrence with y_0 and

(2)
$$\lim_{t \to +\infty} \rho(\pi(t, \alpha_{y_0}), \pi(t, \gamma_{y_0})) = 0.$$

Corollary 3.2. *Under the conditions $(C1) - (C3)$ if the point y_0 is τ-periodic (respectively, quasi periodic, Bohr almost periodic, recurrent, pseudo recurrent and Lagrange stable), then:*

(1) the point u_{y_0} is so;

(2) the point α_{y_0} is asymptotically τ-periodic (respectively, asymptotically quasi periodic, asymptotically Bohr almost periodic, asymptotically recurrent, pseudo recurrent).

Remark 3.6. 1. If the point y_0 is recurrent (in the sense of Birkhoff), then Corollary 3.2 coincides with the results of the work of J. Jiang and X.-Q. Zhao [24].

2. In the works of B. A. Shcherbakov [33]-[35], [36, ChV, Example 5.2.1] were constructed examples of pseudo recurrent and Lagrange stable motions which are not recurrent (in the sense of Birkhoff).

Theorem 3.5. [13] *Assume that the cocycle $\langle E, \varphi, (Y, \mathbb{T}, \sigma)\rangle$*

1. *is monotone;*
2. *admits a compact global attractor* $\mathbf{I} := \{I_y \mid y \in Y\}$;
3. *is positively uniformly Lyapunov stable and denote by* $\alpha(y)$ *(respectively, by* $\beta(y)$*) the greatest lower bound of the set* I_y *(respectively, the least upper bound of* I_y*)*

and the point $y \in Y$ *is positively Poisson stable, i.e.,* $y \in \omega_y$.

Then the following statements hold:

(1) $\alpha(y) \le u \le \beta(y)$ *for any* $u \in I_y$ *and* $y \in Y$;
(2) $\alpha(y), \beta(y) \in I_y$ *and, consequently,* $I_y \subseteq [\alpha(y), \beta(y)]$;
(3) $\varphi(t, \alpha(y), y) = \alpha(\sigma(t, y))$ *(respectively,* $\varphi(t, \beta(y), y) = \beta(\sigma(t, y))$*) for any* $t \ge 0$;
(4) *the point* $\gamma_*(y) := (\alpha(y), y) \in X = E \times Y$ *(respectively,* $\gamma^*(y) := (\beta(y), y) \in X$*) is comparable by character of recurrence with the point* y;
(5) *if* $u \in E$ *and* $u \le \alpha(y)$ *(respectively,* $u \ge \beta(y)$*), then* $\omega_x \bigcap X_y = \{\gamma_*(y)\}$ *(respectively,* $\omega_x \bigcap X_y = \{\gamma^*(y)\}$*), where* $x := (u, y)$;
(6) *if* $u \le \alpha(y)$ *(respectively,* $u \ge \beta(y)$*),then*

$$\lim_{t \to +\infty} \rho(\varphi(t, u, y).\gamma_*(\sigma(t, y))) = 0$$

(respectively,

$$\lim_{t \to +\infty} \rho(\varphi(t, u, y).\gamma^*(\sigma(t, y))) = 0);$$

(7) *if* y *is strongly Poisson stable, then the point* $\gamma_*(y) := (\alpha(y), y) \in X = E \times Y$ *(respectively,* $\gamma^*(y) := (\beta(y), y) \in X$*) is strongly comparable by character of recurrence with the point* y.

Corollary 3.3. [13] *Under the conditions of Theorem 3.5 the following statements take place:*

(1) *if the point* y *is* τ*-periodic (respectively, Levitan almost periodic, almost recurrent, almost automorphic, recurrent, Poisson stable), then the full trajectory* γ_y *passing through the point* $(\alpha(y), y)$ *(respectively, through the point* $(\beta(y), y)$*) is so;*
(2) *if the point* y *is quasi periodic (respectively, Bohr almost periodic, almost automorphic, recurrent, pseudo recurrent and Lagrange stable, uniformly Poisson stable and stable in the sense of Lagrange), then the full trajectory* γ_y *passing through the point* $(\alpha(y), y)$ *(respectively, through the point* $(\beta(y), y)$*) is so.*

Proof. This statement follows from the Theorems 3.5, 2.1 and 2.2. □

Theorem 3.6. [7],[11, Ch.III] *Let* $(X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma)\rangle$ *be a non-autonomous dynamical system with the following properties:*

(1) *it admits a conditionally relatively compact invariant set* J;
(2) *the non-autonomous dynamical system* $\langle (X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma), h\rangle$ *is positively uniformly stable on* J;
(3) *every point* $y \in Y$ *is two-sided Poisson stable.*

Then

(1) *all motions on* J *may be continued uniquely to the left and define on* J *a two-sided dynamical system* $(J, \mathbb{T}, \pi)$;
(2) *for every* $y \in Y$ *with* $J_y \neq \emptyset$ *there are two sequences* $\{t_n^1\} \to +\infty$ *and* $\{t_n^2\} \to -\infty$ *such that*

$$\pi(t_n^i, x) \to x \ (i = 1, 2)$$

as $n \to \infty$ *for all* $x \in J_y$.

Let $U \subseteq \mathbb{R}^n$, $V \in C^1(U, \mathbb{R})$ and denote by $\nabla V := \big(\frac{\partial V}{\partial x_1}, \dots, \frac{\partial V}{\partial x_n}\big)$.

Lemma 3.3. *Assume that the following conditions are fulfilled:*

(i) $\langle \mathbb{R}^n_+, \varphi, (Y, \mathbb{R}, \sigma)\rangle$ *is a monotone cocycle;*

(ii) there exists a first integral $H \in C^1(\mathbb{R}^n_+, \mathbb{R})$ *for cocycle* φ *with* $\nabla H(x) \gg 0$ *for any* $x \in \mathbb{R}^n_+$*;*

(iii) the family of motions $\varphi(t, u_0, y_0)$ *(*$(u_0, y_0) \in F \subseteq \mathbb{R}^n_+ \times Y$*) of the cocycle* φ *is uniformly bounded.*

Then the family of motions $\varphi(t, u_0, y_0)$ *(*$(u_0, y_0) \in F$*) is positively uniformly stable, i.e., for arbitrary* $\varepsilon > 0$ *there exists a positive number* $\delta = \delta(\varepsilon)$ *such that* $|\varphi(t_0, u_1, y_0) - \varphi(t_0, u_2, y_0)| < \delta$ *implies* $|\varphi(t, u_1, y_0) - \varphi(t, u_2, y_0)| < \varepsilon$ *for any* $t \geq t_0$ *and* $(u_1, y_0), (u_2, y_0) \in F$.

Proof. This statement may be proved using the same ideas as in the proof of Lemma 3.1 from [40] (see also [12]). Below we will present the details of this proof. Let $e := (1, \dots, 1) \in \mathbb{R}^n_+$. Since the family of motions $\varphi(t, u_0, y_0)$ ($(u_0, y_0) \in F$) is uniformly bounded on $\mathbb{T}_+$, we can choose a sufficiently large real number $r > 0$ such that

$$0 \leq \varphi(t, u_0, y_0) \leq q_0 := re \tag{3.3}$$

for any $t \geq 0$ and $(u_0, y_0) \in F$. Denote by

$$M := \max_{1 \leq i \leq n} \big\{ \max_{0 \leq x \leq q_0 + e} H_{x_i}(x) \big\},\ m := \min_{1 \leq i \leq n} \big\{ \min_{0 \leq x \leq q_0 + e} H_{x_i}(x) \big\}$$

Condition (ii) implies that $M, m > 0$. From the equality

$$H(y) - H(z) = \sum_{i=1}^n \int_0^1 H_{x_i}(z + s(y - z)) ds (y_i - z_i) \ \ (\forall\, y, z \in \mathbb{R}^n_+)$$

it follows that

$$|H(y) - H(z)| \leq nM \|y - z\|,\ \ \forall\, 0 \leq z, y \leq q_0 + e, \tag{3.4}$$

and

$$|H(y) - H(z)| \geq m \|y - z\|,\ \ \forall\, 0 \leq z \leq y \leq q_0 + e, \tag{3.5}$$

where $\|x\| := \sum_{i=1}^n |x_i|$.

Let $\varepsilon_0 := \min\{1, \frac{m}{2nM}\}$. For any given $0 < \varepsilon \leq \varepsilon_0$, there is $0 < \delta(\varepsilon) \leq \varepsilon/2$ such that

$$\varphi(\tau, u_0, y_0) - \delta(\varepsilon) e \leq \varphi(\tau, u_0, y_0) \leq \varphi(\tau, u_0, y_0) + \delta(\varepsilon) e \leq q_0 + e \tag{3.6}$$

for any $\tau \geq 0$ and $(u_0, y_0) \in F$. Put

$$p(\varepsilon, \tau, (u_0, y_0)) := (\max(\varphi_1(\tau, u_0, y_0) - \delta(\varepsilon), 0), \dots, \max(\varphi_n(\tau, u_0, y_0) - \delta(\varepsilon), 0))$$

and $q(\varepsilon, \tau, (u_0, y_0)) := \varphi(\tau, u_0, y_0) + \delta(\varepsilon) e$ ($(u_0, y_0) \in F$). Note that

$$0 \leq q(\varepsilon, \tau, (u_0, y_0)) - p(\varepsilon, \tau, (u_0, y_0)) =$$
$$[\varphi(\tau, u_0, y_0) - \delta(\varepsilon) e] - p(\varepsilon, \tau, (u_0, y_0)) + 2\delta(\varepsilon) e \leq 2\delta(\varepsilon) e$$

for all $\tau \geq 0$ and $(u_0, y_0) \in F$. Taking into consideration (3.4) and (3.6) we obtain

$$|H(p(\varepsilon, \tau, (u_0, y_0))) - H(q(\varepsilon, \tau, (u_0, y_0)))| \leq nM\varepsilon \tag{3.7}$$

for all $\tau \geq 0$ and $(u_0, y_0) \in F$. For given $\tau \geq 0$ an $(u_0, y_0) \in F$, let

$$U(\varepsilon, \tau, (u_0, y_0)) := \{z \in \mathbb{R}^n_+ |\ p(\varepsilon, \tau, (u_0, y_0)) \leq z \leq q(\varepsilon, \tau, (u_0, y_0))\}.$$

Note that $\varphi_i(\tau, u_0, y_0) \geq \max(\varphi_i(\tau, u_0, y_0) - \delta(\varepsilon), 0) = p_i(\varepsilon, \tau, (u_0, y_0))$ for any $i = 1, 2, \dots, n$ and, consequently, $\varphi(\tau, u_0, y_0) \in U(\varepsilon, \tau, (u_0, y_0))$. Since the cocycle φ is monotone, then we will have

$$\varphi(t, p(\varepsilon, \tau, (u_0, y_0)), \sigma(\tau, y_0)) \leq \varphi(t, \varphi(\tau, u_0, y_0), \sigma(\tau, y_0)) \leq$$
$$\varphi(t, q(\varepsilon, \tau, (u_0, y_0)), \sigma(\tau, y_0))$$

and

$$\varphi(t, p(\varepsilon, \tau, (u_0, y_0)), \sigma(\tau, y_0)) \le \varphi(t, z, \sigma(\tau, y_0)) \le \varphi(t, q(\varepsilon, \tau, (u_0, y_0)), \sigma(\tau, y_0))$$

for all $t \ge 0$ and $z \in U(\varepsilon, \tau, (u_0, y_0)))$. Taking into consideration that H is a first integral for the cocycle φ and inequality (3.7) we obtain

$$|H(\varphi(t, q(\varepsilon, \tau, (u_0, y_0), \sigma(\tau, y_0))) - H(\varphi(t, \varphi(\tau, u_0, y_0), \sigma(\tau, y_0)))| =$$
$$|H(q(\varepsilon, \tau, (u_0, y_0))) - H(\varphi(\tau, u_0, y_0))| \le$$
$$|H(q(\varepsilon, \tau, (u_0, y_0))) - H(p(\varepsilon, \tau, (u_0, y_)))| \le nM\varepsilon$$

for all $t \ge 0$.

