CARPATHIAN J. MATH.
Volume **38** (2022), No. 2,
Pages 315 - 326

Online version at https://semnul.com/carpathian/
Print Edition: ISSN 1584 - 2851; Online Edition: ISSN 1843 - 4401
DOI: https://doi.org/10.37193/CJM.2022.02.04

Ćirić type cyclic contractions and their best cyclic periodic points

MUSTAFA ASLANTAS, HAKAN SAHIN and ISHAK ALTUN

ABSTRACT. In the present paper, by introducing a new notion named as nonunique cyclic contractions, we give some best proximity point results for such mappings. Then, we indicate the shortcoming of the concept of best periodic proximity point which is defined for cyclic mapping by giving a simple example. To overcome this deficiency, we give a more suitable definition named as best cyclic periodic point. Finally, we obtain some best cyclic periodic point theorems, including the famous periodic point result of Ćirić [8], for nonunique cyclic contractions. We also provide some illustrative and comparative examples to support our results.

1. INTRODUCTION

Metric fixed point theory started with a result known as Banach contraction principle in 1922 [4]. Then, a great number of results has been proved to obtain existence and uniqueness of fixed points in this field [12, 14, 19]. However, especially in nonlinear systems which is one of the important application areas of fixed point theory the solution may not be unique. Therefore, some results were obtained by Ćirić including two concepts so called nonunique fixed point and periodic point [8]. In these results, Ćirić used the following contraction conditions to obtain fixed point and periodic point results for the self mapping $\mathcal{F}$ on the metric space $(\mho, \rho)$: for all $\varsigma, \xi \in \mho$

$$P(\varsigma, \xi) - R(\varsigma, \xi) \leq k\rho(\varsigma, \xi) \tag{1.1}$$

and

$$0 < \rho(\varsigma, \xi) < \varepsilon \ implies \ P(\varsigma, \xi) \leq k\rho(\varsigma, \xi) \tag{1.2}$$

respectively, where k in $[0, 1)$, $\varepsilon > 0$,

$$P(\varsigma, \xi) = \min \{\rho(\mathcal{F}\varsigma, \mathcal{F}\xi), \rho(\varsigma, \mathcal{F}\varsigma), \rho(\xi, \mathcal{F}\xi)\}$$

and

$$R(\varsigma, \xi) = \min \{\rho(\varsigma, \mathcal{F}\xi), \rho(\xi, \mathcal{F}\varsigma)\}.$$

Moreover, the mapping $\mathcal{F}$ may not be continuous unlike existing many results in the literature [3, 16]. Because of these reasons, Ćirić's results have been studied to generalize and extend in different ways [2, 9, 13, 17, 18].

On the other hand, recently, a different generalization of fixed point theory has been obtained by taking into account nonself mappings. Consider the nonself mapping $\mathcal{F} : \wp \to \Re$ where $\wp, \Re \subseteq \mho$. If $\wp \cap \Re = \emptyset$, then $\mathcal{F}$ cannot have a fixed point. That is, there is no point in $\mho$ such that $\rho(\varsigma, \mathcal{F}\varsigma) = 0$. In this case, since $\rho(\varsigma, \mathcal{F}\varsigma) \geq \rho(\wp, \Re)$ for each point ς in $\mho$, it makes sense to search the existence of a point ς such that $\rho(\varsigma, \mathcal{F}\varsigma) = \rho(\wp, \Re)$. This point is called a best proximity point which was introduced Basha and Veeramani [5]. It can be easily seen that if we take $\wp = \Re = \mho$, every best proximity point result becomes

Received: 11.03.2021. In revised form: 21.07.2021. Accepted: 28.07.2021
2010 *Mathematics Subject Classification.* 54H25, 47H10.
Key words and phrases. *best proximity point, best cyclic periodic point, nonunique contraction.*
Corresponding author: Ishak Altun; ishakaltun@yahoo.com

a fixed point result. Therefore, there are many authors studying to show the existence of best proximity point [20, 21]. In this sense, defining the cyclic contraction mapping, Eldred and Veeramani obtained a best proximity point theorem for such mappings [10]. Thus, they generalized a number of fixed point and best proximity point results. After that, the result of Eldred and Veeramani has been extended in various ways [1, 11]. Now, we recall some basic definitions and properties which will be used in the rest of paper:

