CARPATHIAN J. MATH.
Volume **38** (2022), No. 2,
Pages 281 - 297

Online version at `https://semnul.com/carpathian/`
Print Edition: ISSN 1584 - 2851; Online Edition: ISSN 1843 - 4401
DOI: https://doi.org/10.37193/CJM.2022.02.02

An accelerated Visco-Cesaro means Tseng Type splitting method for fixed point and monotone inclusion problems

YASIR ARFAT[1], POOM KUMAM[1,2,3], MUHAMMAD AQEEL AHMAD KHAN[4] and PARINYA SA NGIAMSUNTHORN[5]

ABSTRACT. In this paper, we study a variant of Tseng's splitting method for monotone inclusion problem and fixed point problem associated with a finite family of η-demimetric mappings in Hilbert spaces. The proposed algorithm is based on the combination of classical Tseng's method together with the viscosity Cesáro means method and the Nesterov's acceleration method. The proposed iterative method exhibits accelerated strong convergence characteristics under suitable set of control conditions in such framework. Finally, we provide a numerical example to illustrate the applicability of the proposed algorithm as well as some useful abstract applications.

1. INTRODUCTION AND PRELIMINARIES

Let $\mathcal{H}$ be a real Hilbert space with the inner product $< \cdot, \cdot >$ and the associated norm $\| \cdot \|$. The classical monotone inclusion problem aims to find

$$x^* \in \mathcal{H} \quad \text{such that } \ 0 \in Ax^* + Bx^*, \tag{1.1}$$

where $A : \mathcal{H} \to 2^{\mathcal{H}}$ is a multi-valued operator and $B : \mathcal{H} \to \mathcal{H}$ is a single-valued operator.

The problem (1.1), in the context of monotone operator theory, has been largely considered for modeling various real world as well as theoretical problems in the field of convex optimization, subgradients, partial differential equations, variational inequalities and image processing, evolution equations and inclusions, see for instance, [15, 14, 17, 32] and the references cited therein.

Since (1.1) is complex in nature and therefore requires sophisticated tools and iterative algorithms for the consequent analysis. The elegant forward-backward (FB) iterative algorithm [27, 30] is prominent among various iterative algorithms to solve (1.1). It is worth mentioning that FB iterative algorithm exhibits weak convergence even assuming the stronger conditions on the operators A and B. Later, Tseng [35] modified the FB iterative algorithm for weak convergence results in Hilbert spaces. Recently, Gibali and Thong [21] considered a modified variant of the Tseng's splitting method to establish strong convergence results in Hilbert spaces. We remark that the several general splitting algorithms are available in the literature with specific limitations. However, new splitting algorithms are formulated in such a way to unify and/or combine the existing splitting algorithms with enhanced intrinsic properties. We, therefore, propose and analyze a splitting method comprises of forward-backward-forward (FBF) iterates or Tseng's splitting method in Hilbert spaces.

We now elaborate some necessary concepts of fixed point theory as follows:

Received: 27.04.2021. In revised form: 28.10.2021. Accepted: 06.11.2021

2010 *Mathematics Subject Classification.* 47H05, 47H06, 47H09, 47H10, 65Y05, 65K15, 68W10.

Key words and phrases. *monotone inclusion problem, forward-backward-forward method, viscosity approximation method, Cesáro means method, Nesterov's accelerated method, strong convergence.*

Corresponding author: Poom Kumam; poom.kum@kmutt.ac.th

Let C be a nonempty subset of a real Hilbert space $\mathcal{H}$. For an operator $S : C \to C$, the set $Fix(S) = \{x \in C \mid x = Sx\}$ denotes the set of fixed points of the operator S. Recall that the operator S is called (i) nonexpansive, if $\|Sx - Sy\| \leq \|x - y\|$, for all $x, y \in C$; (ii) quasi-nonexpansive, if $Fix(S) \neq \emptyset$ and $\|Sx - y\| \leq \|x - y\|$, for all $x \in C$ and $y \in Fix(S)$; (iii) firmly nonexpansive if for each $x, y \in \mathcal{H}$ such that $\|Sx - Sy\|^2 \leq \langle Sx - Sy, x - y\rangle$ (iv) η-strict pseudocontraction [11], if there exists $\eta \in [0, 1)$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + \eta\|x - y - (Sx - Sy)\|^2, \text{ for all } x, y \in C;$$

(v) η-demicontractive if $Fix(S) \neq \emptyset$ and there exists $\eta \in [0, 1)$ such that

$$\|Sx - y\|^2 \leq \|x - y\|^2 + \eta\|x - Sx\|^2, \text{ for all } x \in C \text{ and } y \in Fix(S);$$

(vi) η-demimetric [33] where $\eta \in (-\infty, 1)$, if $Fix(S) \neq \emptyset$ such that

$$\langle x - y, (Id - S)x\rangle \geq \frac{1}{2}(1 - \eta)\|(Id - S)x\|^2, \text{ for all } x \in C \text{ and } y \in Fix(S),$$

where Id denotes the identity operator. Note that

$$\|Sx - y\|^2 \leq \|x - y\|^2 + \eta\|x - Sx\|^2, \text{ for all } x \in C \text{ and } y \in Fix(S).$$

is an equivalent representation of an η-demimetric operator. It is evident that the class of η-demimetric operators contains the operators defined in (i)-(iv).

Fixed point theory of nonlinear operators is a fertile field of research and emerged as a powerful tool to solve a variety of problems arising in various branches of sciences [15, 22, 23]. In 1975, Baillon[9] established the first nonlinear ergodic theorem as follows:

Theorem 1.1 ([9]). *Let C be a nonempty, closed and convex subset of a real Hilbert space $\mathcal{H}$ and let $S : C \to C$ be a nonexpansive operator such that $Fix(S) \neq \emptyset$ then for all $x \in C$, the Cesáro means*

$$S_n x = \frac{1}{N}\sum_{i=0}^{N-1} S_i x, \ \forall N \geq 1,$$

weakly converges to a fixed point of S.

Since then the classical Cesáro means method have been considered for various classes of nonlinear operators, see [16, 25, 26] and the references cited therein. It is worth mentioning that the Cesáro means method fails to converge strongly for the class of nonexpansive operators[20]. In order to obtain strong convergence results, one has to impose additional requirements on the iterative algorithm. In 1967, Halpern[24] introduced and analyzed an iterative algorithm which strongly converges to the closest fixed point of the nonexpansive operator. It is remarked that the Halpern iterative algorithm coincides with the Cesáro means method for linear operators. In 2000, Moudafi [29] proposed and analyzed the strongly convergent viscosity iterative algorithm by utilizing a strict contraction operator instead of the anchor point in the Halpern iterative algorithm. In this paper, we are going to study an algorithm based on the combination of classical Tseng's method associated with the monotone inclusion problem together with the viscosity Cesáro means method for η-demimetric operators. In order to enhance the speed of convergence of the proposed iterative algorithm, we also utilize the inertial extrapolation technique essentially due to Polyak [31], see also[1, 2, 3, 4, 5, 6, 7].

The rest of the paper is organized as follows: Section 2 contains preliminary concepts and results regarding monotone operator theory and fixed point problem theory. Section 3 comprises strong convergence results of the proposed algorithm whereas Section 4 deals with the efficiency of the proposed algorithm and its comparison with the existing algorithm by numerical experiments. Section 5 provides various abstract applications of of the proposed algorithm in minimization problems, split feasibility problems and image processing.

2. PRELIMINARIES

We start this section with the mathematical preliminary concepts required in the sequel. An operator $P_C^{\mathcal{H}}$ is said to be metric projection of $\mathcal{H}$ onto nonempty, closed and convex subset C if for every $x \in \mathcal{H}$, there exists a unique nearest point in C denoted by $P_C^{\mathcal{H}}x$ such that

$$\|x - P_C^{\mathcal{H}}x\| \leq \|x - z\|, \text{ for all } z \in C.$$

It is remarked that the metric projection operator satisfies firmly nonexpansiveness and can be characterized as:

$$\langle x - P_C^{\mathcal{H}}x, P_C^{\mathcal{H}}x - y\rangle \geq 0, \text{ for all } x \in \mathcal{H} \text{ and } y \in C.$$

Recall that a set-valued operator $A : \mathcal{H} \to 2^{\mathcal{H}}$ is called monotone, if for all $x, y \in \mathcal{H}$, $u \in Ax$ and $v \in Ay$ then $\langle x - y, u - v\rangle \geq 0$. Moreover, a monotone operator A is said to be maximal monotone if there is no proper monotone extension of A. For a monotone operator A, the associated resolvent operator J_m^A of index $m > 0$ is defined as

$$J_m^A = (Id + mA)^{-1},$$

where $(\cdot)^{-1}$ denotes the inverse operator.

Note that the resolvent operator J_m^A is well defined everywhere on Hilbert space $\mathcal{H}$. Further, J_m^A is single valued and satisfies the firmly nonexpansiveness. Furthermore, $x \in A^{-1}(0)$ if and only if $x = J_m^A(x)$.