By (3.5) and (3.6), we have

$$\|\varphi(t, q(\varepsilon, \tau, (u_0, y_0)), \sigma(\tau, y_0)) - \varphi(t, \varphi(\tau, u_0, y_0), \sigma(\tau, y_0))\| \le \frac{nM}{m}\varepsilon \tag{3.8}$$

for all $t \ge 0$ with $\varphi(t, q(\varepsilon, \tau, (u_0, y_0)), \sigma(\tau, y_0)) \in [0, q_0 + e]$. We will show that

$$\varphi(t, q(\varepsilon, \tau, (u_0, y_0)), \sigma(\tau, y_0)) \in [0, q_0 + e] \tag{3.9}$$

for all $t \ge 0$ and $(u_0, y_0) \in F$. If it is not true, then there exist $(u_0, y_0) \in F$ and a real number $t^* > 0$ ($t^* = t^*(u_0, y_0)$) such that $\varphi(t, q(\varepsilon, \tau, (u_0, y_0)), \sigma(\tau, y_0)) \in [0, q_0 + e]$ for any $t \in [0, t^*)$ and

$$\|\varphi(t^*, q(\varepsilon, \tau, (u_0, y_0)), \sigma(\tau, y_0))\| \ge \|q_0\| + 1. \tag{3.10}$$

On the other hand from (3.3) and (3.8) we have

$$\|\varphi(t^*, q(\varepsilon, \tau, (u_0, y_0)), \sigma(\tau, y_0))\| \le$$
$$\|\varphi(t^*, q(\varepsilon, \tau, (u_0, y_0)), \sigma(\tau, y_0)) - \varphi(t^*, \varphi(\tau, u_0, y_0), \sigma(\tau, y_0))\| +$$
$$\|\varphi(t^*, \varphi(\tau, u_0, y_0), \sigma(\tau, y_0))\| \le \|q_0\| + \tfrac{nM}{m}\varepsilon \le \|q_0\| + \tfrac{1}{2} \tag{3.11}$$

for any sufficiently small $\varepsilon > 0$. The inequalities (3.10) and (3.11) are contradictory. The obtained contradiction proves our statement.

It then follows that for every $z \in U(\varepsilon, \tau, (u_0, y_0))$ we have $\varphi(t, z, \sigma(\tau, y_0)) \in [0, q_0 + e]$ for any $t \ge 0$. Similarly, we can show that for any $z \in U(\varepsilon, \tau, (u_0, y_0))$ we have

$$\|\varphi(t, q(\varepsilon, \tau, (u_0, y_0)), \sigma(\tau, y_0)) - \varphi(t, z, \sigma(\tau, y_0))\| \le \frac{nM}{m}\varepsilon, \tag{3.12}$$

for any $t \ge 0$. Then (3.9) and (3.12) imply that for any $z \in U(\varepsilon, \tau, (u_0, y_0))$ and $t \ge 0$,

$$\|\varphi(t, \varphi(\tau, u_0, y_0), \sigma(\tau, y_0)) - \varphi(t, z, \sigma(\tau, y_0))\| \le$$
$$\|\varphi(t, \varphi(\tau, u_0, y_0), \sigma(\tau, y_0)) - \varphi(t, q(\varepsilon, \tau, (u_0, y_0)), \sigma(\tau, y_0))\| +$$
$$\|\varphi(t, q(\varepsilon, \tau, (u_0, y_0)), \sigma(\tau, y_0)) - \varphi(t, z, \sigma(\tau, y_0))\| \le \tfrac{nM}{m}\varepsilon + \tfrac{nM}{m}\varepsilon = \tfrac{2nM}{m}\varepsilon. \tag{3.13}$$

For any $\varepsilon \in (0, \varepsilon_0]$, $\tau \ge 0$, and $(u_1, y_0), (u_2, y_0) \in F$ with

$$\|\varphi(\tau, u_1, y_0) - \varphi(\tau, u_2, y_0)\| \le \delta(\varepsilon),$$

we have $\varphi(\tau, u_1, y_0) \in U(\varepsilon, \tau, (u_2, y_0))$. Then from (3.13) we obtain

$$\|\varphi(t + \tau, u_2, y_0) - \varphi(t + \tau, u_1, y_0)\| =$$
$$\|\varphi(t, \varphi(\tau, u_2, y_0), \sigma(\tau, y_0)) - \varphi(t, \varphi(\tau, u_1, y_0), \sigma(\tau, y_0))\| \le \tfrac{2nM}{m}\varepsilon$$

for any $t \ge 0$. Thus, the family of motions $\varphi(t, u_0, y_0)$ ($(u_0, y_0) \in F$) is uniformly stable. □

Lemma 3.4. *Assume that $\langle W, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ is a compact dissipative cocycle, then for any compact subsets $M \subset W$ and $N \subset Y$ the set $\varphi(\mathbb{T}_+, M, N) := \{\varphi(t, u, y) |\ t \in \mathbb{T}_+,\ u \in M,\ y \in N\}$ is relatively compact.*

Proof. Let ε be an arbitrary positive number, $M \in C(W)$, $N \in C(Y)$ and K be a nonempty compact subset from W figuring in (2.2). Then there exists a positive number $L(\varepsilon)$ such that

$$\sup_{y\in N} \beta(\varphi(t, M, y), K) < \varepsilon \tag{3.14}$$

for any $t \geq L(\varepsilon)$. Since the map $\varphi : \mathbb{T}_+ \times W \times Y \to W$ is continuous, then the set $K_\varepsilon := \varphi([0, L(\varepsilon)], M, N)$ is compact and, consequently, by (3.14) the set $\tilde{K} := K_\varepsilon \bigcup K$ is a compact ε-net for $\varphi(\mathbb{T}_+, M, N)$. Since the metric space W is complete then by Hausdorff's theorem (see, for example, [28, Ch.V]) the set $\varphi(\mathbb{T}_+, M, N)$ is relatively compact. Lemma is proved. □

Lemma 3.5. *Assume that the following conditions are fulfilled:*

(i) $\langle \mathbb{R}^n_+, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ *is a monotone cocycle;*
(ii) there exists a first integral $H \in C^1(\mathbb{R}^n_+, \mathbb{R})$ *for cocycle* φ *with* $\nabla H(x) \gg 0$ *for any* $x \in \mathbb{R}^n_+$*;*
(iii) the cocycle $\langle \mathbb{R}^n_+, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ *is compact dissipative and* $\boldsymbol{I} = \{I_y|\ y \in Y\}$ *is its Levinson center.*

Then the following statement hold:

(1) for any compact subsets $M \in C(\mathbb{R}^n_+)$ *and* $N \in C(Y)$ *the family of motions* $\varphi(t, u_0, y_0)$ *(*$(u_0, y_0) \in M \times N$*) is positively uniformly stable, i.e., for arbitrary* $\varepsilon > 0$ *there exists a positive number* $\delta = \delta(\varepsilon)$ *such that* $|\varphi(t_0, u_1, y_0) - \varphi(t_0, u_2, y_0)| < \delta$ *implies* $|\varphi(t, u_1, y_0) - \varphi(t, u_2, y_0)| < \varepsilon$ *for any* $t \geq t_0$ *and* $(u_1, y_0), (u_2, y_0) \in M \times N$*;*
(2) Levinson center $\boldsymbol{I} = \{I_y|\ y \in Y\}$ *of the cocycle* φ *is positively uniformly stable.*

Proof. The first statement follows from Lemma 3.3. To this end it is sufficient to take as F (figuring in Lemma 3.3) the set $M \times N$ and apply Lemma 3.4.

To prove the second statement we take in the quality of F the set $J := \bigcup\{J_y|\ y \in Y\}$, where $J_y = I_y \times \{y\}$, then the family of motions $\{\varphi(t, u, y)|\ (u, y) \in J \subset \mathbb{R}^n_+ \times Y\}$ is uniformly bounded, because $\varphi(t, u, y) \in I_{\sigma(t,y)} \subset \bigcup\{I_y|\ y \in Y\} \subseteq K$ for any $t \geq 0$ and $(u, y) \in J$. By Lemma 3.3 the family of motions $\{\varphi(t, u, y)|\ (u, y) \in J \subset \mathbb{R}^n_+ \times Y\}$ is positively uniformly stable. □

Let $\langle \mathbb{R}^n_+, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ (or shortly φ) be a cocycle under $(Y, \mathbb{T}, \sigma)$ with the fiber $\mathbb{R}^n_+$.

Theorem 3.7. [12] *Assume that (**C1**) and (**C4**) hold and the following conditions are fulfilled:*

(1) for every $u \in \mathbb{R}^n_+$ *and* $y \in Y$ *the semi-trajectory* $\varphi(\mathbb{T}_+, u, y)$ *is relatively compact;*
(2) $\langle \mathbb{R}^n_+, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ *is a monotone cocycle with the fiber* $\mathbb{R}^n_+$ *over dynamical system* $(Y, \mathbb{T}, \sigma)$*;*
(3) there exists a first integral $V \in C^1(\mathbb{R}^n_+, \mathbb{R})$ *for the cocycle* φ *with* $\nabla V(x) \gg 0$ *for any* $x \in \mathbb{R}^n_+$*;*
(4) the point y *is Poisson stable (in the both direction).*

Then

(1) for any x *(*$x = (u, y)$ *and* $X = \mathbb{R}^n_+ \times Y$*) the set* $\omega^y_x = \omega_x \bigcap X_y$ *consists of a single point* $x^* = (u^*, y)$*;*
(2) the point x^* *is comparable by character of recurrence with* y*, i.e.,* $\mathfrak{N}^{+\infty}_y \subseteq \mathfrak{N}^{+\infty}_{x^*}$*;*
(3)

$$\lim_{t\to+\infty} \rho(\pi(t, x), \pi(t, x^*)) = 0\ . \tag{3.15}$$

Theorem 3.8. *Assume that (**C1**)) and (**C4**) hold and the cocycle* $\langle \mathbb{R}^n_+, \varphi, (Y, \mathbb{T}, \sigma)\rangle$

1. *is monotone;*
2. *admits a compact global attractor* $\boldsymbol{I} := \{I_y|\ y \in Y\}$*;*

3. *there exists a first integral* $V \in C^1(\mathbb{R}^n_+, \mathbb{R})$ *for the cocycle* φ *with* $\nabla V(x) \gg 0$ *for any* $x \in \mathbb{R}^n_+$ *and*
4. *every point* $y \in Y$ *is positively Poisson stable, i.e.,* $y \in \omega_y$ *for any* $y \in Y$.

Then

(1) *for any* $x \in X$ ($x = (u, y)$ *and* $X = \mathbb{R}^n_+ \times Y$) *the set* $\omega^y_x = \omega_x \bigcap X_y$ *consists of a single point* $x^* = (u^*, y)$;

(2) *the point* x^* *is strongly comparable by character of recurrence with* y, *i.e.,* $\mathfrak{N}^{+\infty}_y \subseteq \mathfrak{N}^{+\infty}_{x^*}$;

(3)
$$\lim_{t\to+\infty} \rho(\pi(t, x), \pi(t, x^*)) = 0 \,;$$

(4) *for any* $y \in Y$ *and* $x \in J_y$ *the point* x *is strongly comparable by character of recurrence with* y.

Proof. The first three statements of Theorem follow from Theorem 3.7. Let now y be an arbitrary point from Y and $x \in J_y := I_y \times \{y\}$.

Since the cocycle φ admits a compact global attractor $\mathbf{I} = \{I_y |\ y \in Y\}$, then every semitrajectory $\varphi(\mathbb{T}_+, u, y)$ ($x := (u, y) \in X = \mathbb{R}^n_+ \times Y$) is relatively compact and $\varphi(\mathbb{T}_+, u, y) \subseteq I = \bigcup\{I_y |\ y \in Y\}$ for any $x = (u, y) \in J = \bigcup\{J_y |\ y \in Y\}$, where $J_y := I_y \times \{y\}$. By Lemma 3.3 the cocycle φ is positively uniformly stable on J. According to Theorem 3.7 the first three statement of Theorem take place. Thus to finish the proof of Theorem it is sufficient to establish the fourth statement of Theorem. Let y be an arbitrary point from Y and $x = (u, y) \in J_y$. By third statement of Theorem there exist a point $x^* \in \omega^y_x = \omega_x \bigcap X_y$ which is strongly comparable and satisfies the condition (3.15). We will show that $x = x^*$. In fact, if we suppose that it is not so, then $x \neq x^*$. By Theorem 3.6 there exists a sequence $\{t_k\} \in \mathfrak{N}^{+\infty}_y$ such that

$$\lim_{k\to\infty} \pi(t_k, x) = x \text{ and } \lim_{k\to\infty} \pi(t_k, x^*) = x^*$$

and, consequently,

$$\rho(x, x^*) = \lim_{k\to\infty} \rho(\pi(t_k, x), \pi(t_k, x^*)). \tag{3.16}$$

On the other hand from (3.15), taking in consideration that $t_k \to +\infty$ as $k \to \infty$, we obtain

$$\lim_{k\to\infty} \rho(\pi(t_k, x), \pi(t_k, x^*)) = 0. \tag{3.17}$$

From (3.16) and (3.17) we obtain $x = x^*$ which contradicts to our assumption. Theorem is completely proved. □

Corollary 3.4. *Under the conditions of Theorem 3.8 if the point* y *is stationary (respectively,* τ*-periodic, quasi-periodic with the spectrum* $\{\nu_1, \nu_2, \ldots, \nu_m\}$*, Bohr almost periodic, almost automorphic, recurrent, Levitan almost periodic, almost recurrent, pseudo recurrent, uniformly Poisson stable, strongly Poisson stable), then*

(1) *every motion* $\varphi(t, u, y)$ *(for any* $y \in Y$ *and* $u \in I_y$*) is so;*

(2) *for any* $x = (u, y) \in \mathbb{R}^n_+ \times Y$ *there exists a point* $u_y \in I_y$ *such that*

 (a) *the motion* $\varphi(t, u_y, y)$ *is stationary (respectively,* τ*-periodic, quasi-periodic with the frequency spectrum* $\{\nu_1, \nu_2, \ldots, \nu_m\}$*, Bohr almost periodic, almost automorphic, recurrent, Levitan almost periodic, almost recurrent, pseudo recurrent, uniformly Poisson stable, strongly Poisson stable);*

 (b)
$$\lim_{t\to+\infty} \|\varphi(t, u, y) - \varphi(t, u_y, y)\| = 0,$$

i.e., $\varphi(t, u, y)$ is asymptotically stationary (respectively, τ-periodic, quasi-periodic with the frequency spectrum $\{\nu_1, \nu_2, \dots, \nu_m\}$, Bohr almost periodic, almost automorphic, recurrent, Levitan almost periodic, almost recurrent, pseudo recurrent, uniformly Poisson stable, strongly Poisson stable).