Definition 1.1. Let $\mho$ be a nonempty set, $\varsigma_0 \in \mho$ and $\mathcal{F} : \mho \to \mho$ be a mapping. Then, the set $O_{\mathcal{F}}(\varsigma_0) = \{\varsigma_0, \mathcal{F}\varsigma_0, \mathcal{F}^2\varsigma_0, \cdots\}$ is said to be orbit of ς_0.

Definition 1.2 ([7])**.** Let $(\mho, \rho)$ be a metric space and $\mathcal{F} : \mho \to \mho$ be a mapping. If $\mathcal{F}\varsigma_n \to \mathcal{F}\varsigma^*$for every sequence $\{\varsigma_n\}$ in $O_{\mathcal{F}}(\varsigma)$ for all $\varsigma \in \mho$ such that $\varsigma_n \to \varsigma^*$, then $\mathcal{F}$ is called an orbitally continuous mapping at $\varsigma^* \in \mho$. If $\mathcal{F}$ is orbitally continuous at every point of $\mho$, then $\mathcal{F}$ is said to be orbitally continuous on $\mho$.

Definition 1.3. Let $(\mho, \rho)$ be a metric space, $\varsigma \in \mho$ and $\mathcal{F} : \mho \to \mho$ be a mapping. Then the point ς is said to be a periodic point of $\mathcal{F}$ with period $m \in \mathbb{N}$, if $\mathcal{F}^m\varsigma = \varsigma$ where $\mathcal{F}^0\varsigma = \varsigma$ and $\mathcal{F}^m\varsigma = \mathcal{F}\mathcal{F}^{m-1}\varsigma$.

Definition 1.4 ([15])**.** Let $(\mho, \rho)$ be a metric space, $\wp$, $\Re$ be nonempty subsets of $\mho$ and $\mathcal{F} : \wp \cup \Re \to \wp \cup \Re$ be a mapping. If the mapping $\mathcal{F}$ satisfies $\mathcal{F}(\wp) \subseteq \Re$ and $\mathcal{F}(\Re) \subseteq \wp$, then it is called a cyclic mapping.

Definition 1.5 ([10])**.** Let $(\mho, \rho)$ be a metric space, $\wp$, $\Re$ be nonempty subsets of $\mho$ and $\mathcal{F} : \wp \cup \Re \to \wp \cup \Re$ be a cyclic mapping. If there exists a k in $(0, 1)$ such that

$$\rho(\mathcal{F}\varsigma, \mathcal{F}\xi) \leq k\rho(\varsigma, \xi) + (1-k)\rho(\wp, \Re)$$

for all $\varsigma \in \wp$ and $\xi \in \Re$, then $\mathcal{F}$ is called a cyclic contraction mapping.

In this paper, we generalize some results in literature by combining the contractions (1.1) and (1.2) of some nonunique fixed point and periodic point results defined by Ćirić [8] with the cyclic contraction mappings for some best proximity point results given by Eldred and Veeramani [10]. Firstly, we give a definition of nonunique cyclic contraction mapping. Then, we obtain some best proximity point results for such mappings. We also introduce another notion, the best cyclic periodic point, to prove some periodic point results. Finally, we present some illustrative and comparative examples to support and show the meaningful of our results.

2. Best Proximity point results

Let's start to this section with the following definition:

Definition 2.6. Let $(\mho, \rho)$ be a metric space, $\wp$, $\Re$ be nonempty subsets of $\mho$ and $\mathcal{F} : \wp \cup \Re \to \wp \cup \Re$ be a cyclic mapping. If there exists a k in $[0, 1)$ such that

$$P(\varsigma, \xi) - R(\varsigma, \xi) \leq k\rho(\varsigma, \xi) + (1-k)\rho(\wp, \Re) \tag{2.3}$$

for all $\varsigma \in \wp$ and $\xi \in \Re$, then $\mathcal{F}$ is called a nonunique cyclic contraction mapping.