Definition 2.1. Let $S : C \to C$ be a nonexpansive operator defined on a nonempty, closed and convex subset of a real Hilbert space $\mathcal{H}$. The operator $Id - S$ is said to be demiclosed at the origin provided that for any sequence (x_n) in C that converges weakly to some x and if the sequence $((Id - S)x_n)$ converges strongly to 0, then $(Id - S)(x) = 0$.

The rest of this section is organized with the celebrated results required in the sequel.

Lemma 2.1. *Let $x, y, z \in \mathcal{H}$ and $\alpha, \beta, \gamma \in [0, 1] \subset \mathbb{R}$ and $\alpha + \beta + \gamma = 1$ then we have*
(1) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$;
(2) $\|\beta x + (1 - \beta)y\|^2 = \beta\|x\|^2 + (1 - \beta)\|y\|^2 - \beta(1 - \beta)\|x - y\|^2$.
(3) $\|\alpha x + \beta y + \gamma z\|^2 = \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta\|x - y\|^2 - \alpha\gamma\|x - z\|^2 - \beta\gamma\|y - z\|^2$

Lemma 2.2 ([33]). *Let C be a nonempty, closed and convex subset of a Hilbert space $\mathcal{H}$ and let $S : C \to \mathcal{H}$ be an η-demimetric operator with $\eta \in (-\infty, 1)$. Then $Fix(S)$ is closed and convex.*

Lemma 2.3 ([34]). *Let C be a nonempty, closed and convex subset of a Hilbert space $\mathcal{H}$ and let $S : C \to \mathcal{H}$ be an η-demimetric operator with $\eta \in (-\infty, 1)$ and $Fix(S) \neq \emptyset$. Let γ be a real number with $0 < \gamma < 1 - \eta$ and set $L = (1 - \gamma)Id + \gamma S$, then L is a quasi-nonexpansive operator of C into $\mathcal{H}$.*

Lemma 2.4 ([12]). *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space and $S : C \to C$ be a nonexpansive operator. For each $x \in C$ and the Cesáro means $S_n x = \frac{1}{N}\sum_{i=0}^{N-1} S_i x$, then $\limsup_{n\to\infty} \|S_n x - S(S_n x)\| = 0$.*

Lemma 2.5 ([10]). *Let $A : \mathcal{H} \to 2^{\mathcal{H}}$ be a maximal monotone operator and $B : \mathcal{H} \to \mathcal{H}$ be a Lipschitz continuous and monotone operator. Then the operator $A + B$ is a maximal monotone operator.*

Lemma 2.6 ([21]). *Let $A : \mathcal{H} \to 2^{\mathcal{H}}$ be a maximal monotone operator and $B : \mathcal{H} \to \mathcal{H}$ be a mapping on $\mathcal{H}$. Define $S_\mu := (Id + \mu A)^{-1}(Id - \mu B), \mu > 0$. Then we have $Fix(S_\mu) = (A + B)^{-1}(0)$, for all $\mu > 0$.*

Lemma 2.7 ([37]). *Let (b_n) be a sequences of nonnegative real numbers and there exists $n_0 \in \mathbb{N}$ such that:*

$$b_{n+1} \leq (1 - \psi_n)b_n + \psi_n c_n + d_n, \ \forall\, n \geq n_0,$$

where $(\psi_n) \subset (0, 1)$ and (c_n), (d_n) with the following conditions hold:

(I) $\sum_{n=1}^{\infty} \psi_n = \infty$;
(II) $\limsup_{n\to\infty} c_n \leq 0$;
(III) $\sum_{n=1}^{\infty} d_n < \infty$, $\forall\, 0 \leq d_n(0 \leq n)$,

then $\lim_{n\to\infty} b_n = 0$.

Lemma 2.8 ([28]). *Let (q_n) be a sequence of nonnegative real numbers. Suppose that there is a subsequence (q_{n_j}) of (q_n) such that $q_{n_j} < q_{n_j+1}$ for all $j \in \mathbb{N}$, then there exists a nondecreasing sequence (ε_k) of $\mathbb{N}$ such that $\lim_{k\to\infty} \varepsilon_k = \infty$ and satisfy the following properties such that:*

$$q_{\varepsilon_k} \leq q_{\varepsilon_k+1} \text{ and } q_k \leq q_{\varepsilon_k+1},$$

for some large number $k \in \mathbb{N}$. Thus, ε_k is the largest number n in the set $\{1, 2, \cdots, k\}$ such that $q_n < q_{n+1}$.

3. Main results

In this section, we prove the following strong convergence result.

Theorem 3.2. *Let $A : \mathcal{H} \to 2^{\mathcal{H}}$ be a maximally monotone operator and let $B : \mathcal{H} \to \mathcal{H}$ be a monotone and ρ-Lipschitz operator for some $\rho > 0$ defined on a real Hilbert space $\mathcal{H}$. For all $i \in \{1, 2, \cdots, N\}$, let $S_i : \mathcal{H} \to \mathcal{H}$ be a finite family of η-demimetric operators with $\eta \in (-\infty, 1)$ such that $Id - S_i$ is demiclosed at the origin and let $h : \mathcal{H} \to \mathcal{H}$ be a contraction mapping with constant $\lambda \in [0, 1)$. Assume that $\Gamma = (A + B)^{-1}(0) \cap \bigcap_{i=1}^{N} Fix(S_i) \neq \emptyset$, $\mu_1 > 0$, $\sigma \in (0, 1)$, $(\xi_n) \subset [0, 1)$ and $(\alpha_n), (\beta_n)$ are sequences in $(0, 1)$. For given $x_0, x_1 \in \mathcal{H}$, let the iterative sequence (x_n) be generated by*

$$(3.2)\quad \begin{cases} u_n = x_n + \xi_n(x_n - x_{n-1}); \\ v_n = J^A_{\mu_n}(Id - \mu_n B)u_n; \\ w_n = v_n - \mu_n(Bv_n - Bu_n); \\ x_{n+1} = \alpha_n h(x_n) + (1 - \alpha_n - \beta_n)x_n + \beta_n(\frac{1}{N}\sum_{i=0}^{N-1}((1 - \gamma_n)Id + \gamma_n S_i))w_n. \end{cases}$$

Assume that the following step size rule

$$\mu_{n+1} = \begin{cases} \min\left\{\frac{\sigma\|u_n - v_n\|}{\|Bu_n - Bv_n\|}, \mu_n\right\}, \text{ if } Bu_n - Bv_n \neq 0; \\ \mu_n, \ otherwise, \end{cases}$$

and the conditions hold:

(C1) $\sum_{n=1}^{\infty} \xi_n\|x_n - x_{n-1}\| < \infty$;
(C2) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$, *and for each* $n \in \mathbb{N}$, $0 < a^* < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < b^* < 1 - \alpha_n$, *where* a^*, b^* *be positive real numbers.*

Then the sequence (x_n) generated by (3.2) converge strongly to a point $\bar{x} = P_\Gamma \circ h(\bar{x})$.

In order to prove Theorem 3.2, we need the following results from [21].

Lemma 3.9 ([21]). *The sequence (μ_n) generated by (3.2) is a nonincreasing sequence with a lower bound of* $\min\{\mu_1, \frac{\sigma}{\rho}\}$.

Lemma 3.10 ([21]). *Assume that Conditions (C1)–(C2) hold and let (w_n) be any sequence generated by (3.2), we have*

$$\|w_n - \bar{x}\|^2 \leq \|x_n - \bar{x}\|^2 - \Big(1 - \sigma^2 \frac{\mu_n^2}{\mu_{n+1}^2}\Big)\|x_n - v_n\|^2, \tag{3.3}$$

and

$$\|w_n - v_n\| \leq \sigma \frac{\mu_n}{\mu_{n+1}}\|x_n - v_n\|. \tag{3.4}$$

Lemma 3.11. *Assume that Conditions (C1)–(C2) hold and suppose that*

$$\lim_{n\to\infty}\|x_n - u_n\| = \lim_{n\to\infty}\|x_n - v_n\| = \lim_{n\to\infty}\|x_n - w_n\| =$$
$$\lim_{n\to\infty}\Big\|w_n - \Big(\frac{1}{N}\sum_{i=0}^{N-1}((1-\gamma_n)Id + \gamma_n S_i)\Big)w_n\Big\| = 0.$$

Let (x_n) and (u_n) be two sequences generated by (3.2). If a subsequence (x_{n_t}) of x_n converges weakly to some $x^* \in \mathcal{H}$ then $x^* \in \Gamma$.