Proof. This statement follows from the Theorems 3.8, 2.1 and 2.2. $\square$

4. APPLICATIONS

Let X, W be two metric space. Denote by $C(\mathbb{T}\times W, X)$ the space of all continuous mappings $f : \mathbb{T} \times W \mapsto X$ equipped with the compact-open topology and σ be the mapping from $\mathbb{T}\times C(\mathbb{T}\times W, X)$ into $C(\mathbb{T}\times W, X)$ defined by the equality $\sigma(\tau, f) := f_\tau$ for all $\tau \in \mathbb{T}$ and $f \in C(\mathbb{T} \times W, X)$, where f_τ is the τ-translation (shift) of f with respect to variable t, i.e., $f_\tau(t, x) = f(t+\tau, x)$ for all $(t, x) \in \mathbb{T}\times W$. Then [9, Ch.I] the triplet $(C(\mathbb{T}\times W, X), \mathbb{T}, \sigma)$ is a dynamical system on $C(\mathbb{T} \times W, X)$ which is called a *shift dynamical system* (*dynamical system of translations* or *Bebutov's dynamical system*).

Recall that the function $\varphi \in C(\mathbb{T}, \mathbb{R}^n)$ (respectively, $f \in C(\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^n)$) possesses the property (A), if the motion $\sigma(\cdot, \varphi)$ (respectively, $\sigma(\cdot, f)$) generated by the function φ (respectively, f) possesses this property in the dynamical system $(C(\mathbb{T}, \mathbb{R}^n), \mathbb{T}, \sigma)$ (respectively, $(C(\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^n), \mathbb{T}, \sigma)$).

In the quality of the property (A) there can stand stability in the sense of Lagrange (st. L), uniform stability (un. st. $\mathcal{L}^+$) in the sense of Lyapunov, periodicity, almost periodicity, asymptotical almost periodicity and so on.

For example, a function $f \in C(\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^n)$ is called almost periodic (respectively, recurrent etc) in $t \in \mathbb{T}$ uniformly with respect to (w.r.t.) w on every compact subset from $\mathbb{R}^n$, if the motion $\sigma(\cdot, f)$ is almost periodic (respectively, recurrent) in the dynamical system $(C(\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^n), \mathbb{T}, \sigma)$.

4.1. **Nonlinear Differential Equations.**

We will give below an example of a skew-product dynamical system which plays an important role in the study of non-autonomous differential equations.

Example 4.2. Consider the differential equation

$$u' = f(t, u), \tag{4.18}$$

where $f \in C(\mathbb{R} \times \mathbb{R}^d, \mathbb{R}^d)$. Along with the equation (4.18) we consider its H-class [4],[17], [27], [36],[39], i.e., the family of the equations

$$v' = g(t, v), \tag{4.19}$$

where $g \in H(f) = \overline{\{f_\tau : \tau \in \mathbb{R}\}}$ and $f_\tau(t, u) = f(t+\tau, u)$, where the bar indicating closure in the compact-open topology.

Condition (A1). The function $f \in C(\mathbb{R} \times \mathbb{R}^d, \mathbb{R}^d)$ is said to be regular [32] if for every equation (4.19) the conditions of existence, uniqueness and extendability on $\mathbb{R}_+$ are fulfilled.

We will suppose that the function f is regular. Denote by $\varphi(\cdot, v, g)$ the solution of (4.19) passing through the point $v \in \mathbb{R}^d$ for $t = 0$. Then the mapping $\varphi : \mathbb{R}_+ \times \mathbb{R}^d \times H(f) \to \mathbb{R}^d$ satisfies the following conditions (see, for example, [4],[30],[31]):

1) $\varphi(0, v, g) = v$ for all $v \in \mathbb{R}^d$ and $g \in H(f)$;
2) $\varphi(t, \varphi(\tau, v, g), g_\tau) = \varphi(t + \tau, v, g)$ for each $v \in \mathbb{R}^d$, $g \in H(f)$ and $t, \tau \in \mathbb{R}_+$;
3) $\varphi : \mathbb{R}_+ \times E^n \times H(f) \to \mathbb{R}^d$ is continuous.

Denote by $Y := H(f)$ and $(Y, \mathbb{R}, \sigma)$ a dynamical system of translations on Y, induced by the dynamical system of translations $(C(\mathbb{R} \times \mathbb{R}^d, \mathbb{R}^d), \mathbb{R}, \sigma)$. The triple $\langle \mathbb{R}^d, \varphi, (Y, \mathbb{R}, \sigma)\rangle$ is a cocycle over $(Y, \mathbb{R}_+, \sigma)$ with the fiber $\mathbb{R}^d$. Hence, the equation (4.18) generates a cocycle $\langle \mathbb{R}^d, \varphi, (Y, \mathbb{R}, \sigma)\rangle$ and the non-autonomous dynamical system $\langle (X, \mathbb{R}_+, \pi),\ (Y, \mathbb{R}, \sigma), h\rangle$, where $X := \mathbb{R}^d \times Y$, $\pi := (\varphi, \sigma)$ and $h := pr_2 : X \to Y$.

Condition (A2). Equation (4.18) is monotone. This means that the cocycle $\langle \mathbb{R}^n, \varphi, (H(f), \mathbb{R}, \sigma)\rangle$ (or shortly φ) generated by (4.18) is monotone, i.e., if $u, v \in \mathbb{R}^d$ and $u \le v$ then $\varphi(t, u, g) \le \varphi(t, v, g)$ for all $t \ge 0$ and $g \in H(f)$.

Let K be a closed cone in $\mathbb{R}^d$. The dual cone to K is the closed cone K^* in the dual space $(\mathbb{R}^d)^*$ of linear functions on $\mathbb{R}^d$, defined by

$$K^* := \{\lambda \in \left(\mathbb{R}^d\right)^* : \ \langle \lambda, x\rangle \ge 0 \text{ for any } x \in K\},$$

where $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathbb{R}^d$.

Recall [42],[43, ChV] that the function $f \in C(\mathbb{R} \times \mathbb{R}^d, \mathbb{R}^d)$ is said to be quasimonotone if for any $(t, u), (t, v) \in \mathbb{R} \times \mathbb{R}^d$ and $\phi \in \left(\mathbb{R}^d_+\right)^*$ we have: $u \le v$ and $\phi(u) = \phi(v)$ implies $\phi(f(t, u)) \le \phi(f(t, v))$.

Lemma 4.6. [14] *Let $f \in C(\mathbb{R} \times \mathbb{R}^d, \mathbb{R}^d)$ be a regular and quasimonotone function, then the following statements hold:*

(1) if $u \le v$, then $\varphi(t, u, f) \le \varphi(t, v, f)$ for any $t \ge 0$;
(2) any function $g \in H(f)$ is quasimonotone;
(3) $u \le v$ implies $\varphi(t, u, g) \le \varphi(t, v, g)$ for any $t \ge 0$ and $g \in H(f)$;
(4) equation (4.18) is monotone.

Let $f \in C(\mathbb{R} \times \mathbb{R}^d, \mathbb{R}^d)$, $\sigma(t, f)$ be the motion (in the shift dynamical system $(C(\mathbb{R} \times \mathbb{R}^d, \mathbb{R}^d), \mathbb{R}, \sigma)$) generated by f, $u_0 \in \mathbb{R}^d$, $\varphi(t, u_0, f)$ be the solution of equation (4.18), $x_0 := (u_0, f) \in X := \mathbb{R}^d \times H(f)$ and $\pi(t, x_0) := (\varphi(t, u_0, f), \sigma(t, f))$ the motion of skew-product dynamical system $(X, \mathbb{R}_+, \pi)$.

Definition 4.24. A solution $\varphi(t, u_0, f)$ of equation (4.18) is called [8],[36],[39] compatible (respectively, strongly compatible or uniformly compatible) if the motion $\pi(t, x_0)$ is comparable (respectively, strongly comparable or uniformly comparable) by character of recurrence with $\sigma(t, f)$.

Definition 4.25. A function f is said to be Poisson stable (respectively, strongly Poisson stable) in $t \in \mathbb{T}$ uniformly with respect to u on every compact subset of $\mathbb{R}^d$ if the point $f \in C(\mathbb{T} \times \mathbb{R}^d, \mathbb{R}^d)$ is Poisson stable (respectively, strongly Poisson stable) in shift dynamical system $(C(\mathbb{T} \times \mathbb{R}^d, \mathbb{R}^d), \mathbb{T}, \sigma)$.

Theorem 4.9. *Suppose that the following assumptions are fulfilled:*

- *the function $f \in C(\mathbb{R} \times \mathbb{R}^d, \mathbb{R}^d)$ is strongly Poisson stable in $t \in \mathbb{R}$ uniformly with respect to u on every compact subset from $\mathbb{R}^n$;*
- *the cocycle φ, generated by equation (4.18), admits a compact global attractor and $\boldsymbol{I} := \{I_g |\ g \in H(f)\}$ is its Levinson center;*
- *the cocycle φ is monotone;*
- *equation (4.18) has a first integral $V \in C^1(\mathbb{R}^n, R)$ such that $\nabla V(x) \gg 0$.*

Then under condition $(A1) - (A2)$ the following statement hold:

(1) for any $g \in H(f)$ and $v \in \mathbb{R}^n$ there exists a point $p_{(v,g)} \in I_g$ such that the solution $\varphi(t, p_{(v,g)}, g)$ of equation (4.19) is strongly compatible and

$$\lim_{t\to\infty} \|\varphi(t, v, g) - \varphi(t, p_{(v,g)}, g)\| = 0;$$

(2) *for any $g \in H(f)$ and $v \in I_g$ the solution $\varphi(t, v, g)$ of equation (4.19) is strongly compatible;*

(3) *if the function g is stationary (respectively, τ-periodic, quasi-periodic with the frequency spectrum $\{\nu_1, \nu_2, \dots, \nu_m\}$, Bohr almost periodic, almost automorphic, recurrent, Levitan almost periodic, almost recurrent, pseudo recurrent, uniformly Poisson stable, strongly Poisson stable), then every solution $\varphi(t, v, g)$ ($v \in I_g$) of equation (4.19) is so.*

Proof. Let $f \in C(\mathbb{R} \times \mathbb{R}^d, \mathbb{R}^d)$ and $(C(\mathbb{R} \times \mathbb{R}^d, \mathbb{R}^n d), \mathbb{R}, \sigma)$ be the shift dynamical system no $C(\mathbb{R} \times \mathbb{R}^d, \mathbb{R}^d)$. Denote by $Y := H(f)$ and $(Y, \mathbb{R}, \sigma)$ the shift dynamical system on $H(f)$ induced by $(C(\mathbb{R} \times \mathbb{R}^d, \mathbb{R}^d), \mathbb{R}, \sigma)$. Consider the cocycle $\langle \mathbb{R}^d, \varphi, (Y, \mathbb{R}, \sigma)\rangle$ generated by equation (4.18) (see Condition (A1)). Now to finish the proof of Theorem it is sufficient to apply Theorem 3.5 and Corollary 3.3. Theorem is proved. □

4.2. **Linear Differential Equations.** Let $A(t) = (a_{ij}(t))_{i,j=1}^n$ ($t \in \mathbb{R}$) be a matrix, satisfying the following **Condition (L1)**

$$a_{ij}(t) \geq 0 \ \text{ and } \ \sum_{i=1}^n a_{ij}(t) = 0$$

for any $i, j = 1, \dots, n$ and $t \in \mathbb{R}$.