Proposition 2.1. *Let $(\mho, \rho)$ be a metric space, $\wp$, $\Re$ be nonempty subsets of $\mho$ and $\mathcal{F} : \wp \cup \Re \to \wp \cup \Re$ be a nonunique cyclic contraction mapping. For any sequence $\{\varsigma_n\}$ defined by $\varsigma_{n+1} = \mathcal{F}\varsigma_n$ with initial point $\varsigma_0 \in \wp \cup \Re$, if there exists $n_0 \in \mathbb{N}$ such that*

$$\rho(\varsigma_{n_0}, \varsigma_{n_0+1}) \leq \rho(\varsigma_{n_0+1}, \varsigma_{n_0+2}),$$

then $\mathcal{F}$ has a best proximity point in $\wp \cup \Re$.

Proof. Without loss of generality assume $\varsigma_0 \in \wp$. Since $\mathcal{F}$ is a cyclic mapping, we have $\{\varsigma_{2n}\} \subseteq \wp$ and $\{\varsigma_{2n+1}\} \subseteq \Re$ for constructed the sequence $\{\varsigma_n\}$ by $\varsigma_{n+1} = \mathcal{F}\varsigma_n$. Assume that there exists $n_0 \in \mathbb{N}$ such that

$$\rho(\varsigma_{n_0}, \varsigma_{n_0+1}) \leq \rho(\varsigma_{n_0+1}, \varsigma_{n_0+2}).$$

Now we consider the following cases:

Case 1. Let n_0 be odd. Since $\mathcal{F}$ is a nonunique cyclic contraction, for $\varsigma = \varsigma_{n_0+1}$ and $\xi = \varsigma_{n_0}$, we have

$$P(\varsigma_{n_0+1}, \varsigma_{n_0}) - R(\varsigma_{n_0+1}, \varsigma_{n_0}) \leq k\rho(\varsigma_{n_0+1}, \varsigma_{n_0}) + (1-k)\rho(\wp, \Re)$$

which implies that

$$\min\{\rho(\varsigma_{n_0+1}, \varsigma_{n_0+2}), \rho(\varsigma_{n_0}, \varsigma_{n_0+1})\} \leq k\rho(\varsigma_{n_0}, \varsigma_{n_0+1}) + (1-k)\rho(\wp, \Re).$$

Because of $\rho(\varsigma_{n_0}, \varsigma_{n_0+1}) \leq \rho(\varsigma_{n_0+1}, \varsigma_{n_0+2})$, we get

$$\rho(\varsigma_{n_0}, \varsigma_{n_0+1}) \leq k\rho(\varsigma_{n_0}, \varsigma_{n_0+1}) + (1-k)\rho(\wp, \Re)$$

and so,

$$(1-k)\rho(\varsigma_{n_0}, \varsigma_{n_0+1}) \leq (1-k)\rho(\wp, \Re).$$

Thus, we have

$$\rho(\varsigma_{n_0}, \varsigma_{n_0+1}) \leq \rho(\wp, \Re).$$

On the other hand, since $\rho(\wp, \Re) \leq \rho(\varsigma_{n_0}, \varsigma_{n_0+1})$, we conclude that

$$\rho(\varsigma_{n_0}, \mathcal{F}\varsigma_{n_0}) = \rho(\varsigma_{n_0}, \varsigma_{n_0+1}) = \rho(\wp, \Re).$$

So, ς_{n_0} is a best proximity point of $\mathcal{F}$.