Proof. Let $x^* \in \mathcal{H}$ such that $x_{n_t} \rightharpoonup x^*$ then $x^* \in (A+B)^{-1}(0)$ follows from [21, Lemma 7]. Since $\lim_{n\to\infty}\|x_n - w_n\| = 0$ and $x_{n_t} \rightharpoonup x^*$ therefore we have $w_{n_t} \rightharpoonup x^*$. Since

$$\lim_{n\to\infty}\Big\|w_n - \Big(\frac{1}{N}\sum_{i=0}^{N-1}((1-\gamma_n)Id + \gamma_n S_i)\Big)w_n\Big\| = 0,$$

therefore, utilizing Lemma 2.1, we get $x^* \in Fix(S_i)$ and hence $x^* \in \Gamma$.

Now we are able to prove the main result of this section.

Proof of Theorem 3.2. For simplicity, the proof is divided into the following steps.

Step 1. Show that the sequence (x_n) is bounded.

Let $\bar{x} \in \Gamma$, then for each $n \in \mathbb{N}$ we have

$$\begin{aligned}\|u_n - \bar{x}\|^2 &= \|x_n - \bar{x} + \xi_n(x_n - x_{n-1})\|^2 \\ &\leq \|x_n - \bar{x}\|^2 + \xi_n^2\|x_n - x_{n-1}\|^2 + 2\xi_n\langle x_n - \bar{x}, x_n - x_{n-1}\rangle.\end{aligned} \tag{3.5}$$

Now for all $i \in \{1, 2, \cdots, N\}$, set $S_n = \frac{1}{N}\sum_{i=0}^{N-1}((1-\gamma_n)Id + \gamma_n S_i)$. Utilizing Lemma 2.3 for any $\bar{x} \in \Gamma$, we observe that

$$\begin{aligned}\|S_n x - \bar{x}\| &= \Big\|\frac{1}{N}\sum_{i=0}^{N-1}((1-\gamma_n)Id + \gamma_n S_i)x - \bar{x}\Big\| \\ &\leq \frac{1}{N}\sum_{i=0}^{N-1}\|((1-\gamma_n)Id + \gamma_n S_i)x - \bar{x}\| \\ &\leq \frac{1}{N}\sum_{i=0}^{N-1}\|x - \bar{x}\| = \|x - \bar{x}\|.\end{aligned}$$

It follows from the above estimate that S_n is a quasi-nonexpansive operator. Since $\lim_{n\to\infty}(1-\sigma^2\frac{\mu_n^2}{\mu_{n+1}^2}) = 1 - \sigma^2 > 0$, therefore for each $n \geq n_0$ where $n_0 \in \mathbb{N}$, we have that

$$1 - \sigma^2\frac{\mu_n^2}{\mu_{n+1}^2} > 0. \tag{3.6}$$

From (3.3) and (3.6), we obtain

$$\|w_n - \bar{x}\| \leq \|x_n - \bar{x}\|. \tag{3.7}$$

Further, from (C2) and (3.7), we have

$$\begin{aligned}
\|x_{n+1}-\bar{x}\| &= \|\alpha_n(h(x_n)-\bar{x})+(1-\alpha_n-\beta_n)(x_n-\bar{x})+\beta_n(S_nw_n-\bar{x})\| \\
&\leq \alpha_n\|h(x_n)-\bar{x}\|+(1-\alpha_n-\beta_n)\|x_n-\bar{x}\|+\beta_n\|S_nw_n-\bar{x}\| \\
&\leq \alpha_n\|h(x_n)-h(\bar{x})\|+\alpha_n\|h(\bar{x}-\bar{x})\|+(1-\alpha_n-\beta_n)\|x_n-\bar{x}\| \\
&\quad +\beta_n\|w_n-\bar{x}\| \\
&\leq \alpha_n\lambda\|x_n-\bar{x}\|+\alpha_n\|h(\bar{x})-\bar{x}\|+(1-\alpha_n)\|x_n-\bar{x}\| \\
&= (1-\alpha_n(1-\lambda))\|x_n-\bar{x}\|+\alpha_n(1-\lambda)\frac{\|h(\bar{x})-\bar{x}\|}{1-\lambda} \\
&\leq \max\Big\{\|x_n-\bar{x}\|,\frac{\|h(\bar{x})-\bar{x}\|}{1-\lambda}\Big\}.
\end{aligned}$$

Thus, for all $n\geq n_0$, $\|x_{n+1}-\bar{x}\|\leq\max\{\|x_{n_0}-\bar{x}\|,\frac{\|h(\bar{x})-\bar{x}\|}{1-\lambda}\}$. This implies that $(\|x_n-\bar{x}\|)$ is bounded.

Step 2. Compute the following two estimates:

$$(i):\beta_n\Big(1-\sigma^2\frac{\mu_n^2}{\mu_{n+1}^2}\Big)\|x_n-v_n\|^2+\beta_n(1-\alpha_n-\beta_n)\|x_n-S_nw_n\|^2 \quad\leq$$

$$\|x_n-\bar{x}\|^2-\|x_{n+1}-\bar{x}\|^2+\alpha_n\|h(x_n)-\bar{x}\|^2; \tag{3.8}$$

$$(ii):\|x_{n+1}-\bar{x}\|^2\leq[1-\alpha_n(1-\lambda)]\|x_n-\bar{x}\|^2$$

$$+\alpha_n(1-\lambda)\Big(\frac{2}{1-\lambda}(\beta_n\|x_n-S_nw_n\|\|x_{n+1}-\bar{x}\|+\langle h(\bar{x})-\bar{x},x_{n+1}-\bar{x}\rangle)\Big)\Big). \tag{3.9}$$

Utilizing Lemma 2.1(iii), we obtain

$$\begin{aligned}
\|x_{n+1}-\bar{x}\|^2 &= \|\alpha_n(h(x_n)-\bar{x})+(1-\alpha_n-\beta_n)(x_n-\bar{x})+\beta_n(S_nw_n-\bar{x})\|^2 \\
&= \alpha_n\|h(x_n)-\bar{x}\|^2+(1-\alpha_n+\beta_n)\|x_n-\bar{x}\|^2+\beta_n\|(S_nw_n-\bar{x})\|^2 \\
&\quad -\alpha_n(1-\alpha_n-\beta_n)\|h(x_n)-x_n\|^2 \\
&\quad -\beta_n(1-\alpha_n-\beta_n)\|x_n-S_nw_n\|^2-\alpha_n\beta_n\|h(x_n)-S_nw_n\|^2 \\
&\leq \alpha_n\|h(x_n)-\bar{x}\|^2+(1-\alpha_n-\beta_n)\|x_n-\bar{x}\|^2+\beta_n\|w_n-\bar{x}\|^2 \\
&\quad -\beta_n(1-\alpha_n-\beta_n)\|x_n-S_nw_n\|^2.
\end{aligned}$$

Now utilizing (3.3) in the above estimate, we get

$$\begin{aligned}
\|x_{n+1}-\bar{x}\|^2 &\leq \alpha_n\|h(x_n)-\bar{x}\|^2+(1-\alpha_n)\|x_n-\bar{x}\|^2-\beta_n(1-\alpha_n-\beta_n)\|x_n-S_nw_n\|^2 \\
&\quad -\beta_n\Big(1-\sigma^2\frac{\mu^2}{\mu_{n+1}^2}\Big)\|x_n-v_n\|^2 \\
&\leq \alpha_n\|h(x_n)-\bar{x}\|^2+\|x_n-\bar{x}\|^2-\beta_n(1-\alpha_n-\beta_n)\|x_n-S_nw_n\|^2 \\
&\quad -\beta_n\Big(1-\sigma^2\frac{\mu^2}{\mu_{n+1}^2}\Big)\|x_n-v_n\|^2.
\end{aligned}$$

Simplifying the above estimate, we have the desired estimate (3.8).
Next, by using (C2) and setting $j_n=(1-\beta_n)x_n+\beta_nS_nw_n$, we get

$$\|j_n-\bar{x}\| \quad\leq\quad \|x_n-\bar{x}\|, \tag{3.10}$$

and

$$\|x_n-j_n\|=\beta_n\|x_n-S_nw_n\|. \tag{3.11}$$

Utilizing (3.10), (3.11) and Lemma 2.1(ii)-(iii), the desired estimate (3.9) follows from the following calculation:

$$\begin{aligned}
&\|x_{n+1}-\bar{x}\|^2\\
&= \|(1-\alpha_n)(j_n-\bar{x})+\alpha_n(h(x_n)-h(\bar{x}))-\alpha_n(x_n-j_n)-\alpha_n(\bar{x}-h(\bar{x}))\|^2\\
&\leq \|(1-\alpha_n)(j_n-\bar{x})+\alpha_n(h(x_n)-h(\bar{x}))\|^2-2\alpha_n\langle x_n-j_n+\bar{x}-h(\bar{x}),x_{n+1}-\bar{x}\rangle\\
&\leq (1-\alpha_n)\|j_n-\bar{x}\|^2+\alpha_n\|h(x_n)-h(\bar{x})\|^2-2\alpha_n\langle x_n-j_n+\bar{x}-h(\bar{x}),x_{n+1}-\bar{x}\rangle\\
&\leq (1-\alpha_n)\|x_n-\bar{x}\|^2+\alpha_n\lambda\|x_n-\bar{x}\|^2+2\alpha_n\langle x_n-j_n,\bar{x}-x_{n+1}\rangle\\
&\quad +2\alpha_n\langle h(\bar{x})-\bar{x},x_{n+1}-\bar{x}\rangle\\
&\leq (1-\alpha_n(1-\lambda))\|x_n-\bar{x}\|^2+2\alpha_n\|x_n-j_n\|\|x_{n+1}-\bar{x}\|+2\alpha_n\langle h(\bar{x})-\bar{x},x_{n+1}-\bar{x}\rangle\\
&= (1-\alpha_n(1-\lambda))\|x_n-\bar{x}\|^2+2\alpha_n\beta_n\|x_n-S_nw_n\|\|x_{n+1}-\bar{x}\|\\
&\quad +2\alpha_n\langle h(\bar{x})-\bar{x},x_{n+1}-\bar{x}\rangle\\
&= (1-\alpha_n(1-\lambda))\|x_n-\bar{x}\|^2+\alpha_n(1-\lambda)\Big(\frac{2}{1-\lambda}(\beta_n\|x_n-S_nw_n\|\|x_{n+1}-\bar{x}\|\\
&\quad +\langle h(\bar{x})-\bar{x},x_{n+1}-\bar{x}\rangle)\Big).
\end{aligned}$$

Step 3. Show that $\lim_{n\to\infty}\|x_n-\bar{x}\|=0$.
We consider the two possible cases on the sequence $(\|x_n-\bar{x}\|)$.
Case A For all $n\geq n_0$, $\|x_{n+1}-\bar{x}\|^2\leq\|x_n-\bar{x}\|^2$ and $n_0\in\mathbb{N}$. This implies that $\lim_{n\to\infty}\|x_n-\bar{x}\|$ exists. Since $\lim_{n\to\infty}(1-\sigma^2\frac{\mu_n^2}{\mu_{n+1}^2})=1-\sigma^2>0$. By using (C2) and (3.8), we have

$$\lim_{n\to\infty}\|x_n-v_n\|=\lim_{n\to\infty}\|x_n-S_nw_n\|=0. \tag{3.12}$$

From (3.4), we get

$$\lim_{n\to\infty}\|w_n-v_n\|=0. \tag{3.13}$$

By the definition of (u_n) and (C1), we have

$$\lim_{n\to\infty}\|u_n-x_n\|=\lim_{n\to\infty}\xi_n\|x_n-x_{n-1}\|=0. \tag{3.14}$$

By using the triangle inequality, we obtain the following estimates:

$$\begin{aligned}
\|u_n-v_n\| &\leq \|u_n-x_n\|+\|x_n-v_n\|\to 0,\ \text{as } n\to\infty;\\
\|u_n-w_n\| &\leq \|u_n-v_n\|+\|v_n-w_n\|\to 0,\ \text{as } n\to\infty;\\
\|x_n-w_n\| &\leq \|x_n-v_n\|+\|v_n-w_n\|\to 0,\ \text{as } n\to\infty;\\
\|w_n-S_nw_n\| &\leq \|x_n-w_n\|+\|x_n-S_nw_n\|\to 0,\ \text{as } n\to\infty.
\end{aligned}$$

By using Lemma 2.4, we have

$$\limsup_{n\to\infty}\|S_nw_n-S(S_nw_n)\|=0. \tag{3.15}$$

Note that for all $n\in\mathbb{N}$, we get

$$\begin{aligned}
\|x_{n+1}-x_n\| &\leq \|x_{n+1}-S_nw_n\|+\|x_n-S_nw_n\|\\
&\leq \alpha_n\|h(x_n)-x_n\|+(2-\beta_n)\|x_n-S_nw_n\|.
\end{aligned} \tag{3.16}$$

From (3.12) and (C2), the estimate (3.16) implies that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{3.17}$$

Similarly, from (3.14), (3.17) and the following triangle inequality, we have

$$\|x_{n+1}-u_n\|\leq\|x_{n+1}-x_n\|+\|x_n-u_n\|\to 0,\ \text{as } n\to\infty.$$

Since (x_n) is bounded, then there exists a subsequence (x_{n_t}) of (x_n) with $x_{n_t} \rightharpoonup x^* \in \mathcal{H}$. Now utilizing Lemma 3.11 we have $x^* \in \Gamma$.
By making use of the estimate (3.17), we get

$$\begin{aligned}\limsup_{n\to\infty}\langle h(\bar{x}) - \bar{x}, x_{n+1} - \bar{x}\rangle \leq\ & \limsup_{n\to\infty}\langle h(\bar{x}) - \bar{x}, x_{n+1} - x_n\rangle + \\ & \limsup_{n\to\infty}\langle h(\bar{x}) - \bar{x}, x_n - \bar{x}\rangle \leq 0.\end{aligned} \tag{3.18}$$

From the estimate (3.18) and Lemma 2.7, we get $\lim_{n\to\infty}\|x_n - \bar{x}\| = 0$.
Case B There exists a subsequence $(\|x_{n_k} - \bar{x}\|^2)$ of $(\|x_n - \bar{x}\|^2)$ such that $\|x_{n_k} - \bar{x}\| < \|x_{n_k+1} - \bar{x}\|$ for all $k \in \mathbb{N}$.
It follows from Lemma 2.8 that there exists a nondecreasing sequence $(b_m) \in \mathbb{N}$ such that $\lim_{m\to\infty} b_m = \infty$, for all $m \in \mathbb{N}$ with the inequality $\|x_{b_m} - \bar{x}\|^2 \leq \|x_{b_m+1} - \bar{x}\|^2$ holds. In a similar fashion from (3.8), we obtain

$$\begin{aligned}&\beta_{b_m}\Big(1 - \sigma^2\frac{\mu_{b_m}^2}{\mu_{b_m+1}^2}\Big)\|x_{b_m} - v_{b_m}\|^2 + \beta_{b_m}(1 - \alpha_{b_m} - \beta_{b_m})\|x_{b_m} - S_{b_m}w_{b_m}\|^2 \\ &\leq \|x_{b_m} - \bar{x}\|^2 - \|x_{b_m+1} - \bar{x}\|^2 + \alpha_{b_m}\|h(x_{b_m}) - \bar{x}\|^2 \\ &\leq \alpha_{b_m}\|h(x_{b_m}) - \bar{x}\|^2.\end{aligned}$$

Since $\lim_{n\to\infty}\alpha_n = 0$, so we get

$$\lim_{m\to\infty}\|x_{b_m} - v_{b_m}\| = \lim_{m\to\infty}\|x_{b_m} - S_{b_m}w_{b_m}\| = 0.$$

Similarly from Case A, we have

$$\limsup_{m\to\infty}\langle h(\bar{x}) - \bar{x}, x_{b_m+1} - \bar{x}\rangle \leq 0.$$

Using (3.9) for $n \geq \max\{n^*, n_0\}$, we have the following estimate:

$$\begin{aligned}&\|x_{b_m+1} - \bar{x}\|^2 \\ &\leq (1 - \alpha_{b_m}(1-\lambda))\|x_{b_m} - \bar{x}\|^2 + \alpha_{b_m}(1-\lambda)\Big(\frac{2}{1-\lambda}(\beta_{b_m}\|x_{b_m} - S_{b_m}w_{b_m}\|\|x_{b_m+1} - \bar{x}\| \\ &\quad + \langle h(\bar{x}) - \bar{x}, x_{b_m+1} - \bar{x}\rangle)\Big) \\ &\leq (1 - \alpha_{b_m}(1-\lambda))\|x_{b_m+1} - \bar{x}\|^2 + \alpha_{b_m}(1-\lambda)\Big(\frac{2}{1-\lambda}(\beta_{b_m}\|x_{b_m} - S_{b_m}w_{b_m}\|\|x_{b_m+1} - \bar{x}\| \\ &\quad + \langle h(\bar{x}) - \bar{x}, x_{b_m+1} - \bar{x}\rangle)\Big).\end{aligned}$$

The above estimate yields that

$$\begin{aligned}&\|x_{b_m+1} - \bar{x}\|^2 \\ &\leq \frac{2}{1-\lambda}(\beta_{b_m}\|x_{b_m} - S_{b_m}w_{b_m}\|\|x_{b_m+1} - \bar{x}\| + \langle h(\bar{x}) - \bar{x}, x_{b_m+1} - \bar{x}\rangle).\end{aligned} \tag{3.19}$$

Therefore, $\limsup_{m\to\infty}\|x_{b_m} - \bar{x}\|^2 \leq 0$. Therefore, $x_n \to \bar{x} \in \Gamma$ and this completes the proof. □

Replacing viscosity iteration in sequence (3.2) by Halpern's one, we have the following iterates:

Theorem 3.3. *Let $A : \mathcal{H}_1 \to 2^{\mathcal{H}}$ be a maximally monotone operator and let $B : \mathcal{H} \to \mathcal{H}_1$ be a monotone and ρ-Lipschitz operator for some $\rho > 0$ defined on a real Hilbert space $\mathcal{H}$. For all $i \in \{1, 2, \cdots, N\}$, $S_i : \mathcal{H} \to \mathcal{H}$ be a finite family of η-demimetric operators with $\eta \in (-\infty, 1)$ such that $Id - S_i$ is demiclosed at the origin. Assume that $\Gamma = (A+B)^{-1}(0) \cap \bigcap_{i=1}^N Fix(S_i) \neq \emptyset$,*

$(\mu_1) > 0$, $\sigma \in (0,1)$, $(\xi_n) \subset [0,1)$, and $(\alpha_n), (\beta_n)$ are sequences in $(0,1)$. For given $p, x_0, x_1 \in \mathcal{H}$, let the iterative sequence (x_n) be generated by

$$
\left\{
\begin{array}{l}
u_n = x_n + \xi_n(x_n - x_{n-1}); \\
v_n = J^A_{\mu_n}(Id - \mu_n B)u_n; \\
w_n = v_n - \mu_n(Bv_n - Bu_n); \\
x_{n+1} = \alpha_n p + (1-\alpha_n-\beta_n)x_n + \beta_n(\frac{1}{N}\sum_{i=0}^{N-1}((1-\gamma_n)Id + \gamma_n S_i))w_n.
\end{array}
\right. \tag{3.20}
$$

Assume that the following step size rule

$$
\mu_{n+1} = \left\{
\begin{array}{ll}
\min\left\{\frac{\sigma\|u_n - v_n\|}{\|Bu_n - Bv_n\|}, \mu_n\right\}, & \text{if } Bu_n - Bv_n \neq 0; \\
\mu_n, & otherwise.
\end{array}
\right.
$$

and the conditions hold:

(C1) $\sum_{n=1}^{\infty} \xi_n \|x_n - x_{n-1}\| < \infty$;

(C2) $\lim_{n\to\infty} \frac{\alpha_n}{\beta_n} = 0$, $\lim_{n\to\infty}(1-\alpha_n-\beta_n) = 0$ *and* $\sum_{n=1}^{\infty} \frac{\alpha_n}{\beta_n} = \infty$.

(C3) *For each* $n \in \mathbb{N}$, $0 < a^* < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < b^* < 1 - \alpha_n$, *where* a^*, b^* *be positive real numbers.*

Then the sequence (x_n) generated by (3.20) converge strongly to a point $\bar{x} = P_\Gamma \circ h(\bar{x})$.

Remark 3.1. In order to obtain the desired result, we have to assume a stopping criteria for (3.20) like that if $n > n_{max}$ for some chosen sufficiently large number n_{max}.

Proof. Observe that for each $n \geq 1$, arguing similarly as in the proof of Theorem 3.2 (Steps 1-3), we deduce that Γ is well-defined, closed and bounded. Furthermore, the sequence (x_k) is bounded and

$$
\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.21}
$$

Let $x_{n+1} = \alpha_n p + (1-\alpha_n-\beta_n)x_n + \beta_n S_n w_n$. An easy calculation along (3.20) and (C2)-(C3) implies that

$$
\|S_n w_n - x_n\| \leq \frac{1}{\beta_n}\|x_{n+1} - x_n\| + \frac{\alpha_n}{\beta_n}\|h(p) - x_n\| + \frac{1-\alpha_n-\beta_n}{\beta_n}\|p - x_n\|.
$$

Hence, the above estimate implies that

$$
\lim_{n\to\infty} \|S_n w_n - x_n\| = 0.
$$

The rest of the proof follows immediately from the proof of Theorem 3.2 and is therefore omitted. □

Remark 3.2. The condition (C1) is easily applicable in numerical calculation since the value of $\|x_n - x_{n-1}\|$ is known before choosing (ξ_n) which satisfies $0 \leq \xi_n \leq \widehat{\xi_n}$

$$
\widehat{\xi_n} = \left\{
\begin{array}{ll}
\min\left\{\frac{\Theta_n}{\|x_n - x_{n-1}\|}, \xi\right\} & if\ x_n \neq x_{n-1}; \\
\xi & \text{otherwise},
\end{array}
\right.
$$

where (Θ_n) is a positive sequence such that $\sum_{n=1}^{\infty} \Theta_n < \infty$ and $\xi \in [0,1)$.

4. Applications

In this section, we illustrate some theoretical results as an application of our main result in Section 3.

Split Convex Feasibility Problem. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces and $\hbar : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator. Let C and Q be nonempty, closed and convex subsets of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. The formalism $\bar{x} \in C$ such that $\hbar\bar{x} \in Q$ is referred as the split convex feasibility problem (SCFP) where as the set $\omega := C \cap \hbar^{-1}(Q) = \{\bar{x} \in C : \hbar\bar{x} \in Q\}$ denotes the corresponding solutions of SCFP.

In the sequel, we recall the indicator function b_C associated with the set C as

$$b_C(\bar{x}) := \begin{cases} 0, \ \bar{x} \in C; \\ \infty, \ otherwise. \end{cases}$$

It is well-known that the proximal operator of b_C is the metric projection on C

$$prox_{b_C} \quad = \quad argmin_{\bar{p} \in C} \|\bar{p} - \bar{x}\| = P_C(\bar{x}).$$

Setting $B(\bar{x}) = \hbar^*(Id - P_Q)\hbar\bar{x}$, where P_Q is the metric projection onto Q and $A(\bar{x}) = prox_{b_C}(\bar{x}) = \partial_{b_C}(\bar{x})$ then the SCFP has the inclusion structure as defined in (1.1). Since B is ρ-Lipschitz continuous, where $\rho = \|\hbar\|^2 = 1$ and A is maximal monotone, (see [8]), therefore, we compute the SCFP from the following result:

Theorem 4.4. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces and let $\hbar : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator. Let C and Q be nonempty, closed and convex subsets of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Assume that $\Gamma = \omega \cap \bigcap_{i=1}^N Fix(S_i) \neq \emptyset$ and (ξ_n) is a bounded real sequence. For given $x_0, x_1 \in \mathcal{H}_1$, let the iterative sequence (x_n) be generated by*

$$\text{(4.22)} \quad \begin{cases} u_n = x_n + \xi_n(x_n - x_{n-1}); \\ v_n = P_C(Id - \mu_n \hbar^*(Id - P_Q)\hbar)u_n; \\ w_n = v_n - \mu_n((\hbar^*(Id - P_Q)\hbar)v_n - (\hbar^*(Id - P_Q)\hbar)u_n); \\ x_{n+1} = \alpha_n h(x_n) + (1 - \alpha_n - \beta_n)x_n + \beta_n(\frac{1}{N}\sum_{i=0}^{N-1}((1-\gamma_n)Id + \gamma_n S_i))w_n. \end{cases}$$

Assume that the following step size rule

$$\mu_{n+1} = \begin{cases} \min\{\frac{\sigma\|u_n - v_n\|}{\|(\hbar^*(Id-P_Q)\hbar)u_n - (\hbar^*(Id-P_Q)\hbar)v_n\|}, \mu_n\}, \\ \text{if } (\hbar^*(Id - P_Q)\hbar)u_n - (\hbar^*(Id - P_Q)\hbar)v_n \neq 0; \\ \mu_n, \ otherwise, \end{cases}$$

and the conditions (C1)-(C2) hold. Then the sequence (x_n) generated by (4.22) converges strongly to an element $\bar{x} = P_\Gamma \circ h(\bar{x})$.