Let $[\mathbb{R}^n]$ be the family of all matrices $A = (a_{ij})_{ij=1}^n$ with real coefficients $a_{ij} \in \mathbb{R}$ and $C(\mathbb{R}, [\mathbb{R}^n])$ be the space of all matrix-functions $A(t) = (a_{ij}(t))_{i,j=1}^n$ equipped with the distance

$$d(A, B) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{d_k(A, B)}{1 + d_k(A, B)},$$

where $d_k(A, B) := \max\{||A(t) - B(t)|| : \ |t| \leq k\}$. Denote by $(C(\mathbb{R}, [\mathbb{R}^n]), \mathbb{R}, \sigma)$ the shift dynamical system on $C(\mathbb{R}, [\mathbb{R}^n])$, i.e., $\sigma(a, \tau) = A_\tau$ and $A_\tau(t) := A(t + \tau)$ for any $t, \tau \in \mathbb{R}$ and $A \in C(\mathbb{R}, [\mathbb{R}^n])$ and by $H(A)$ denote the closure of the set $\{A_\tau | \ \tau \in \mathbb{R}\}$ in the space $C(\mathbb{R}, [\mathbb{R}^n])$.

Remark 4.7. If the matrix $A \in C(\mathbb{R}, [\mathbb{R}^n])$ satisfies Condition (**L1**), then every matrix $B \in H(A)$ satisfies Condition (**L1**).

Consider the differential equation

$$x'(t) = A(t)x(t) \tag{4.20}$$

and its H-class

$$y'(t) = B(t)y(t) \ \ (B \in H(A)). \tag{4.21}$$

Lemma 4.7. [12] *Suppose that the matrix $A \in C(\mathbb{R}, [\mathbb{R}^n])$ satisfies condition (**L1**). Then the function $V : \mathbb{R}^n_+ \to \mathbb{R}$ defined by equality*

$$V(x_1, x_2, \dots, x_n) = x_1 + x_2 + \dots + x_n \tag{4.22}$$

is a first integral for equation (4.20).

Corollary 4.5. *The set $M := \{x \in \mathbb{R}^n_+ | \ \sum_{i=1}^n x_i = 1\}$ is invariant with respect to cocycle $\langle \mathbb{R}^n_+, \varphi, (H(A), \mathbb{R}, \sigma)\rangle$.*

Proof. Let $v \in M$ and $B \in H(A)$, then by Lemma 4.7 we have

$$1 = \sum_{i=1}^n v_i = \sum_{i=1}^n \varphi_i(t, v, B)$$

for any $t \geq 0$. □

Lemma 4.8. *Let $A_1, A_2 \in C(\mathbb{R}, [\mathbb{R}^n])$. The following statements hold:*

(1) *if $A_1(t) \geq 0$ and $A_2(t) \geq 0$, then $A_1(t) + A_2(t) \geq 0$ and $A_1(t)A_2(t) \geq 0$ for any $t \in \mathbb{R}_+$;*
(2) *if $A(t) \geq 0$ for any $t \in \mathbb{R}_+$, then $\int_0^t A(s)ds \geq 0$ for all $t \in \mathbb{R}_+$;*
(3) *if $A_1(t) \geq A_2(t) \geq 0$ and $B_1(t) \geq B_2(t) \geq 0$, then $A_1(t)B_1(t) \geq A_2(t)B_2(t)$ for any $t \in \mathbb{R}_+$;*
(4) *if $A_k \in C(\mathbb{R}, [\mathbb{R}^n])$ ($k \in \mathbb{N}$), $A_k \to A$ as $k \to \infty$ and $A_k(t) \geq 0$ for any $k \in \mathbb{N}$ and $t \in \mathbb{R}_+$, then $A(t) \geq 0$.*

Proof. The first two and fourth statements are obvious.

The third statement follows from the relation

$$A_1(t)B_1(t) - A_2(t)B_2(t) = (A_1(t) - A_2(t))B_1(t) + A_2(t)(B_1(t) - B_2(t))$$

for all $t \in \mathbb{R}$. □

Let $\varphi(t, u, A)$ be the solution of equation (4.20) passing through the point $u \in \mathbb{R}^n$ at the initial moment and by $U(t, A)$ the Cauchy operator of equation (4.20), i.e., $U(t, A) := \varphi(t, \cdot, A)$ for any $t \in \mathbb{R}$.

Recall (see, for example, [16, Ch.III]) the operator $U(t, A)$ can be constructed as follow

$$U(t, A) = \lim_{k\to\infty} U_k(t, A) \tag{4.23}$$

and the limit above is uniform with respect to t on every compact from $\mathbb{R}$, where

$$\begin{gathered} U_0(t, A) = E, \\ U_1(t, A) = E + \int_0^t A(s)ds, \\ U_2(t, A) = E + \int_0^t A(s)U_1(s, A)ds, \\ \dots\dots\dots\dots\dots\dots\dots, \\ U_k(t, A) = E + \int_0^t A(s)U_{k-1}(s, A)ds \end{gathered}$$

for any $k \in \mathbb{N}$ and $t \in \mathbb{R}$.

Lemma 4.9. *Assume that $A_1, A_2 \in C(\mathbb{R}, [\mathbb{R}^n])$. If $0 \leq A_1(t) \leq A_2(t)$ for any $t \in \mathbb{R}$, then $0 \leq U(t, A_1) \leq U(t, A_2)$ for any $t \in \mathbb{R}_+$.*

Proof. To prove this statement we note that

$$\begin{gathered} U_0(t, A_1) = U_0(t, A_2), \\ 0 \leq U_1(t, A_1) = E + \int_0^t A_1(s)ds \leq E + \int_0^t A_2(s)ds = U_2(t, A_2) \end{gathered}$$

for any $t \in \mathbb{R}_+$. Assume that

$$0 \leq U_k(t, A_1) \leq U_k(t, A_2)$$

for any $t \in \mathbb{R}_+$ and $k = 2, \dots, m$. Taking in consideration Lemma 4.8 and the assumption above we will have

$$\begin{gathered} 0 \leq U_{m+1}(t, A_1) = E + \int_0^t A_1(s)U_m(s, A_1)ds \leq \\ E + \int_0^t A_2(s)U_m(s, A_2)ds = U_{m+1}(t, A_2) \end{gathered}$$

for any $t \in \mathbb{R}_+$ and, consequently, we have

$$0 \leq U_k(t, A_1) \leq U_k(t, A_2) \tag{4.24}$$

for any $k \in \mathbb{N}$ and $t \in \mathbb{R}_+$. Passing to the limit in (4.24) as $k \to \infty$ and taking into account (4.23) we obtain $0 \leq U(t, A_1) \leq U(t, A_2)$ for any $t \in \mathbb{R}_+$. Lemma is proved. □

Corollary 4.6. *Assume that $A \in C(\mathbb{R}, [\mathbb{R}^n])$, $A(t) = (a_{ij}(t))_{i,j=1}^n$ and there are $\alpha_{ij} \in \mathbb{R}_+$ $(i, j = 1, 2, \dots, n)$ such that $a_{ij}(t) \geq \alpha_{ij}$ for any $t \in \mathbb{R}$ and $i, j = 1, , \dots, n$, then*

$$U(t, A) \geq e^{\mathcal{A}t}$$

for any $t \in \mathbb{R}_+$, where $\mathcal{A} := (\alpha_{ij})_{i,j=1}^n$.

Proof. This statement directly follows from Lemma 4.9 because $A(t) \geq \mathcal{A} \geq 0$ for any $t \in \mathbb{R}$ and $U(t, \mathcal{A}) = e^{\mathcal{A}t}$. □

Corollary 4.7. *Assume that $A \in C(\mathbb{R}, [\mathbb{R}^n])$ and $A(t) \geq 0$ for any $t \in \mathbb{R}_+$, then*

(1) $\varphi(t, u, A) = U(t, A)u \geq 0$ for any $t \geq 0$ and $u \in \mathbb{R}^n_+$;
(2) for any $u, v \in \mathbb{R}^n_+$ with $u \leq v$ we have $\varphi(t, u, A) \leq \varphi(t, v, A)$ for any $t \geq 0$.

Proof. From Corollary 4.6 ($\mathcal{A} = 0$) we obtain $U(t, A)u \geq u \geq 0$ for any $t \geq 0$ and $u \in \mathbb{R}^n_+$ because $e^{\mathcal{A}t} = E$ if $\mathcal{A} = 0$.

Let $u, v \in \mathbb{R}^n_+$ with $u \leq v$ and $A(t) \geq 0$ for any $t \geq 0$, then $\varphi(t, v, A) - \varphi(t, u, A) = U(t, A)v - U(t, A)u = U(t, A)(v - u) \geq 0$ for any $t \geq 0$ because $v - u \geq 0$. □

Remark 4.8. Let $A \in C(\mathbb{R}, [\mathbb{R}^n])$ and $A(t) \geq 0$ for any $t \in \mathbb{R}$, then

(1) for every $B \in H(A)$ we have $B(t) \geq 0$ for any $t \in \mathbb{R}$;
(2) for any $u, v \in \mathbb{R}^n_+$ with $v_1 \leq v_2$ we have $\varphi(t, v_1, B) \leq \varphi(t, v_2, B)$ for any $t \geq 0$ and $B \in H(A)$.

Condition (L2). A matrix $A(t) = (a_{ij}(t))_{i,j=1}^n$ satisfies the following conditions:

$$a_{ij}(t) \geq 0$$

for any $t \geq 0$ and $i, j = 1, , \dots, n$ with $i \neq j$.

Remark 4.9. If the matrix-function $A \in C(\mathbb{R}, [\mathbb{R}^n])$ satisfies Condition (**L2**), then every matrix-function $B \in H(A)$ possesses the same property.

Lemma 4.10. [12] *The following statements hold*

(1) if the matrix $A(t) \geq 0$ for any $t \in \mathbb{R}$, then the cocycle φ, generated by equation (4.20), is monotone, i.e., $\varphi(t, u, B) \leq \varphi(t, v, B)$ for any $t \in \mathbb{R}_+$ and $B \in H(A)$ whenever $u \leq v$ $(u, v \in \mathbb{R}^n_+)$;
*(2) if the matrix $A(t)$ satisfies Condition (**L2**), then the cocycle φ is componentwise monotone, i.e., $\varphi_i(t, u, B) < \varphi_i(t, v, B)$ for any $(t, B) \in \mathbb{R}_+ \times H(A)$ whenever $u \leq v$ and $u_i < v_i$ $(i = 1, 2, \dots, n)$.*

Corollary 4.8. [12] *If the matrix $A(t) \geq 0$ for any $t \in \mathbb{R}$, then the cone $\mathbb{R}^n_+$ is positively invariant with respect to cocycle φ, generated by equation (4.20). This means that $\varphi(t, v, B) \in \mathbb{R}^n_+$ for any $t \in \mathbb{R}_+$ whenever $(v, B) \in \mathbb{R}^n_+ \times H(A)$.*

Theorem 4.10. [12] *Assume that $A \in C(\mathbb{R}, [\mathbb{R}^n])$ be a matrix satisfying Conditions (**L1**)–(**L2**) and it is strongly Poisson stable in $t \in \mathbb{R}$.*

Then the following statements hold:

(1) the cone $\mathbb{R}^n_+$ is positively invariant with respect to cocycle $\langle \mathbb{R}^n, \varphi, (H(A), \mathbb{R}, \sigma)\rangle$ (or shortly φ), generated by equation (4.20) and its H-class (4.21);
(2) the cocycle φ is monotone with respect to spacial variable;
(3) the cocycle φ is componentwise monotone;
(4) the function $V : \mathbb{R}^n_+ \to \mathbb{R}$, defined by equality (4.22), is a first integral for non-autonomous dynamical system, generated by equation (4.20);
(5) every solution $\varphi(t, v, B)$ of every equation (4.21) is bounded on $\mathbb{R}_+$ and positively uniformly stable;