Case 2. Let n_0 be even. In this case by taking $\varsigma = \varsigma_{n_0}$ and $\xi = \varsigma_{n_0+1}$ in the nonunique contractive condition, it can be shown ς_{n_0} is a best proximity point of $\mathcal{F}$. □

Remark 2.1. The sequence $\{\varsigma_n\}$ mentioned in Proposition 2.1 is called a Picard sequence in literature. Note that $\mathcal{F}$ has a best proximity point in $\wp \cup \Re$ under conditions of Proposition 2.1. Hence, for every Picard sequence $\{\varsigma_n\}$ in $\wp \cup \Re$, we investigate the inequality

$$\rho(\varsigma_{n+1}, \varsigma_{n+2}) < \rho(\varsigma_n, \varsigma_{n+1})$$

for all $n \in \mathbb{N}$ in the rest of the paper.

Proposition 2.2. *Let $(\mho, \rho)$ be a metric space, $\wp$, $\Re$ be nonempty subsets of $\mho$ and $\mathcal{F} : \wp \cup \Re \to \wp \cup \Re$ be a nonunique cyclic contraction mapping. Then, for every Picard sequence $\{\varsigma_n\}$ in $\wp \cup \Re$, we have*

$$\rho(\varsigma_n, \varsigma_{n+1}) \to \rho(\wp, \Re) \text{ as } n \to +\infty.$$

Proof. Without loss of the generality, we assume that ς_0 is an arbitrary point in $\wp$. Since $\mathcal{F}$ is a cyclic mapping, we have $\{\varsigma_{2n}\} \subseteq \wp$ and $\{\varsigma_{2n+1}\} \subseteq \Re$ for constructed the sequence $\{\varsigma_n\}$ by $\varsigma_{n+1} = \mathcal{F}\varsigma_n$. Since $\mathcal{F}$ is a nonunique cyclic contraction mapping, for $\varsigma = \varsigma_0$ and $\xi = \varsigma_1$, we have

$$P(\varsigma_0, \varsigma_1) - R(\varsigma_0, \varsigma_1) \leq k\rho(\varsigma_0, \varsigma_1) + (1-k)\rho(\wp, \Re)$$

and so,

$$\min\{\rho(\varsigma_1, \varsigma_2), \rho(\varsigma_0, \varsigma_1)\} \leq k\rho(\varsigma_0, \varsigma_1) + (1-k)\rho(\wp, \Re).$$

Thus, from Remark 2.1, we obtain

$$\rho(\varsigma_1, \varsigma_2) \leq k\rho(\varsigma_0, \varsigma_1) + (1-k)\rho(\wp, \Re).$$

Similarly, from the inequality (2.3) and Remark 2.1, we get

$$P(\varsigma_2, \varsigma_1) - R(\varsigma_2, \varsigma_1) \leq k\rho(\varsigma_1, \varsigma_2) + (1-k)\rho(\wp, \Re)$$

for $\varsigma = \varsigma_2$, $\xi = \varsigma_1$ and so,

$$\rho(\varsigma_2, \varsigma_3) \leq k\rho(\varsigma_1, \varsigma_2) + (1-k)\rho(\wp, \Re).$$

Continuing this process, we have

$$\rho(\varsigma_n, \varsigma_{n+1}) \leq k\rho(\varsigma_{n-1}, \varsigma_n) + (1-k)\rho(\wp, \Re).$$

for all $n \in \mathbb{N}$. Thus, we obtain

$$\begin{array}{rcl}
\rho(\wp, \Re) & \leq & \rho(\varsigma_n, \varsigma_{n+1}) \\
& \leq & k\rho(\varsigma_{n-1}, \varsigma_n) + (1-k)\rho(\wp, \Re) \\
& \leq & k\left(k\rho(\varsigma_{n-2}, \varsigma_{n-1}) + (1-k)\rho(\wp, \Re)\right) + (1-k)\rho(\wp, \Re) \\
& = & k^2\rho(\varsigma_{n-2}, \varsigma_{n-1}) + (1+k)(1-k)\rho(\wp, \Re) \\
& & \vdots \\
& \leq & k^n\rho(\varsigma_0, \varsigma_1) + (1+k+k^2+\cdots+k^{n-1})(1-k)\rho(\wp, \Re) \\
& = & k^n\rho(\varsigma_0, \varsigma_1) + (1-k^n)\rho(\wp, \Re)
\end{array}$$

for all $n \in \mathbb{N}$. Therefore, $\rho(\varsigma_n, \varsigma_{n+1}) \to \rho(\wp, \Re)$ as $n \to +\infty$. □