Convex Minimization Problems. Let $\mathfrak{f} : \mathcal{H} \to \mathbb{R}$ be a convex differentiable function and $\mathfrak{g} : \mathcal{H} \to \mathbb{R}$ be a convex lower semicontinuous function defined on a real Hilbert space $\mathcal{H}$. We consider the following convex minimization problem of finding $\bar{x} \in \mathcal{H}$ such that

$$\text{(4.23)} \quad \mathfrak{f}(\bar{x}) + \mathfrak{g}(\bar{x}) = \min_{x \in \mathcal{H}} \{\mathfrak{f}(x) + \mathfrak{g}(x)\}.$$

In view of the Fermat's rule, the problem (4.23) is equivalent to the following problem of finding $\bar{x} \in \mathcal{H}$ such that

$$\text{(4.24)} \quad 0 \in \nabla\mathfrak{f}(\bar{x}) + \partial\mathfrak{g}(\bar{x}),$$

where the subdifferential $\partial\mathfrak{g}$ is a maximal monotone operator and the gradient $\nabla\mathfrak{f}$ is ρ-Lipschitz continuous [8, 32]. Assume that ω, the set of solutions of problem (4.23), is nonempty and setting $B := \nabla\mathfrak{f}$ and $A := \partial\mathfrak{g}$ in Theorem 3.2, we compute the following result:

Theorem 4.5. *Let $\mathfrak{f} : \mathcal{H} \to \mathbb{R}$ be a convex differentiable function such that the gradient $\nabla\mathfrak{f}$ is ρ-Lipschitz continuous and $\mathfrak{g} : \mathcal{H} \to \mathbb{R}$ be a convex lower semicontinuous function defined on a real*

Hilbert space $\mathcal{H}$. Assume that $\Gamma = \omega \cap \bigcap_{i=1}^{N} Fix(S_i) \neq \emptyset$ and (ξ_n) is a bounded real sequence. For given $x_0, x_1 \in \mathcal{H}$, let the iterative sequences (x_n) be generated by

$$(4.25)\quad \begin{cases} u_n = x_n + \xi_n(x_n - x_{n-1}); \\ v_n = J_{\mu_n}^{\partial \mathfrak{g}}(Id - \mu_n \nabla \mathfrak{f})u_n; \\ w_n = v_n - \mu_n(\nabla \mathfrak{f} v_n - \nabla \mathfrak{f} u_n); \\ x_{n+1} = \alpha_n h(x_n) + (1 - \alpha_n - \beta_n)x_n + \beta_n(\frac{1}{N}\sum_{i=0}^{N-1}((1-\gamma_n)Id + \gamma_n S_i))w_n. \end{cases}$$

Assume that the following step size rule

$$\mu_{n+1} = \begin{cases} \min\left\{\frac{\sigma\|u_n - v_n\|}{\|\nabla \mathfrak{f} u_n - \nabla \mathfrak{f} v_n\|}, \mu_n\right\}, \text{ if } \nabla \mathfrak{f} u_n - \nabla \mathfrak{f} v_n \neq 0; \\ \mu_n, \ otherwise, \end{cases}$$

and the conditions (C1)-(C2) hold. Then the sequence (x_n) generated by (4.25) converges strongly to an element $\bar{x} = P_\Gamma \circ h(\bar{x})$.

Application to Image Processing Problems: Images are a main source of human information about the world. The theory of image processing deals with the restoration and enhancement of the original noisy and blurred images. For a given matrix $\hbar \in \mathbb{R}^{n \times n}$ describing a blur operator and a given vector $w \in \mathbb{R}^n$ representing the blurred and noisy image, the task is to estimate the unknown original image $z \in \mathbb{R}^n$ via the following convex minimization problem:

$$(4.26)\quad \min_{z \in \mathbb{R}^n} \left\{\frac{1}{2}\|\hbar z - w\|_2^2 + \Bbbk\|z\|_1\right\},$$

where $\Bbbk > 0$ is a regularization parameter.

In connection with Theorem 4.4, we set $A(z) = \|z\|_1$, $B(z) = \frac{1}{2}\|\hbar z - w\|_2^2$ and $\Bbbk = 0.7875$. Also fix $\mu = 0.001$, $\xi_n = \frac{1}{(100*n+1)^2}$, $\alpha_n = \frac{1}{n}$, $\beta_n = \frac{1}{58*n+1}$. The quality of the the restored images are analyzed on the following scale of signal to noise ratio (SNR) defined as $SNR = 20\log_{10}\frac{\|z\|^2}{\|z - z_n\|^2}$, where z and z_n are the original and estimated images at iteration n, respectively. We compare the performance of the algorithms abbreviated as Thm. 4.4, $\xi_n \neq 0$, Thm. 4.4, $\xi_n = 0$ and Theorem 2 of Gibali et. al [21] on the test images (Mona Lisa and Cameraman) via the image restoration experiment for motion operator, respectively.

(A) Original image (B) Blurred and noisy image (C) Reconstructed image

FIGURE 1. (A) Original image (182 × 276) with a motion length 21 and an angle 31 (B) Blurred and noisy image, degraded by motion (C) Reconstructed image

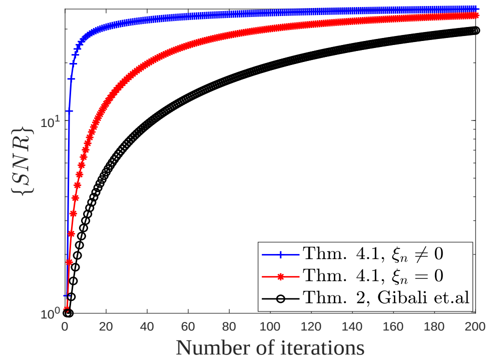


FIGURE 2. Comparison of Thm. 4.4, $\xi_n \neq 0$, Thm. 4.4, $\xi_n = 0$ and Theorem 2 of Gibali and Thong [21]

(A) Original image (B) Blurred and noisy image (C) Reconstructed image

FIGURE 3. (A) Original image (256 × 256)with Gaussian blur of size 9×9 and standard deviation $\sigma = 4$ (B) Blurred and noisy image, degraded by Gaussian (C) Reconstructed image

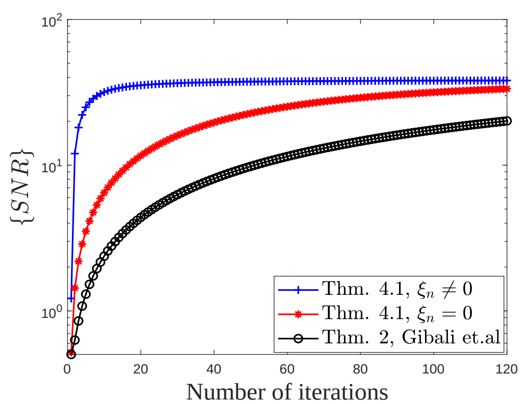


FIGURE 4. Comparison of Thm. 4.4, $\xi_n \neq 0$, Thm. 4.4, $\xi_n = 0$ and Theorem 2 of Gibali and Thong [21]

TABLE 1. The SNR in decibel(dB) values and average per iteration computation time of the two optimization algorithms

Algorithms	**Mona Lisa**		**Cameraman**	
	SNR	CPU(sec)	SNR	CPU(sec)
1.Thm. 4.22, $\xi_n \neq 0$	38.249952	12.943470	38.148891	10.433934
2.Thm. 4.22, $\xi_n = 0$	35.364615	13.002959	36.732162	15.617105
3.Thm. 2 of Gibali and Thong	29.491617	17.632131	28.380412	20.112824

It can be observed from Figure 2 and Figure 4 that the larger SNR values infer the better restored images. We can see from Table 1 that the Theorem 4.4 with $\xi_n \neq 0$ performs better as compared to the Theorem 4.4 with $\xi_n = 0$ and Theorem 2 of Gibali and Thong [21].

5. EXAMPLE AND NUMERICAL RESULTS

This section shows effectiveness to our algorithm by following given examples and numerical results.

Example 5.1. Let $\mathcal{H} = \mathbb{R}$,the set of all real numbers, with the inner product defined by $\langle x, y\rangle = xy$, for all $x, y \in \mathbb{R}$ and induced usual norm $|\cdot|$. For $\mu > 0$, we define three operators $h, A, B : \mathbb{R} \to \mathbb{R}$ as $h(x) = \frac{x}{8}$, $Ax = 4x$ and $Bx = 3x$ for all $x \in \mathbb{R}$. For each $i \in \{1, 2, \cdots, N\}$, let the operator $S_i : \mathbb{R} \to \mathbb{R}$ be defined by

$$S_i(x) = \begin{cases} -\frac{3x}{i}, x \in (-\infty, 1); \\ x, x \in (1, \infty). \end{cases}$$

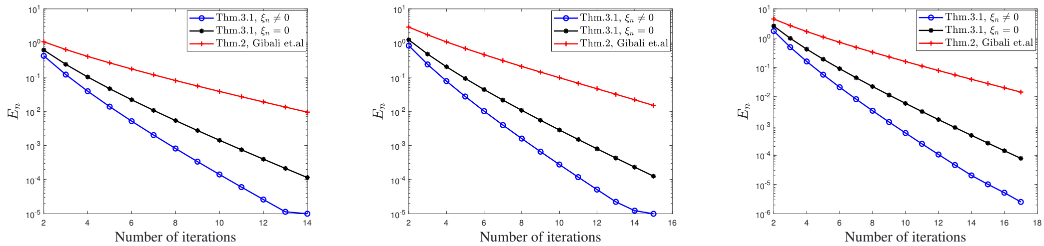
For all $x = x_0, x_1 \in \mathbb{R}$, then there exists unique sequence (x_n) generated by the iterative method (3.2) converges strongly to a point in $P_\Gamma \circ h(\bar{x})$.
Now, observe that, $h : \mathcal{H} \to \mathcal{H}$ is a contraction operator with constant $\lambda \in [0, 1)$, B a monotone and ρ-Lipschitz operator for some $\rho > 0$ and A a maximal monotone operator such that $(A+B)^{-1}(0) = \{0\}$. Note that S_i is a finite family of $\frac{3-i^2}{(3+i)^2}$-demimetric operators with $\bigcap_{i=1}^N Fix(S_i) = \{0\}$. Hence $\Gamma = (A + B)^{-1}(0) \cap \bigcap_{i=1}^N Fix(S_i)(0) = 0$.
In order to compute the numerical values of (x_n), we choose $\Theta = 0.5$, $\alpha_n = \frac{1}{n}$, $\beta_n = \frac{n}{2(n+1)}$, $\mu_1 = 7.45$, $\sigma = 0.785$.