(6) *for every solution $\varphi(t, v, B)$ of every equation (4.21) there exists a unique solution $\varphi(t, \bar{v}, B)$ defined and bounded on $\mathbb{R}$;*
(7) *the solution $\varphi(t, \bar{v}, B)$ is strongly compatible and*

$$\lim_{t\to\infty} |\varphi(t, u, B) - \varphi(t, \bar{u}, B)| = 0;$$

(8) *if the matrix-function $A \in C(\mathbb{R}, [\mathbb{R}^n])$ is stationary (respectively, τ-periodic, Bohr almost periodic, almost automorphic, recurrent, strongly Poisson stable) in $t \in \mathbb{R}$, then $\varphi(t, \bar{v}, B)$ is also stationary (respectively, τ-periodic, Bohr almost periodic, almost automorphic, recurrent, strongly Poisson stable) and*
(9) *$\varphi(t, v, B)$ is asymptotically stationary (respectively, asymptotically τ-periodic, asymptotically Bohr almost periodic, asymptotically almost automorphic, asymptotically recurrent, asymptotically strongly Poisson stable).*

Theorem 4.11. *Assume that $A \in C(\mathbb{R}, [\mathbb{R}^n])$ be a matrix satisfying Conditions (**L1**)–(**L2**) and it is strongly Poisson stable in $t \in \mathbb{R}$.*

Then the following statements hold:

(1) *the set $M := \{x \in \mathbb{R}^n_+ |\ \sum_{i=1}^n x_i = 1\}$ is positively invariant with respect to cocycle $\langle \mathbb{R}^n, \varphi, (H(A), \mathbb{R}, \sigma)\rangle$ and, consequently, it is defined the cocycle $\langle M, \varphi, (H(A), \mathbb{R}, \sigma)\rangle$;*
(2) *the cocycle $\langle M, \varphi, (H(A), \mathbb{R}, \sigma)\rangle$ is compactly dissipative and it admits a compact global attractor $\textbf{I} = \{I_B |\ B \in H(A)\}$;*
(3) *for any $(v, B) \in X := M \times H(A)$ there exists a point $p_v \in I_B$ such that the solution $\varphi(t, p_v, B)$ defined and bounded on $\mathbb{R}$;*
(4) *the solution $\varphi(t, p_v, B)$ is strongly compatible and*

$$\lim_{t\to\infty} |\varphi(t, v, B) - \varphi(t, p_v, B)| = 0;$$

(5) *for any $B \in H(A)$ and $v \in I_B$ the solution $\varphi(t, v, B)$ is strongly compatible;*
(6) *if the matrix-function $A \in C(\mathbb{R}, [\mathbb{R}^n])$ is stationary (respectively, τ-periodic, Bohr almost periodic, almost automorphic, recurrent, strongly Poisson stable) in $t \in \mathbb{R}$, then for any $B \in H(A)$ and $v \in I_B$ the solution $\varphi(t, v, B)$ is also stationary (respectively, τ-periodic, quasi-periodic with the spectrum $\{\nu_1, \nu_2, \ldots, \nu_m\}$, Bohr almost periodic, almost automorphic, recurrent, Levitan almost periodic, almost recurrent, pseudo recurrent, uniformly Poisson stable, strongly Poisson stable) and*
(7) *if $B \in H(A)$ and $v \in M \setminus I_B$, then the solution $\varphi(t, v, B)$ is asymptotically stationary (respectively, τ-periodic, quasi-periodic with the spectrum $\{\nu_1, \nu_2, \ldots, \nu_m\}$, Bohr almost periodic, almost automorphic, recurrent, Levitan almost periodic, almost recurrent, pseudo recurrent, uniformly Poisson stable, strongly Poisson stable).*

Proof. The first statement follows from Corollary 4.5.

The second statement follows from the compactness of M and Theorem 2.3.

The statements (iii)-(iv) and (vi)-(vii) follow from Theorem 4.10.

The statement (v) follows from Theorem 3.8 and Corollary 3.4. □

Remark 4.10. Under the conditions of Theorem 4.10 for any $B \in H(A)$ the subset I_B of M is convex.

This statement follows from the fact that $\varphi(t, \lambda u_1 + (1-\lambda)u_2, B) = \lambda\varphi(t, u_1, B) + (1-\lambda)\varphi(t, u_2, B)$ for any $\lambda \in [0, 1]$

A matrix $\mathcal{A} \in [\mathbb{R}^n]$ satisfies **Condition** (**L3**) if the following take hold

(1) $\mathcal{A} \geq 0$, i.e., $\alpha_{ij} \geq 0$ for any $i, j = 1, 2, \ldots, n$;
(2) $\alpha_{ij} > 0$ for any $i \neq j$ and $i, j = 1, 2, \ldots, n$.

Note that

$$e^X = E + X + \frac{X^2}{2!} + \ldots + \frac{X^n}{n!} + \ldots = E + X + R(X), \tag{4.25}$$

where

$$R(X) = \frac{X^2}{2!} + \ldots + \frac{X^n}{n!} + \ldots \tag{4.26}$$

for any $X \in [\mathbb{R}^n]$ and $E \in [\mathbb{R}^n]$ is the unite matrix. From (4.25) and (4.26) it follows that

(1)
$$\|R(X)\| \leq e^{\|X\|} - 1 - \|X\| = o(\|X\|)\|X\|$$
as $\|X\| \to 0$;

(2) the corresponding elements of the matrices e^X and $E + X$ have the same signs, if the norm $\|X\|$ is sufficiently small.

Lemma 4.11. *Assume that the matrix $\mathcal{A}$ satisfies Condition (**L3**), then the matrix $B := e^{\mathcal{A}} = (b_{ij})_{i,j=1}^n$ is positive, i.e., $b_{ij} > 0$ for any $i, j = 1, 2, \ldots, n$.*

Proof. Let now $k \in \mathbb{N}$ be a naturale number and $B_k := e^{\mathcal{A}/k}$. Since

$$B_k = E + \mathcal{A}/k + \ldots,$$

then for sufficiently large k the matrix B_k is positive. Taking into consideration that

$$B = (B_k)^k$$

for any $k \in \mathbb{N}$ we conclude that the matrix B is also positive. Lemma is proved. □

Denote by $Int(M)$ the interior of the set M.

Lemma 4.12. *Let $A(t) \geq \mathcal{A}$ for any $t \in \mathbb{R}$. If the matrix $\mathcal{A}$ satisfies Condition (**L3**), then $\varphi(t, \tilde{M}, B) \subset Int(M)$ for any subset $\tilde{M} \subseteq M$, $B \in H(A)$ and $t > 0$.*

Proof. To prove the first statement we note that $\varphi(t, v, B) = U(t, B)v$. Since the matrix $\mathcal{A}$ satisfies Condition (**L3**) with the constants $\alpha_{ij} \geq 0$ $(i, j = 1, , \ldots, n)$, then by Remark 4.9 every matrix $B \in H(A)$ satisfies the same condition with the same constants α_{ij}. According to Corollary 4.6 we have

$$U(t, B) \geq e^{\mathcal{A}t} \tag{4.27}$$

for any $t \geq 0$ and $B \in H(A)$, where $\mathcal{A} = (\alpha_{ij})_{i,j=1}^n$. Let $\tilde{M}$ be a subset of M and $v \in \tilde{M}$. We will show that $U(t, B)v \in Int(M)$ for any $t > 0$. To this end it is sufficient to prove that every component of the vector $U(t, B)v$ is positive. Since $v \in \tilde{M} \subseteq M$, then at least one component v_{j_0} of the vector v is positive. Then by (4.27) for any $i = 1, 2, \ldots, n$ we obtain

$$(U(t, B)v)_i \geq ((e^{\mathcal{A}t})v)_i \geq (e^{\mathcal{A}t})_{ij_0} v_{j_0} > 0 \tag{4.28}$$

for any $t > 0$ because the matrix $e^{\mathcal{A}t}$ (by Lemma 4.11) is positive, where by $(v)_i$ is denoted the i-th component of the vector $v \in \mathbb{R}^n_+$ and $(A)_{ij}$ the element of the matrix A located on the place with the index ij. This means that $U(t, B)\tilde{M} \subset Int(M)$ for any $t > 0$ and $B \in H(A)$. Lemma is proved. □

A matrix-function $A \in C(\mathbb{R}, [\mathbb{R}^n])$ satisfies **Condition (L4)** if there exists a positive matrix $\mathcal{A} = (\alpha_{ij})_{i,j=1}^n$ (i.e., $\alpha_{ij} > 0$ for any $i, j = 1, 2, \ldots, n$) such that $A(t) \geq \mathcal{A}$ for any $t \in \mathbb{R}$.

Let E be a real Banach space with a nonempty closed convex cone P such that $P\bigcap(-P) = \{0\}$ and $Int(P) \neq \emptyset$. For $u_1, u_2 \in E$ we will write $u_1 \leq u_2$ (respectively, $u_1 < u_2$ or $u_1 \ll u_2$) if $u_2 - u_1 \in P$ (respectively, $u_2 - u_1 \in P$ and $u_1 \neq u_2$ or $u_2 - u_1 \in Int(P)$).

We define the part (Birkhoff) metric on $Int(P)$ by

$$p(u_1, u_2) := \inf\{\ln\alpha|\ \alpha \geq 1, \text{ and } \alpha^{-1}u_1 \leq u_2 \leq \alpha u_1\}.$$

Then $(Int(P), p)$ is a metric space (see, for example, [15, 25, 29] and the bibliography therein).

Theorem 4.12. *Assume that $A \in C(\mathbb{R}, [\mathbb{R}^n])$ be a stochastic matrix satisfying Condition (**L4**) and it is strongly Poisson stable in $t \in \mathbb{R}$.*

Then I_B consists of a single point $\{v_B\}$ for any $B \in H(A)$.

Proof. Suppose that this statement is not true, then there exists at least one $B_0 \in H(A)$ such that I_{B_0} contains at least two different points $v_1, v_2 \in I_{B_0}$. Denote by $\phi(t) := p(\varphi(t, v_1, B_0), \varphi(t, v_2, B_0))$ for any $t \in \mathbb{R}$, where p is the part metric on $Int(\mathbb{R}^n_+)$. Note that the function $\phi : \mathbb{R} \to \mathbb{R}_+$ possesses the following properties:

(1) $\phi(t) > 0$ for any $t \in \mathbb{R}$;
(2) it is bounded on $\mathbb{R}$;
(3) it is Poisson stable in the both direction (see Theorem 4.11), i.e., there are the sequences $\{t^{\pm}_k\} \to \pm\infty$ as $k \to \infty$ such that $\phi(t + t^{\pm}_k) \to \phi(t)$ as $k \to \infty$ in the space $C(\mathbb{R}, \mathbb{R}_+)$;
(4) ϕ is monotone increasing.

The first two statements are evident. To prove the third statement we note that the cocycle $\langle \mathbb{R}^n_+, \varphi, (H(A), \mathbb{R}, \sigma)\rangle$, generated by equation (4.20), is monotone (order-preserving) and homogeneous (this means that $\varphi(t, \lambda v, B) = \lambda\varphi(t, v, B)$ for any $\lambda > 0$, $t \in \mathbb{R}+$, $v \in \mathbb{R}^n_+$ and $B \in H(A)$). Let $B \in H(A)$, $v_1, v_2 \in I_B$ with $v_1 \neq v_2$ and $p(v_1, v_2) = \ln\alpha > 0$, then $\alpha > 1$ and $\alpha^{-1} \in (0, 1)$. Since

$$\alpha^{-1}v_1 \leq v_2 \leq \alpha v_1 \tag{4.29}$$

and φ is monotone and homogeneous we have

$$\alpha^{-1}\varphi(t, v_1, B) \leq \varphi(t, v_2, B) \leq \alpha\varphi(t, v_1, B) \tag{4.30}$$

for any $t \geq 0$. From (4.30) it follows that

$$p(\varphi(t, v_1, B), \varphi(t, v_2, B)) \leq p(v_1, v_2)$$

for any $t \geq 0$, $B \in H(A)$ and $v_1, v_2 \in I_B$ with $v_1 \neq v_2$. Let now $t_1, t_2 \in \mathbb{R}$ with $t_2 \geq t_1$, then

$$\begin{gathered}\phi(t_2) = p(\varphi(t_2, v_1, B_0), \varphi(t_2, v_2, B_0)) = \\ p(\varphi(t_2 - t_1, \varphi(t, v_1, B_0), \sigma(t_1, B_0)), \varphi(t_2 - t_1, \varphi(t, v_2, B_0), \sigma(t_1, B_0))) \leq \\ = p(\varphi(t_1, v_1, B_0), \varphi(t_1, v_2, B_0)) = \phi(t_1)\end{gathered}$$

because $v_1 \neq v_2$ ($v_1, v_2 \in I_{B_0}$) and, consequently, $\varphi(t, v_1, B_0) \neq \varphi(t, v_2, B_0)$ ($\varphi(t, v_i, B_0) \in I_{\sigma(t,B_0)}$, $i = 1, 2$) for any $t \in \mathbb{R}$.