Theorem 2.1. *Let $(\mho, \rho)$ be a metric space, $\wp$, $\Re$ be nonempty subsets of $\mho$ and $\mathcal{F} : \wp \cup \Re \to \wp \cup \Re$ be a nonunique cyclic contraction mapping. Then we have the following:*

(i) if the sequence $\{\varsigma_{2n}\}$ has a convergent subsequence in $\wp$ for every Picard sequence $\{\varsigma_n\}$ with the initial point $\varsigma_0 \in \wp$ and $f : \wp \to \mathbb{R}$ defined by $f(\varsigma) = \rho(\varsigma, \mathcal{F}\varsigma)$ for all $\varsigma \in \wp$ is lower semicontinuous, then $\mathcal{F}$ has a best proximity point in $\wp$.

(ii) if the sequence $\{\varsigma_{2n}\}$ has a convergent subsequence in $\Re$ for every Picard sequence $\{\varsigma_n\}$ with the initial point $\varsigma_0 \in \Re$ and $f : \Re \to \mathbb{R}$ defined by $f(\varsigma) = \rho(\varsigma, \mathcal{F}\varsigma)$ for all $\varsigma \in \Re$ is lower semicontinuous, then $\mathcal{F}$ has a best proximity point in $\Re$.

Proof. Assume that the condition (i) holds. Let $\varsigma_0 \in \wp$ be an arbitrary point and consider the Picard sequence $\{\varsigma_n\}$ with the initial point ς_0. Because of the condition (i), there exists a subsequence $\{\varsigma_{2n_k}\}$ of $\{\varsigma_{2n}\}$ such that $\varsigma_{2n_k} \to \varsigma^* \in \wp$ as $k \to +\infty$. Moreover, since $f(\varsigma) = \rho(\varsigma, \mathcal{F}\varsigma)$ is lower semicontinuous and from (2.2), we have

$$\begin{array}{rcl}
\rho(\wp, \Re) & \leq & \rho(\varsigma^*, \mathcal{F}\varsigma^*) \\
& = & f(\varsigma^*) \\
& \leq & \lim\limits_{k\to+\infty} \inf f(\varsigma_{2n_k}) \\
& = & \lim\limits_{k\to+\infty} \inf \rho(\varsigma_{2n_k}, \mathcal{F}\varsigma_{2n_k}) \\
& = & \rho(\wp, \Re)
\end{array}$$

Hence, $\rho(\varsigma^*, \mathcal{F}\varsigma^*) = \rho(\wp, \Re)$ and so, ς^* is a best proximity point of $\mathcal{F}$. Note that, if we assume that the condition (ii) holds, then choosing initial point $\varsigma_0 \in \Re$, we show that $\mathcal{F}$ has a best proximity point in $\Re$ by the similar way as above. □

Theorem 2.2. *Let $(\mho, \rho)$ be a metric space, $\wp$, $\Re$ be nonempty subsets of $\mho$ and $\mathcal{F} : \wp \cup \Re \to \wp \cup \Re$ be an orbitally continuous nonunique cyclic contraction mapping. Then we have the following:*

(i) if the sequence $\{\varsigma_{2n}\}$ has a convergent subsequence in $\wp$ for every Picard sequence $\{\varsigma_n\}$ with the initial point $\varsigma_0 \in \wp$, then $\mathcal{F}$ has a best proximity point in $\wp$.

(ii) if the sequence $\{\varsigma_{2n}\}$ has a convergent subsequence in $\Re$ for every Picard sequence $\{\varsigma_n\}$ with the initial point $\varsigma_0 \in \Re$, then $\mathcal{F}$ has a best proximity point in $\Re$.