Since $\begin{cases} \min\{\frac{1}{n^2\|x_n - x_{n-1}\|}, 0.5\} & if \quad x_n \neq x_{n-1}; \\ 0.5 & \text{otherwise.} \end{cases}$

Now, we provide a numerical test for a comparison between our accelerated Tseng type splitting method defined in (3.2) (i.e Thm. 3.2, $\xi_n \neq 0$), Tseng type splitting method (i.e Thm. 3.2, $\xi_n = 0$) and Theorem 2 of Gibali and Thong [21]. The stopping criteria is defined as $E_n = \|x_n - x_{n-1}\| < 10^{-5}$. The values of the sequence (3.2) in these cases have been computed for different choices of x_0 and x_1 in the following table:

TABLE 2. Numerical results for Example 5.1

	Thm.3.1,	$\xi_n \neq 0$	Thm.3.1,	$\xi_n = 0$	Thm. 2[21]	
No. of Choices	Iter.	CPU(s)	Iter.	CPU(s)	Iter.	CPU(s)
1.$x_0 = 3, x_1 = 2$	13	0.057912	17	0.061491	33	0.071583
2.$x_0 = -6, x_1 = 4$	14	0.050666	20	0.054804	35	0.061804
3.$x_0 = -1.7, x_1 = -8.3$	15	0.051221	22	0.055838	37	0.062315



(A) Choice 1 (B) Choice 2 (C) Choice 3

FIGURE 5. Comparison of Thm. 3.2, $\xi_n \neq 0$, $\xi_n = 0$ and Theorem 2 of Gibali and Thong

.

Now, we provide the numerical example for Theorem 4.4 to solve the SCFP in an infinite dimensional space $L_2([0, 2\pi])$.

Example 5.2. Let $\mathcal{H}_1 = L^2([0, 2\pi]) = \mathcal{H}_2$ with induced norm $\|x\| = (\int_0^{2\pi} |x(s)|^2 ds)^{\frac{1}{2}}$ and inner product $\langle x, y\rangle = \int_0^{2\pi} x(s)y(s)ds$, for all $x, y \in L^2([0, 2\pi])$. The feasible set C and Q are given by: $C = \{x \in \{x \in \mathcal{H}_1 : \int_0^{2\pi} x(s)ds \leq 1\}$ and now let the closed ball centered at sin $\in L^2([0, 2\pi]$ with radius 4, that $Q = \{x \in \{x \in \mathcal{H}_2 : \int_0^{2\pi} |x(s) - \sin(s)|^2 ds \leq 16\}$. Let $\hbar : L^2([0, 2\pi]) \to L^2([0, 2\pi])$ be a bounded linear operator such that $(\hbar x)(s) = x(s)$, for all $x \in L^2([0, 2\pi])$. Then $(\hbar^* x)(s) = x(s)$ and $\|\hbar\| = 1$. Further, $S_i = S$ for each $i = 1, 2, \cdots, N$, then the operator $S : \mathcal{H}_1 \to \mathcal{H}_1$ is defined by

$$S(x) = P_C(x) = \begin{cases} \frac{x}{\|x\|}, \|x\| > 1; \\ x, \|x\| \leq 1. \end{cases}$$

Consider the following problem:

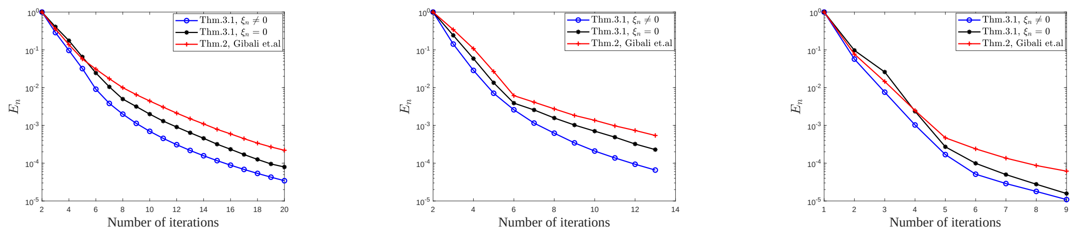
$$Find\ \bar{x} \in \Gamma = \omega \cap Fix(S) \neq \emptyset.$$

It is noted that ω is the convex feasibility problem is a problem of finding a point $\bar{x} \in \mathcal{H}$ such that $\bar{x} \in C \cap Q$. It is clear that S is an η-demimetric operator and $h : \mathcal{H}_1 \to \mathcal{H}_2$ is a contraction operator with constant $\lambda \in [0, 1)$ defined as $h(x)(s) = \frac{x(s)}{8}$ for all $x \in L^2([0, 2\pi])$, $s \in [0, 2\pi]$. Hence $\omega \cap Fix(S) = 0$. Choose $\alpha_n = \frac{1}{15 \times n}$. The values of the Thm. 4.4 with $\xi_n \neq 0$, Thm. 4.4 with $\xi_n = 0$ and Thm. 2 of Gibali and Thong [21] have been computed for different choices of $x_0(s)$ and $x_1(s),\ s \in [0, 2\pi]$ in the following table:
Choice 1. Choose $x_0 = 4(s^2 - 2s)e^{2s} + 2e^{4s}$, $x_1 = \frac{e^t}{\sin(s)}$
Choice 2. Choose $x_0 = (s^3 - 3s)\cos(4s) + 3e^{2s}$, $x_1 = (s^2 - e^s)\cos(s)$
Choice 3. Choose $x_0 = \frac{3\sin(s)}{8}$, $x_1 = 2e^s s^5$.

The tolerance plotting (E_n) against the Thm. 3.2-4.4 with $\xi_n \neq 0$, Thm. 3.2-4.4 with $\xi_n = 0$ and Thm. 2 of Gibali and Thong [21] for each choices in Tables 2-3 and has shown in Figures 5-6.
We can see from Tables 2-3 and Figures 5-6, that the Thm. 3.2-4.4 with $\xi_n \neq 0$ performs better as compared to the Thm. 3.2-4.4 with $\xi_n = 0$ and Thm. 2 of Gibali and Thong [21]. Elaborating the behaviour of these algorithms with respect to the Figures 5-6, the number

TABLE 3. Comparison of Theorem 4.4, $\xi_n \neq 0$, Theorem 4.4, $\xi_n = 0$ and Theorem 2 of Gibali and Thong [21]).

	No. of Iterations			CPU Time(Sec)		
	Choice 1	Choice 2	Choice 3	Choice 1	Choice 2	Choice 3
Thm.4.4,$\xi_n \neq 0$,	20	13	9	3.83664	2.98754	2.35670
Thm.4.4,$\xi_n = 0$,	29	21	17	4.38926	3.39264	2.79551
Thm. 2 [21]	46	37	32	5.08027	4.01928	3.01132



(A) Choice 1 (B) Choice 2 (C) Choice 3

FIGURE 6. Comparison of Thm. 4.4, $\xi_n \neq 0$, Thm. 4.4, $\xi_n = 0$ and Thm. 2 of Gibali and Thong [21]

.

of iterations required to converge to the common solution is expressed in Figures 5-6 (A, B, C). Summarizing these facts, we say that the Thm. 3.2, $\xi_n \neq 0$ exhibits a reduction in the tolerance, time and the number of iterations of the function as compared to the Thm. 3.2-4.4, $\xi_n = 0$ and Thm. 2 of Gibali and Thong [21].

Conclusion. In this paper, we have devised an accelerated visco-Cesáro means Tseng type splitting method for computing a common solution of a monotone inclusion problems and the FPP associated with an η-demimetric operator in Hilbert spaces. The theoretical framework of the algorithm has been strengthened with an appropriate numerical example. Moreover, this framework has also been implemented to various instances of the inverse problems. We would like to emphasize that the above mentioned problems occur naturally in many applications like as mentioned above image processing to illustrate the convergence, therefore, iterative algorithms are inevitable in this field of investigation. As a consequence, our theoretical framework constitutes an important topic of future research.

Acknowledgment. The authors wish to thank the anonymous referees for their comments and suggestions. The authors (Y. Arfat, P. Kumam and P. S. Ngiamsunthorn) acknowledge the financial support provided by the Center of Excellence in Theoretical and Computational Science (TaCS-CoE), KMUTT. Moreover, this project is funded by National Council of Thailand (NRCT) under Research Grants for Talented Mid-Career Researchers (Contract no. N41A640089). The author Y. Arfat was supported by the Petchra Pra Jom Klao Ph.D Research Scholarship from King Mongkut's University of Technology Thonburi, Thailand(Grant No.16/2562).