Taking in consideration that the function ϕ is bounded and decreasing we conclude that there exist constants $c_-, c_+ \in \mathbb{R}_+$ such that

$$\lim_{t\to\pm\infty} \phi(t) = c_{\pm}. \tag{4.31}$$

Since ϕ is Poisson stable in the both directions, then $\phi \in \omega_\phi = \alpha_\phi$, where ω_ϕ (respectively, α_ϕ) is the ω (respectively, α) limit set of ϕ in the shift dynamical system $(C(\mathbb{R}, \mathbb{R}_+),$

$\mathbb{R}$, σ). On the other hand by (4.31) we have $\omega_\phi = \{c_+\}$ (respectively $\alpha_\phi = \{c_-\}$ and, consequently, $c_- = \phi(t) = c_+$ for any $t \in \mathbb{R}$. From this it follows that

$$p(\varphi(t, v_1, B_0), \varphi(t, v_2, B_0)) = p(v_1, v_2) \tag{4.32}$$

for any $t \in \mathbb{R}$.

Further, let $\alpha > 1$ as above. From (4.29) it follows that the relations $\alpha^{-1}v_1 \neq v_2$ and $v_2 \neq \alpha v_1$ take place, because if we suppose that at least one of them is not true then (for example)

$$\alpha^{-1}v_1 = v_2. \tag{4.33}$$

Since $v_1, v_2 \in M$, then from (4.33) we obtain $\alpha = 1$ which contradicts to the choice of the number α.

Thus, we have $\alpha^{-1}v_1 < v_2 < \alpha v_1$, then from (4.29), the last inequality and (4.28) we receive

$$\alpha^{-1}\varphi(t, v_1, B_0) \ll \varphi(t, v_2, B_0) \ll \alpha\varphi(t, v_1, B_0).$$

We fix a positive number $t_0 \in \mathbb{R}$. Since $Int(\mathbb{R}^n_+)$ is an open subset of $\mathbb{R}^n$, then we can take an $\tilde{\alpha} \in (1, \alpha)$ sufficiently close to α such that

$$\alpha_1^{-1}\varphi(t_0, v_1, B_0) \ll \varphi(t_0, v_2, B_0) \ll \alpha_1\varphi(t_0, v_1, B_0).$$

From the last inequality it follows that

$$p(\varphi(t_0, v_1, B_0), \varphi(t_0, v_2, B_0)) \le \ln\alpha_1 < \ln\alpha = p(v_1, v_2),$$

i.e., $\phi(t_0) < \phi(0)$. The last inequality contradicts to relation (4.32). The obtained contradiction proves our statement, i.e., for any $B \in H(A)$ the set I_B consists a single point. □

Corollary 4.9. *Under the conditions of Theorem 4.12 if the matrix-function $A \in C(\mathbb{R}, [\mathbb{R}^n])$ is stationary (respectively, τ-periodic, quasi-periodic with the frequency spectrum $\{\nu_1, \nu_2, \ldots, \nu_m\}$, Bohr almost periodic, almost automorphic, recurrent, Levitan almost periodic, almost recurrent, pseudo recurrent and Lagrange stable, pseudo periodic and Lagrange stable, strongly Poisson stable), then equation (4.20) has a unique bounded on $\mathbb{R}$ solution $p(t)$ with the values from $Int(M)$ ($p(\mathbb{R}) \subset Int(M)$) possessing the following properties:*

1. *$p(t)$ is stationary (respectively, τ-periodic, quasi-periodic with the frequency spectrum $\{\nu_1, \nu_2, \ldots, \nu_m\}$, Bohr almost periodic, almost automorphic, recurrent, Levitan almost periodic, almost recurrent, pseudo recurrent and Lagrange stable, pseudo periodic and Lagrange stable, strongly Poisson stable);*
2. *the solution p is uniformly globally asymptotically stable with respect to M, i.e.,*
   - (a) *for arbitrary positive number ε there exists a positive number $\delta = \delta(\varepsilon)$ such that $\|\varphi(t_0, u, A) - p(t_0)\| < \delta$ ($u \in M$) implies $\|\varphi(t, u, A) - p(t)\| < \varepsilon$ for any $t \ge t_0$;*
   - (b) *$\lim_{t\to+\infty} \|\varphi(t, u, A) - p(t)\| = 0$ for every $u \in M$.*

Proof. This statement follows directly from Theorems 4.12 and 4.10. □

4.3. **Difference Equations.** Consider the difference equation

$$u_{n+1} = f(n, u_n), \tag{4.34}$$

where $f \in C(\mathbb{Z} \times \mathbb{R}^d, \mathbb{R}^d)$. Along with equation (4.34) we will consider H-class of (4.34), i.e., the family of equations

$$v_{n+1} = g(n, v_n), \ (g \in H(f)) \tag{4.35}$$

where $H(f) := \overline{\{f_\tau | \ \tau \in \mathbb{Z}\}}$ and by bar is denoted the closure in the space $C(\mathbb{Z} \times \mathbb{R}^d, \mathbb{R}^d)$.

Denote by $\varphi(n, v, g)$ the solution of equation (4.35) with initial condition $\varphi(0, v, g) = v$. From the general properties of difference equations it follows that:

(1) $\varphi(0, v, g) = v$ for all $v \in \mathbb{R}^d$ and $g \in H(f)$;
(2) $\varphi(n+m, v, g) = \varphi(n, \varphi(m, v, g), \sigma(m, g))$ for all $n, m \in \mathbb{Z}_+$ and $(v, g) \in \mathbb{R}^d \times H(f)$;
(3) the mapping φ is continuous.

Thus every equation (4.34) generate a cocycle $\langle \mathbb{R}^d, \varphi, (H(f), \mathbb{Z}, \sigma)\rangle$ over $(H(f), \mathbb{Z}, \sigma)$ with the fiber $\mathbb{R}^d$.

Lemma 4.13. [10] *Let $f \in C(\mathbb{Z} \times \mathbb{R}^d, \mathbb{R}^d)$. Suppose that the following conditions hold:*

(1) $u_1, u_2 \in \mathbb{R}^d$ and $u_1 \le u_2$;
(2) the function f is monotone non-decreasing with respect to variable $u \in \mathbb{R}^d$, i.e., $f(t, u_1) \le f(t, u_2)$ for any $t \in \mathbb{Z}$.

Then $\varphi(n, v_1, g) \le \varphi(n, v_2, g)$ for any $n \in Z_+$, $v_1, v_2 \in \mathbb{R}^d$ with $v_1 \le v_2$ and $g \in H(f)$.

Condition (D1). Equation (4.34) is monotone. This means that the cocycle $\langle \mathbb{R}^d, \varphi, (H(f), \mathbb{Z}, \sigma)\rangle$ (or shortly φ) generated by (4.34) is monotone, i.e., if $u, v \in \mathbb{R}^n$ and $u \le v$ then $\varphi(t, u, g) \le \varphi(t, v, g)$ for all $t \ge 0$ and $g \in H(f)$.

Let $f \in C(\mathbb{Z} \times \mathbb{R}^d, \mathbb{R}^d)$, $\sigma(t, f)$ be the motion (in the shift dynamical system $(C(\mathbb{Z} \times \mathbb{R}^d, \mathbb{R}^d), \mathbb{Z}, \sigma)$) generated by f, $u_0 \in \mathbb{R}^n$, $\varphi(t, u_0, f)$ be the solution of equation (4.34), $x_0 := (u_0, f) \in X := \mathbb{R}^d \times H(f)$ and $\pi(t, x_0) := (\varphi(t, u_0, f), \sigma(t, f))$ the motion of skew-product dynamical system $(X, \mathbb{Z}_+, \pi)$.

Definition 4.26. A solution $\varphi(t, u_0, f)$ of equation (4.34) is called [8],[36],[39] compatible (respectively, strongly compatible or uniformly compatible) if the motion $\pi(t, x_0)$ is comparable (respectively, strongly comparable or uniformly comparable) by character of recurrence with $\sigma(t, f)$.

Theorem 4.13. *Suppose that the following assumptions are fulfilled:*

- *the function $f \in C(\mathbb{Z} \times \mathbb{R}^d, \mathbb{R}^d)$ is strongly Poisson stable in $t \in \mathbb{Z}$ uniformly with respect to u on every compact subset from $\mathbb{R}^n$;*
- *the cocycle φ, generated by equation (4.34), admits a compact global attractor and $\boldsymbol{I} := \{I_g |\ g \in H(f)\}$ is its Levinson center;*
- *the cocycle φ is monotone;*
- *equation (4.34) has a first integral $V \in C^1(\mathbb{R}^n, R)$ such that $\nabla V(x) \gg 0$.*

*Then under condition **(D1)** the following statements hold:*

(1) for any $g \in H(f)$ and $v \in \mathbb{R}^n$ there exists a point $p_{(v,g)} \in I_g$ such that the solution $\varphi(t, p_{(v,g)}, g)$ of equation (4.35) is strongly compatible and

$$\lim_{t\to\infty} \|\varphi(t, v, g) - \varphi(t, p_{(v,g)}, g)\| = 0;$$

(2) for any $g \in H(f)$ and $v \in I_g$ the solution $\varphi(t, v, g)$ of equation (4.35) is strongly compatible;
(3) if the function g is stationary (respectively, τ-periodic, Bohr almost periodic, almost automorphic, recurrent, pseudo recurrent and Lagrange stable, pseudo periodic and stable in the sense of Lagrange), then every solution $\varphi(t, v, g)$ ($v \in I_g$) of equation (4.35) is so.

Proof. Let $f \in C(\mathbb{Z} \times \mathbb{R}^d, \mathbb{R}^d)$ and $(C(\mathbb{Z} \times \mathbb{R}^d, \mathbb{R}^d), \mathbb{Z}, \sigma)$ be the shift dynamical system no $C(\mathbb{Z} \times \mathbb{R}^d, \mathbb{R}^d)$. Denote by $Y := H(f)$ and $(Y, \mathbb{Z}, \sigma)$ the shift dynamical system on $H(f)$ induced by $(C(\mathbb{Z} \times \mathbb{R}^d, \mathbb{R}^d), \mathbb{Z}, \sigma)$. Consider the cocycle $\langle \mathbb{R}^n, \varphi, (Y, \mathbb{Z}, \sigma)\rangle$ generated by equation (4.34). Now to finish the proof of Theorem it is sufficient to apply Theorem 3.8 and Corollary 3.4. Theorem is proved. □

4.4. **Linear Difference Equations.** Let $A(t) = (a_{ij}(t))_{i,j=1}^n$ $(t \in \mathbb{Z})$ be a stochastic matrix, which means that

$$a_{ij}(t) \geq 0 \ \text{ and } \ \sum_{i=1}^{n} a_{ij}(t) = 1 \tag{4.36}$$

for any $i, j = 1, \dots, n$ and $t \in \mathbb{Z}$. The matrix $A(t)$ can be considered as the transition matrix at the moment $t \in \mathbb{Z}$ for a Markov process acting on a set of n states $\{1, 2, \dots, n\}$.

Let $[\mathbb{R}^n]$ be the family of all matrices $A = (a_{ij})_{ij=1}^n$ with real coefficients $a_{ij} \in \mathbb{R}$ and $C(\mathbb{Z}, [\mathbb{R}^n])$ be the space of all matrix-functions $A(t) = (a_{ij}(t))_{i,j=1}^n$ equipped with the distance

$$d(A, B) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{d_k(A, B)}{1 + d_k(A, B)},$$

where $d_k(A, B) := \max\{||A(t) - B(t)|| : \ |t| \leq k\}$. Denote by $(C(\mathbb{Z}, [\mathbb{R}^n]), \mathbb{Z}, \sigma)$ the shift dynamical system on $C(\mathbb{Z}, [\mathbb{R}^n])$, i.e., $\sigma(a, \tau) = A_\tau$ and $A_\tau(t) := A(t + \tau)$ for any $t, \tau \in \mathbb{Z}$ and $A \in C(\mathbb{Z}, [\mathbb{R}^n])$.

Remark 4.11. If the matrix $A \in C(\mathbb{Z}, [\mathbb{R}^n])$ satisfies condition (4.36), then every matrix $B \in H(A)$ satisfies condition (4.36).

Consider the difference equation

$$x(t + 1) = A(t)x(t) \tag{4.37}$$

and its H-class

$$y(t + 1) = B(t)y(t) \ \ (B \in H(A)). \tag{4.38}$$

Lemma 4.14. [12] *Suppose that the matrix $A \in C(\mathbb{Z}, [\mathbb{R}^n])$ satisfies condition (4.20). Then the function $V : \mathbb{R}^n_+ \to \mathbb{R}$ defined by equality*

$$V(x_1, x_2, \dots, x_n) = x_1 + x_2 + \dots + x_n \tag{4.39}$$

is a first integral for equation (4.20).