Proof. Assume that the condition (i) holds. Let $\varsigma_0 \in \wp$ be an arbitrary point and consider the Picard sequence $\{\varsigma_n\}$ with the initial point ς_0. Then, there exists a subsequence $\{\varsigma_{2n_k}\}$ of $\{\varsigma_{2n}\}$ such that $\varsigma_{2n_k} \to \varsigma^* \in \wp$ as $k \to +\infty$. Since $\mathcal{F}$ is orbitally continuous, we have $\varsigma_{2n_k+1} = \mathcal{F}\varsigma_{2n_k} \to \mathcal{F}\varsigma^*$as $k \to +\infty$. Thus, using the Proposition 2.2, we obtain

$$\begin{aligned} \rho(\wp,\Re) &\leq \rho(\varsigma^*,\mathcal{F}\varsigma^*) \\ &= \lim_{k\to+\infty} \rho(\varsigma_{2n_k},\varsigma_{2n_k+1}) \\ &= \rho(\wp,\Re) \end{aligned}$$

and so, ς^* is a best proximity point of $\mathcal{F}$. Note that, if the condition (ii) holds, then it can be shown that $\mathcal{F}$ has a best proximity point in $\Re$. □

Example 2.1. Let $\mho = \mathbb{R}$ endowed with the usual metric ρ. Let's consider the sets

$$\wp = \{-\frac{1}{2^n} : n \in \mathbb{N}\} \cup \{0\}$$

and

$$\Re = \{1 + \frac{1}{2^n} : n \in \mathbb{N}\} \cup \{1\}.$$

Then, $\rho(\wp,\Re) = 1$. Define a mapping $\mathcal{F} : \wp \cup \Re \to \wp \cup \Re$ by

$$\mathcal{F}\varsigma = \begin{cases} 1+\frac{1}{2^{n+1}} & , \quad \varsigma = -\frac{1}{2^n}, n \in \mathbb{N} \\ -\frac{1}{2^{n+1}} & , \quad \varsigma = 1+\frac{1}{2^n}, n \in \mathbb{N} \\ 1 & , \quad \varsigma = 0 \\ 0 & , \quad \varsigma = 1 \end{cases}.$$

In this case, $\mathcal{F}$ is an orbitally continuous mapping. Now by investigated the following cases we show that $\mathcal{F}$ is a nonunique cyclic contraction mapping for $k = \frac{1}{2}$:

Case 1: Let $\varsigma = -\frac{1}{2^n}$, $\xi = 1 + \frac{1}{2^m}$ with $m > n$. Then, we have

$$\begin{aligned} P(\varsigma,\xi) - R(\varsigma,\xi) &\leq P(\varsigma,\xi) \\ &= \min\{\rho(\mathcal{F}\varsigma,\mathcal{F}\xi),\rho(\varsigma,\mathcal{F}\varsigma),\rho(\xi,\mathcal{F}\xi)\} \\ &= \min\left\{1+\frac{1}{2^{n+1}}+\frac{1}{2^{m+1}}, 1+\frac{1}{2^n}+\frac{1}{2^{n+1}}, 1+\frac{1}{2^m}+\frac{1}{2^{m+1}}\right\} \\ &= 1+\frac{1}{2^m}+\frac{1}{2^{m+1}} \\ &\leq 1+\frac{1}{2^{n+1}}+\frac{1}{2^{m+1}} \\ &= k\rho(\varsigma,\xi)+(1-k)\rho(\wp,\Re). \end{aligned}$$

Case 2: Let $\varsigma = -\frac{1}{2^n}$, $\xi = 1 + \frac{1}{2^n}$. Then, we have

$$\begin{aligned} P(\varsigma,\xi) - R(\varsigma,\xi) &\leq P(\varsigma,\xi) \\ &= \min\{\rho(\mathcal{F}\varsigma,\mathcal{F}\xi),\rho(\varsigma,\mathcal{F}\varsigma),\rho(\xi,\mathcal{F}\xi)\} \\ &= \min\left\{1+\frac{1}{2^{n+1}}+\frac{1}{2^{n+1}}, 1+\frac{1}{2^n}+\frac{1}{2^{n+1}}\right\} \\ &= 1+\frac{1}{2^n} \\ &= k\rho(\varsigma,\xi)+(1-k)\rho(\wp,\Re) \end{aligned}$$