Competing interests. The authors declare that they have no competing interests.

Authors' contributions. All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

Availability of Data and Material. Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

REFERENCES

[1] Alvarez, F.; Attouch, H. An inertial proximal method for monotone operators via discretization of a nonlinear oscillator with damping. *Set-Valued Anal.* **9** (2001), 3–11.
[2] Arfat, Y.; Kumam. P.; Ngiamsunthorn. P. S. et al. An inertial based Forward-Backward algorithm for monotone inclusion problems and split mixed equilibrium problems in Hilbert spaces. *Adv. Differ. Equ.* **2020** (2020), 453.
[3] Arfat, Y.; Kumam, P.; Ngiamsunthorn, P. S. et al., Approximation results for split equilibrium problems and fixed point problems of nonexpansive semigroup in Hilbert spaces. *Adv. Differ. Equ.* **2020** (2020), 512.
[4] Arfat, Y.; Kumam, P.; Khan, M. A. A. et al., An inertially constructed forward-backward splitting algorithm in Hilbert spaces. *Adv. Differ. Equ.* **2021** (2021), 124.
[5] Arfat, Y.; Kumam, P.; Khan, M. A. A.; Ngiamsunthorn, P. S.; Kaewkhao, A. A parallel hybrid accelerated extragradient algorithm for pseudomonotone equilibrium, fixed point, and split null point problems. *Adv. Differ. Equ.* **2021** (2021), 364.
[6] Arfat, Y.; Kumam, P.; Ngiamsunthorn, P. S.; Khan, M. A. A. An accelerated projection based parallel hybrid algorithm for fixed point and split null point problems in Hilbert spaces. *Math. Meth. Appl. Sci.* (2021), 1–19. doi.org/10.1002/mma.7405.
[7] Arfat, Y.; Kumam, P.; Khan, M. A. A.; Ngiamsunthorn, P. S. Parallel shrinking inertial extragradient approximants for pseudomonotone equilibrium, fixed point and generalized split null point problem. *Ricerche Mat.* (2021). https://doi.org/10.1007/s11587-021-00647-4.
[8] Bauschke, H. H.; Combettes, P. L. *Convex Analysis and Monotone Operators Theory in Hilbert Spaces*. CMS Books in Mathematics, Springer, New York, (2011).
[9] Baillon, J. B. Un theorem de type ergodique pour les contractions non lineairs dans un e'spaces de Hilbert. *C.R. Acad. Sci. Paris Ser., A-B* **280** (1975), 1511–1541.
[10] Brézix, H. Chapitre, I. I. Operateurs maximaux monotones. *North-Holland Math. Stud.* **5** (1973), 19–51.
[11] Browder, F. E.; Petryshyn, W. V. Construction of fixed points of nonlinear mappings in Hilbert spaces. *J. Math. Anal. Appl.* **20** (1967), 197–228.
[12] Bruck, R. E. On the convex approximation property and the asymptotic behavior of Nonlinear contractions in Banach spaces. *Israel J. Math.* **38** (1981), 304–314.
[13] Byrne. C. A unified treatment of some iterative algorithms in signal processing and image reconstruction. *Inverse Probl.* **20** (2004), no. 1, 103–120.
[14] Censor, Y.; Elfving, T. A multiprojection algorithm using Bregman projections in a product space. *Numer. Algorithms* **8** (1994), 221–239.
[15] Combettes, P. L. The convex feasibility problem in image recovery. *Adv. Imaging Electron Phys.* **95** (1996), 155–453.
[16] Deepho, J.; Moreno, J. M.; Sitthithakerngkiet, K.; Kumam, P. Convergence analysis of hybrid projection with Ces/' aro mean method for the split equilibrium and general system of finite variational inequalities. *J. Comp. Appl. Math.* **318** (2017), 658–673.
[17] Douglas, J.; Rachford, H. H. On the numerical solution of the heat conduction problem in two and three space variables. *Trans. Amer. Math. Soc.* **82** (1956), 421–439.
[18] Duchi, J.; Shalev-Shwartz, S.; Singer, Y.; Chandra, T. Efficient projections onto the l1-ball for learning in high dimensions. *Proceedings of the 25 th International Conference on Machine Learning*, Helsinki, Finland (2008).
[19] Engl, H. W.; Hanke, M.; Neubauer, A. *Regularization of Inverse Problems*. Kluwer Academic Publishers, Dordrecht, (2000).
[20] Genel, A.; Lindenstrauss, J. An example concerning fixed points. *Israel J. Math.* **22** (1975), 81–86.
[21] Gibali, A.; Thong, D. V. Tseng type methods for solving inclusion problems and its applications. *Calcolo* **55** (2018), 49.
[22] Gibali, A. A new split inverse problem and an application to least intensity feasible solutions. *Pure. Appl. Func. Anal.* **2** (2017), 243–258.
[23] Gibali, A.; Reich, S.; Zalas, R. Outer approximation methods for solving variational inequalities in Hilbert space. *Optimization* **66** (2017), 417–437.
[24] Halpern, B. Fixed points of nonexpansive maps. *Bull. Amer. Math. Soc.* **73** (1967), 957–961.

[25] Harisa, S. A.; Khan, M. A. A.; Mumtaz, F.; Farid, N.; Morsy, A.; Nisar, K. S.; Ghaffar, A. Shrinking Cesáro means method for the split equilibrium and fixed point problems in Hilbert spaces. *Adv. Differ. Equ.* **2020** (2020), 345.
[26] Hirano, N.; Takahashi, W. Nonlinear ergodic theorems for nonexpansive mappings in Hilbert spaces. *Kodai Math. J.* **2** (1979), 11–25.
[27] Lions, P. L.; Mercier, B. Splitting algorithms for the sum of two nonlinear operators. *SIAM J. Numer. Anal.* **16** (1979), 964–979.
[28] Maingé, F. A hybrid extragradient-viscosity method for monotone operators and fixed point problems. *SIAM J. Control Optim.* **47** (2008), 1499–1515.
[29] Moudafi, A. Viscosity Approximation Methods for Fixed-Points Problems. *J. Math. Anal. Appl.* **241** (2000), No. 1, 46–55.
[30] Pasty, G. B. Ergodic Convergence to a Zero of the Sum of Monotone Operators in Hilbert Space. *J. Math. Anal. Appl.* **72** (1979), 383–390.
[31] Polyak, B. T. *Introduction to Optimization, Optimization Software*. New York (1987).
[32] Rockafellar, R. T. On the maximality of sums of nonlinear monotone operators. *Trans. Amer. Math. Soc.* **149** (1970), 75–88.
[33] Takahashi, W. The split common fixed point problem and the shrinking projection method in Banach spaces. *J. Conv. Anal.* **24** (2017), 1015–1028.
[34] Takahashi, W.; Wen, C. F.; Yao, J. C. The shrinking projection method for a finite family of demimetric mappings with variational inequality problems in a Hilbert space. *Fixed Point Theory* **19** (2018), 407–419.
[35] Tseng, P. A modified forward-backward splitting method for maximal monotone mappings. *SIAM J. Control. Optim.* **38** (2000), 431–446.
[36] Vogel, C. R. *Computational Methods for Inverse Problems*. SIAM, Philadelphia, (2002).
[37] Xu, H. K. Iterative algorithms for nonlinear operators. *J. Lond. Math. Soc.* **66** (2002), 240–256.

[1]KMUTT Fixed Point Research Laboratory
KMUTT-Fixed Point Theory and Applications Research Group
Department of Mathematics
Faculty of Science
King Mongkut's University of Technology Thonburi (KMUTT)
126 Pracha-Uthit Road, Bang Mod, Thung Khru, Bangkok 10140, Thailand
Email address: yasir.arfat@mail.kmutt.ac.th
Email address: poom.kum@kmutt.ac.th

[2]Center of Excellence in Theoretical and Computational Science (TaCS-CoE)
SCL 802 Fixed Point Laboratory
Science Laboratory Building
King Mongkut's University of Technology Thonburi (KMUTT)
126 Pracha-Uthit Road, Bang Mod, Thung Khru, Bangkok 10140, Thailand
Email address: poom.kum@kmutt.ac.th

[3]Research Center for Interneural Computing
China Medical University Hospital
China Medical University, Taichung, 40402, Taiwan
Email address: poom.kum@kmutt.ac.th

[4]Department of Mathematics
COMSATS University Islamabad
Lahore Campus, Lahore 54000, Pakistan
Email address: itsakb@hotmail.com

[5]Department of Mathematics
Faculty of Science
King Mongkut's University of Technology Thonburi (KMUTT)
126 Pracha-Uthit Road, Bang Mod, Thung Khru, Bangkok 10140, Thailand
Email address: parinya.san@kmutt.ac.th