Condition (D2). A matrix $A(t) = (a_{ij}(t))_{i,j=1}^n$ satisfies the following condition:

(1) $A(t) \geq 0$ for any $t \in \mathbb{Z}_+$, i.e., $a_{ij}(t) \geq 0$ for any $i, j = 1, 2, \dots, n$ and $t \in \mathbb{Z}_+$;
(2) $a_{ii}(t) \geq \alpha_i > 0$ for any $i = 1, 2, \dots, n$ and $t \in \mathbb{Z}_+$, where α_i $(i = 1, 2, \dots, n)$ are some positive numbers.

Remark 4.12. If the stochastic matrix A satisfies condition (**D2**) with the positive constants $\alpha_1, \alpha_2, \dots, \alpha_n$, then $\alpha_i \in (0, 1]$ for any $i = 1, 2, \dots, n$.

Condition (D3). A matrix $A(t) = (a_{ij}(t))_{i,j=1}^n$ satisfies the following conditions: $A(t) \gg 0$ for any $t \in \mathbb{Z}_+$, i.e., there exist positive number α_{ij} $(i, j = 1, 2, \dots, n)$ such that $a_{ij}(t) \geq \alpha_{ij} > 0$ for any $i, j = 1, 2, \dots, n$ and $t \in \mathbb{Z}$.

Remark 4.13. If the stochastic matrix A satisfies condition (**D3**) with the positive constants α_{ij} $(i, j = 1, 2, \dots, n)$, then $\alpha_{ij} \in (0, 1]$ for any $i, j = 1, 2, \dots, n$.

Lemma 4.15. *1. If the matrix $A(t)$ satisfies Condition (**D3**), then any matrix $B \in H(A)$ satisfies the same condition.*

*2. If the matrices $A'(t)$ and $A''(t)$ satisfy Condition (**D3**) and α'_{ij} (respectively, α''_{ij}) are the constants which figures in Condition (**D3**) for matrix $A'(t)$ (respectively, $A''(t)$), then the matrix $A(t) = A'(t)A''(t)$ possesses the same property with the constant $\alpha_{ij} = \sum_{k=1}^n \alpha'_{ik}\alpha''_{kj}$ $(i, j = 1, 2, \dots, n)$.*

*3. Assume that the matrices $A^k(t)$ ($k = 1, 2, \dots, m$) satisfy Condition (**D3**) with the constants α^k_{ij} ($i, j = 1, 2, \dots, n$), then the matrix $\tilde{A}(t) := \prod_{k=1}^{m} A^k(t)$ satisfies Condition (**D3**) with the constants $\tilde{\alpha}_{ij} \ge (\prod_{k=1}^m \mathcal{A}^k)_{ij}$ ($i, j = 1, 2, \dots, n$), where $\mathcal{A}^k = (\alpha^k_{ij})^n_{i,j=1}$, $k = 1, 2, \dots, m$ and $(\prod_{k=1}^m \mathcal{A}^k)_{ij}$ is the element of the matrix $\prod_{k=1}^m \mathcal{A}^k$ on the place with the index ij.*

Proof. Assume that the matrix $A(t)$ satisfies Condition (**D3**), then there are positive constants $\alpha_{ij}(i, j = 1, 2, \dots, n)$ such that

$$a_{ij}(t) \ge \alpha_{ij} \tag{4.40}$$

for any $t \in \mathbb{Z}$ and $i, j = 1, 2, \dots, n$. If $B \in H(A)$, then there exists a sequence $\{h_k\} \subseteq \mathbb{Z}$ such that $B(t) = \lim_{k\to\infty} A(t + h_k)$ and, consequently, $b_{ij}(t) = \lim_{k\to\infty} a_{ij}(t + h_k)$. From (4.40) we have

$$a_{ij}(t + h_k) \ge \alpha_{ij} \tag{4.41}$$

for any $k \in \mathbb{N}$, $t \in \mathbb{Z}$ and $i, j = 1, 2, \dots, n$. Passing to the limit in inequality (4.41) as $k \to \infty$ we obtain

$$b_{ii}(t) \ge \alpha_{ij}.$$

Let $A'(t) = (a^{'}_{ij})^n_{i,j=1}$ and $A^{''}(t) = (a^{''}_{ij})^n_{i,j=1}$ be two matrices satisfying Condition (**D3**) and $\alpha^{'}_{ij}$ (respectively, $\alpha^{''}_{ij}$) be the constants which figures in Condition (**D3**) for matrix $A'(t)$ (respectively, $A^{''}(t)$) and let $(a_{ij}(t))^n_{i,j=1} = A(t) = A^{'}(t)A^{''}(t)$. Since

$$a_{ij}(t) = \sum_{k=1}^{n} a^{'}_{ik}(t)a^{''}_{kj}(t),$$

then we have

$$a_{ij}(t) = \sum_{k=1}^{n} a^{'}_{ik}(t)a^{''}_{kj}(t) \ge \sum_{k=1}^{m} \alpha^{'}_{ik}\alpha^{''}_{kj}$$

for any $t \in \mathbb{Z}$ and $i, k = 1, 2, \dots, n$ and, consequently, $A'(t)A^{''}(t) \ge \mathcal{A}'\mathcal{A}^{''}$ for any $t \in \mathbb{Z}$.

Finally, the third statement follows from the second one in combination with the method of mathematical induction. Lemma is completely proved. □

Lemma 4.16. [12] *The following statement hold:*

1. *if the matrix $A(t) \ge 0$ for any $t \in \mathbb{Z}_+$, then the cocycle φ, generated by equation (4.20), is monotone, i.e., $\varphi(t, u, B) \le \varphi(t, v, B)$ for any $t \in \mathbb{Z}_+$ and $B \in H(A)$ whenever $u \le v$ ($u, v \in \mathbb{R}^n_+$);*
2. *if the matrix $A(t)$ satisfies Condition (**D2**), then the cocycle φ is componentwise monotone, i.e., $\varphi_i(t, u, B) < \varphi_i(t, v, B)$ for any $(t, B) \in \mathbb{Z}_+ \times H(A)$ whenever $u \le v$ and $u_i < v_i$ ($i = 1, 2, \dots, n$).*

Corollary 4.10. [12] *If the matrix $A(t) \ge 0$ for any $t \in \mathbb{Z}_+$, then the cone $\mathbb{R}^n_+$ is positively invariant with respect to cocycle φ, generated by equation (4.20). This means that $\varphi(t, v, B) \in \mathbb{R}^n_+$ for any $t \in \mathbb{Z}_+$ whenever $(v, B) \in \mathbb{R}^n_+ \times H(A)$.*

Theorem 4.14. [12] *Assume that $A \in C(\mathbb{Z}, [\mathbb{R}^n])$ is a stochastic matrix possessing the property (**D2**) and it is strongly Poisson stable in $t \in \mathbb{Z}$.*

Then the following statements hold:

1. *the cone $\mathbb{R}^n_+$ is positively invariant with respect to cocycle $\langle \mathbb{R}^n, \varphi, (H(A), \mathbb{Z}, \sigma)\rangle$ (or shortly φ), generated by equation (4.20) and its H-class (4.21);*
2. *the cocycle φ is monotone with respect to spacial variable;*
3. *the cocycle φ is componentwise monotone;*

(4) *the function $V : \mathbb{R}^n_+ \to \mathbb{R}$, defined by equality (4.22), is a first integral for non-autonomous dynamical system, generated by equation (4.20);*
(5) *every solution $\varphi(t, v, B)$ of every equation (4.21) is bounded on $\mathbb{Z}_+$ and positively uniformly stable;*
(6) *for every solution $\varphi(t, v, B)$ of every equation (4.21) there exists a unique solution $\varphi(t, \bar{v}, B)$ defined and bounded on $\mathbb{Z}$;*
(7) *the solution $\varphi(t, \bar{v}, B)$ is strongly compatible and*
$$\lim_{t\to\infty} |\varphi(t, u, B) - \varphi(t, \bar{u}, B)| = 0;$$
(8) *if the matrix-function $A \in C(\mathbb{Z}, [\mathbb{R}^n])$ is stationary (respectively, τ-periodic, Bohr almost periodic, almost automorphic, recurrent, strongly Poisson stable) in $t \in \mathbb{Z}$, then $\varphi(t, \bar{v}, B)$ is also stationary (respectively, τ-periodic, Bohr almost periodic, almost automorphic, recurrent, strongly Poisson stable) and*
(9) *$\varphi(t, v, B)$ is asymptotically stationary (respectively, asymptotically τ-periodic, asymptotically Bohr almost periodic, asymptotically almost automorphic, asymptotically recurrent, asymptotically strongly Poisson stable).*

Theorem 4.15. *Assume that $A \in C(\mathbb{Z}, [\mathbb{R}^n])$ is a stochastic matrix possessing the property (**D2**) and it is strongly Poisson stable in $t \in \mathbb{Z}$.*

Then the following statements hold:

(1) *the set $M := \{x \in \mathbb{R}^n_+ |\ \sum_{i=1}^n x_i = 1\}$ is positively invariant with respect to cocycle $\langle \mathbb{R}^n, \varphi, (H(A), \mathbb{Z}, \sigma)\rangle$ and, consequently, it is defined the cocycle $\langle M, \varphi, (H(A), \mathbb{Z}, \sigma)\rangle$;*
(2) *the cocycle $\langle M, \varphi, (H(A), \mathbb{Z}, \sigma)\rangle$ is compactly dissipative and it admits a compact global attractor $\boldsymbol{I} = \{I_B |\ B \in H(A)\}$;*
(3) *for any $(v, B) \in X := M \times H(A)$ there exists a point $p_v \in I_B$ such that the solution $\varphi(t, p_v, B)$ defined and bounded on $\mathbb{Z}$;*
(4) *the solution $\varphi(t, p_v, B)$ is strongly compatible and*
$$\lim_{t\to\infty} |\varphi(t, v, B) - \varphi(t, p_v, B)| = 0;$$
(5) *for any $B \in H(A)$ and $v \in I_B$ the solution $\varphi(t, v, B)$ is strongly compatible;*
(6) *if the matrix-function $A \in C(\mathbb{Z}, [\mathbb{R}^n])$ is stationary (respectively, τ-periodic, Bohr almost periodic, almost automorphic, recurrent, strongly Poisson stable) in $t \in \mathbb{Z}$, then for any $B \in H(A)$ and $v \in I_B$ the solution $\varphi(t, v, B)$ is also stationary (respectively, τ-periodic, Bohr almost periodic, almost automorphic, recurrent, strongly Poisson stable) and*
(7) *if $B \in H(A)$ and $v \in M \setminus I_B$, then the solution $\varphi(t, v, B)$ is asymptotically stationary (respectively, asymptotically τ-periodic, asymptotically Bohr almost periodic, asymptotically almost automorphic, asymptotically recurrent, asymptotically strongly Poisson stable).*

Proof. The first statement follows from Lemma 4.14.

The second statement follows from the compactness of M and Theorem 2.3.

The statements (iii)-(iv) and (vi)-(vii) follow from Theorem 4.14.

The statement (v) follows from Theorem 3.8 and Corollary 3.4. □

Remark 4.14. Under the conditions of Theorem 4.15 for any $B \in H(A)$ the subset I_B of M is convex.

Proof. This statement follows from the fact that $\varphi(t, \lambda u_1 + (1-\lambda)u_2, B) = \lambda\varphi(t, u_1, B) + (1-\lambda)\varphi(t, u_2, B)$, for any $\lambda \in [0, 1]$. □

Denote by $Int(M)$ the interior of the set M.