Case 3: Let $\varsigma = -\frac{1}{2^n}$, $\xi = 1 + \frac{1}{2^m}$ with $m < n$. Then, we have

$$\begin{aligned}
P(\varsigma,\xi) - R(\varsigma,\xi) &\leq P(\varsigma,\xi) \\
&= \min\{\rho(\mathcal{F}\varsigma,\mathcal{F}\xi),\rho(\varsigma,\mathcal{F}\varsigma),\rho(\xi,\mathcal{F}\xi)\} \\
&= \min\left\{1+\frac{1}{2^{n+1}}+\frac{1}{2^{m+1}}, 1+\frac{1}{2^n}+\frac{1}{2^{n+1}}, 1+\frac{1}{2^m}+\frac{1}{2^{m+1}}\right\} \\
&= 1+\frac{1}{2^n}+\frac{1}{2^{n+1}} \\
&\leq 1+\frac{1}{2^{m+1}}+\frac{1}{2^{n+1}} \\
&= k\rho(\varsigma,\xi)+(1-k)\rho(\wp,\Re).
\end{aligned}$$

Case 4: If $\varsigma \in \wp$, $\xi \in \Re$ with $\{\varsigma,\xi\} \cap \{0,1\} \neq \emptyset$, then we have

$$P(\varsigma,\xi) - R(\varsigma,\xi) \leq 1 \leq k\rho(\varsigma,\xi)+(1-k)\rho(\wp,\Re).$$

Finally, the sequence $\{\varsigma_{2n}\}$ is convergent in $\wp \cup \Re$ for every Picard sequence $\{\varsigma_n\}$ with the initial point $\varsigma_0 \in \wp \cup \Re$. Therefore, all hypothesis of Theorem 2.2 are satisfied and so $\mathcal{F}$ has a best proximity point in $\wp \cup \Re$.

If we assume that $\wp$ or $\Re$ are compact subset of $\mho$ in Theorem 2.2, then we obtain the following corollary:

Corollary 2.1. *Let $(\mho,\rho)$ be a metric space, $\wp$ and $\Re$ be nonempty subsets of $\mho$ where $\wp$ or $\Re$ is a compact. If $\mathcal{F} : \wp \cup \Re \to \wp \cup \Re$ is an orbitally continuous nonunique cyclic contraction mapping, then $\mathcal{F}$ has a best proximity point in $\wp \cup \Re$.*

Taking $\wp = \Re = \mho$ in Theorem 2.2 and Corollary 2.1, respectively, we have following fixed point results:

Corollary 2.2. *Let $(\mho,\rho)$ be a metric space and $\mathcal{F} : \mho \to \mho$ be an orbitally continuous mapping. If there exists a k in $[0,1)$ such that*

$$\min\{\rho(\mathcal{F}\varsigma,\mathcal{F}\xi),\rho(\varsigma,\mathcal{F}\varsigma),\rho(\xi,\mathcal{F}\xi)\} - \min\{\rho(\varsigma,\mathcal{F}\xi),\rho(\xi,\mathcal{F}\varsigma)\} \leq k\rho(\varsigma,\xi)$$

for all $\varsigma,\xi \in \mho$ and every Picard sequence in $\mho$ has a convergent subsequence, then $\mathcal{F}$ has a fixed point in $\mho$.

If we take $\wp = \Re = \mho$ in Theorem 2.2, then we can show that every Picard sequence $\{\varsigma_n\}$ in $\mho$ is a Cauchy sequence. Hence, by accepting the orbitally completeness of $\mho$, we have every Picard sequence in $\mho$ has a convergent subsequence. Therefore we obtain the following corollary which is actually main result of Ćirić [8].