Lemma 4.17. *If the matrix $A(t)$ satisfies Condition (**D3**), then $\varphi(t, M', B) \subset Int(M)$, for any subset $M' \subseteq M$, $B \in H(A)$ and $t \in \mathbb{T}_+$.*

Proof. To prove the first statement we note that

$$\varphi(t,v,B)=\prod_{k=0}^{t-1}B(k)v=U(t,B)v,$$

where

$$U(t,B):=\prod_{k=0}^{t-1}B(k).$$

Since the matrix $A(t)$ satisfies Condition (**D3**) with the positive constants α_{ij} $(i,j=1,\dots,n)$, then by Lemma 4.15 (item 1) every matrix $B\in H(A)$ satisfies the same condition with the same constants α_{ij}. According to Lemma 4.15 (item 3) we have

$$U(t,B)\geq(\mathcal{A})^t$$

for any $t\geq 0$ and $B\in H(A)$, where $\mathcal{A}=(\alpha_{ij})_{i,j=1}^n$. Let M' be a subset of M and $v\in M'$. We will show that $U(\tau,B)v\in Int(M)$. To this end it is sufficient to prove that every component of the vector $U(\tau,B)v$ is positive. Since $v\in M'\subseteq M$, then at least one component v_{j_0} of the vector v is positive. Then by (4.27) for any $i=1,2,\dots,n$ we obtain

$$(U(t,B)v)_i\geq((\mathcal{A})^t v)_i\geq(\mathcal{A}^t)_{ij_0}v_{j_0}>0$$

because the matrix $(\mathcal{A})^t$ as the finite product of positive matrix is a positive matrix, where by $(v)_i$ is denoted the i-th component of the vector $v\in\mathbb{R}^n_+$ and $(A)_{ij}$ the element of the matrix A located on the pace with the index ij. This means that $U(t,B)M'\subset Int(M)$ for any $t>0$ and $B\in H(A)$. Lemma is proved. □

Theorem 4.16. *Assume that $A\in C(\mathbb{Z},[\mathbb{R}^n])$ be a stochastic matrix possessing the property (**D3**) and it is strongly Poisson stable in $t\in\mathbb{Z}$.*

Then I_B is a subset of $Int(M)$ for any $B\in H(A)$.

Proof. According to Theorem 4.15 for any $B\in H(A)$ and $t>0$ we have $U(t,B)I_B=I_{\sigma(t,B)}$ and, consequently,

$$I_B=U(t,\sigma(-t,B))I_{\sigma(-t,B)}.$$

Since $I_{\sigma(-t,B)}\subseteq M$, then by Lemma 4.17 $I_B=U(t,\sigma(-t,B))I_{\sigma(-t,B)}\subseteq Int(M)$. □

Theorem 4.17. *Assume that $A\in C(\mathbb{Z},[\mathbb{R}^n])$ be a stochastic matrix possessing the property (**D3**) and it is strongly Poisson stable in $t\in\mathbb{Z}$.*

Then I_B consists a single point $\{v_B\}$ for any $B\in H(A)$.

Proof. This statement can be proved using absolutely the same arguments as in the proof of the Theorem 4.12 so we omit the details. □

Corollary 4.11. *Under the conditions of Theorem 4.17 if the matrix-function $A\in C(\mathbb{Z},[\mathbb{R}^n])$ is stationary (respectively, τ-periodic, Bohr almost periodic, almost automorphic, recurrent, Levitan almost periodic, almost recurrent, pseudo recurrent and Lagrange stable, pseudo periodic and Lagrange stable, strongly Poisson stable), then equation (4.20) has a unique bounded on $\mathbb{Z}$ solution $p(t)$ with the values from $Int(M)$ ($p(\mathbb{Z})\subset Int(M)$) possessing the following properties:*

1. *$p(t)$ is stationary (respectively, τ-periodic, Bohr almost periodic, almost automorphic, recurrent, Levitan almost periodic, almost recurrent, pseudo recurrent and Lagrange stable, pseudo periodic and Lagrange stable, strongly Poisson stable);*
2. *the solution p is uniformly globally asymptotically stable with respect to M, i.e.,*
   - (a) *for arbitrary positive number ε there exists a positive number $\delta=\delta(\varepsilon)$ such that $\|\varphi(t_0,u,A)-p(t_0)\|<\delta$ ($u\in M$) implies $\|\varphi(t,u,A)-p(t)\|<\varepsilon$ for any $t\geq t_0$;*
   - (b) $\lim\limits_{t\to+\infty}\|\varphi(t,u,A)-p(t)\|=0$ *for every $u\in M$.*

Proof. This statement directly follows from Theorems 4.17 and 4.10. □

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REFERENCES

[1] Arnold, L. *Random Dynamical Systems*. Springer-Verlag, 1998, xv+586 pp.

[2] Bebutov, V. M. On the shift dynamical systems on the space of continuous functions. *Bull. of Inst. of Math. of Moscow University* 2:5 (1940), pp. 1–65. (in Russian)

[3] Bohr, H. *Almost Periodic Functions.* Chelsea Publishing Company, New York, 1947. ii+114 pp.

[4] Bronsteyn, I. U. *Extensions of Minimal Transformation Group. Kishinev*, Stiintsa, 1974 (in Russian). [English translation: Extensions of Minimal Transformation Group, Sijthoff & Noordhoff, Alphen aan den Rijn. The Netherlands Germantown, Maryland USA, 1979]

[5] Cheban, D. N. On the comparability of the points of dynamical systems with regard to the character of recurrence property in the limit. *Mathematical Sciences*, issue No. 1, Kishinev, ”Shtiintsa”, 1977, pp. 66–71.

[6] Cheban, D. N. Global Pullback Atttactors of C-Analytic Nonautonomous Dynamical Systems. *Stochastics and Dynamics*. **1** (2001), No. 4, pp. 511–535.

[7] Cheban, D. N. Levitan Almost Periodic and Almost Automorphic Solutions of V-monotone Differential Equations. *J. Dynamics and Differential Equations* **20** (2008), No. 3, pp. 669–697.

[8] Cheban D. N. *Asymptotically Almost Periodic Solutions of Differential Equations.* Hindawi Publishing Corporation, New York, 2009, ix+186 pp.

[9] Cheban, D. N. *Global Attractors of Nonautonomous Dynamical and Control Systems. 2nd Edition.* Interdisciplinary Mathematical Sciences, **18**, River Edge, NJ: World Scientific, 2015, xxv+589 pp.

[10] Cheban, D. N. I. U. Bronshtein’s Conjecture for Monotone Nonautonomous Dynamical Systems. *Discrete and Continuous Dynamical Systems: series B* **24** (2019), 2019, pp. 1095–1113.

[11] Cheban, D. N. *Nonautonomous Dynamics:* Nonlinear Oscillations and Global Attractors. Springer Nature Switzerland AG 2020 xxii+ 434 pp.

[12] Cheban, D. N. Almost Periodic and Almost Automorphic Solutions of Monotone Differential Equations with a Strict Monotone First Integral. *Buletinul Academiei de Ştiinţe a Republicii Moldova.* Matematica, no. 3. 2020, pp. 39–74.

[13] David Cheban, D. N. *On the structure of the Levinson center for monotone dissipative non-autonomous dynamical systems*. Chapter in the book ”Advance in Mathematics Research”,Volume 29. Nova Science Publishers, Inc. 2021, pp. 173–218.

[14] Cheban, D. N.; Liu, Z. Poisson Stable Solutions of Monotone Differential Equations. *Science China Mathematics* **62** (2019), No. 7, pp. 1391–1418.

[15] Chueshov, I. *Monotone Random Dynamical Systems Theory and Applications.* Berlin; Heidelberg: New York: Barselona: Hong Kong: London; Milan: Paris; Tokyo; Springer, 2002, viii+234 pp.

[16] Daletskii, Yu. L. ; Krein, M. G. *Stability of Solutions of Differential Equations in Banach Space*. Moscow, ”Nauka”, 1970, 534 pp. [English transl., Amer. Math. Soc., Providence, RI 1974.]

[17] Demidovich, B. P. *Lectures on Mathematical Theory of Stability.* Moscow, ”Nauka”, 1967. (in Russian)

[18] Ellis, R. *Lectures on Topological Dynamics.* W. A. Benjamin, Inc., New York 1969 xv+211 pp.

[19] Fink, A. M. *Almost Periodic Differential Equations.* Lecture notes in Mathematics, **377**, Springer-Verlag, 1974.

[20] Fink A. M.; Fredericson, P. O. Ultimate Boundedness Does not Imply Almost Periodicity. *Journal of Differential Equations*, **9** (1971), pp. 280–284.

[21] Flandoli, F.; Schmalfuß, B. Random Attractors for the Stochastic 3D Navier–Stokes Equation with Multiplicative White Noise, *Stochastics and Stochastics Reports*, **59** (1996), 21–45.

[22] Husemoller, D. *Fibre Bundles*. Third edition. Graduate Texts in Mathematics, 20. Springer-Verlag, New York, 1994. xx+353 pp.

[23] Hirsch, M.; Smith, H. *Monotone Dynamical Systems.* Handbook of Differential Equations, **2**, Ch. IV. A. Canada, P. Drabek and A. Fonda, editors. Elsevier North Holland, 2005.

[24] Jiang, J.; Zhao, X. Convergence in monotone and uniformly stable skew-product semiflows with applications, *J. Reine Angew. Math.* **589** (2005), 21–55.

[25] Krasnoselskii, M. A.; Lifshits, E. A.; Sobolev, A. V. *Positive Linear Systems. The Method of Positive Operators.* Sigma Series in Applied Mathematics 5. Heldermann Verlag, Berlin, 1989. viii+354 pp.

[26] Kuchment, P. *Floquet Theory for Partial Differential Equations.* Birkhäuser Verlag, Basel · Boston · Berlin, 1993, xiv+350 pp.

[27] Levitan, B. M.; Zhikov V. V. *Almost Periodic Functions and Differential Equations.* Moscow State University Press, Moscow, 1978, 2004 p. (in Russian). [English translation: Almost Periodic Functions and Differential Equations. Cambridge Univ. Press, Cambridge, 1982, xi+211 p.]

[28] Lusternik, L. A.; Sobolev, V. J. *Elements of Functional Analysis.* Hindustan Publishing Corporation (India), 1975, xvi+360 pp.

[29] Nussbaum, R. D. *Hilbert's projective metric and iterated nonlinear maps.* Mem. Amer. Math. Soc. 391, Providence, R.I., 1988, vi+133 pp.

[30] Sell, G. R. Non-Autonomous Differential Equations and Topological Dynamics, I. The basic theory. *Trans. Amer. Math. Soc.*, **127** (1967), pp. 241–262.

[31] Sell, G. R. Non-Autonomous Differential Equations and Topological Dynamics, II. Limiting equations. *Trans. Amer. Math. Soc.*, **127** (1967), pp. 263–283.

[32] Sell, G. R. *Lectures on Topological Dynamics and Differential Equations*, **2** of *Van Nostrand Reinhold math. studies*. Van Nostrand–Reinbold, London, 1971.

[33] Shcherbakov, B. A. Classification of Poisson-stable motions. Pseudo-recurrent motions. *Dokl. Akad. Nauk SSSR*, **146** (1962), pp. 322–324.

[34] Shcherbakov, B. A. On classes of Poisson stability of motion. Pseudorecurrent motions. *Bul. Akad. Stiince RSS Moldoven.* No. 1, 1963, pp. 58–72.

[35] Shcherbakov, B. A. Constituent classes of Poisson-stable motions. *Sibirsk. Mat. Zh.* **5** (1964), no. 6, pp. 1397–1417. (in Russian)

[36] Shcherbakov, B. A. *Topologic Dynamics and Poisson Stability of Solutions of Differential Equations*. Ştiinţa, Chişinău, 1972, 231 p. (in Russian)

[37] Shcherbakov, B. A. The compatible recurrence of the bounded solutions of first order differential equations. *Differencial'nye Uravnenija.* **10** (1974), pp. 270–275.

[38] Shcherbakov, B. A. The comparability of the motions of dynamical systems with regard to the nature of their recurrence. *Differential Equations* **11** (1975), no. 7, pp. 1246–1255. (in Russian) [English translation: Differential Equations, Vol.11. No.7, pp. 937-943].

[39] Shcherbakov, B. A. *Poisson Stability of Motions of Dynamical Systems and Solutions of Differential Equations*. Ştiinţa, Chişinău, 1985, 147 pp. (in Russian)

[40] Wenxian, Shen; Zhao, Xiao-Qiang Convergence in Almost Periodic Cooperative Systems with a First Integral. *Preceedings of the American Mathematical Society.* **133** (2004), No. 1, pp. 203-212.

[41] Sibirsky, K. S. *Introduction to Topological Dynamics.* Kishinev, RIA AN MSSR, 1970, 144 p. (in Russian). [English translationn: Introduction to Topological Dynamics. Noordhoff, Leyden, 1975]

[42] Smith, H. L., *Monotone Semiflows Generated by Functional Differential Equations. Journal of Differential Equations.* **66** (1987), pp. 420-442.

[43] Smith, H. L. *Monotone Dynamical Systems. An Introduction to the Theory of Competitive and Cooperative Systems.* Series: Mathematical surveys and monographs, Volume 41. American Mathematical Society. Providence, Rhode Island, 1995, x+174 p.

[44] Zhikov V. V. On Problem of Existence of Almost Periodic Solutions of Differential and Operator Equations. *Nauchnye Trudy VVPI, Matematika*, **8** (1969), pp. 94–188. (in Russian)

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