Corollary 2.3. *Let $(\mho,\rho)$ be an orbitally complete metric space and $\mathcal{F} : \mho \to \mho$ be an orbitally continuous mapping. If there exists a k in $[0,1)$ such that*

$$\min\{\rho(\mathcal{F}\varsigma,\mathcal{F}\xi),\rho(\varsigma,\mathcal{F}\varsigma),\rho(\xi,\mathcal{F}\xi)\} - \min\{\rho(\varsigma,\mathcal{F}\xi),\rho(\xi,\mathcal{F}\varsigma)\} \leq k\rho(\varsigma,\xi)$$

for all $\varsigma,\xi \in \mho$, then $\mathcal{F}$ has a fixed point in $\mho$.

3. Best cyclic periodic point

In this section, we investigate some periodic point results for cyclic mappings which satisfies inequality (1.2). Before we recall the definition of best periodic proximity point for cyclic mappings by given Chiming and Lin [6]:

Definition 3.7. Let $(\mho, \rho)$ be a metric space, $\wp$, $\Re$ be nonempty subsets of $\mho$ and $\mathcal{F} : \wp \cup \Re \to \wp \cup \Re$ be a cyclic mapping. If there exists a $z \in \wp \cup \Re$ such that

$$\rho(z, \mathcal{F}^{2q+1} z) = \rho(\wp, \Re)$$

for some $q \in \mathbb{N}$, then z is called a best periodic proximity point of $\mathcal{F}$.

In fact, best proximity point theory has emerged by considering nonself mappings in fixed point theory. Therefore, in case of $\wp = \Re = \mho$, the concepts in the best proximity point theory coincide with their counterparts in the fixed point theory. For example, the concepts of fixed point and best proximity point coincide with each other for a self mappings. However, according to Definition 3.7, every periodic point is not a best periodic proximity point for a self mapping. Indeed, let $\mho = \mathbb{R} \backslash \{0\}$ endowed with the usual metric ρ. Define a mapping $\mathcal{F} : \mathbb{R} \to \mathbb{R}$ by $\mathcal{F}\varsigma = -\varsigma$ for all $\varsigma \in \mho$. In this case, for all $\varsigma \in \mho$ and $n \in \mathbb{N}$, we have $\rho(\varsigma, \mathcal{F}^{2n}\varsigma) = 0$, that is, each point in $\mho$ is a periodic point of $\mathcal{F}$. But, $\rho(\varsigma, \mathcal{F}^{2n+1}\varsigma) \neq 0$ for all $\varsigma \in \mho$ and $n \in \mathbb{N}$, hence $\mathcal{F}$ has no best periodic proximity point.

To overcome this problem, we introduce a new concept called best cyclic periodic point of a cyclic mapping by modifying the Definition 3.7:

Definition 3.8. Let $(\mho, \rho)$ be a metric space, $\wp$, $\Re$ be nonempty subsets of $\mho$ and $\mathcal{F} : \wp \cup \Re \to \wp \cup \Re$ be a cyclic mapping. If there exists a $z \in \wp \cup \Re$ such that

$$\rho(z, \mathcal{F}^q z) = \begin{cases} 0 & , \quad q \in \mathbb{Z}_e^+ \\ \\ \rho(\wp, \Re) & , \quad q \in \mathbb{Z}_o^+ \end{cases}$$

for some positive integer q, where $\mathbb{Z}_e^+$ and $\mathbb{Z}_o^+$ are the sets of all positive even and odd integers, respectively, then z is called a best cyclic periodic point of $\mathcal{F}$.

Note that, if we take $\wp = \Re = \mho$ in Definition 3.8, then best cyclic periodic point become a periodic point of $\mathcal{F}$.

Now, we give main result of this section as follows:

Theorem 3.3. *Let $(\mho, \rho)$ be a metric space, $\wp$, $\Re$ be nonempty subsets of $\mho$, $\mathcal{F} : \wp \cup \Re \to \wp \cup \Re$ be an orbitally continuous cyclic mapping and $\varepsilon > 0$. Assume, there exists a k in $[0, 1)$ such that, for all $(\varsigma, \xi) \in \wp^2 \cup \Re^2$*

$$0 < \rho(\varsigma, \xi) < \varepsilon \text{ implies } P(\varsigma, \xi) \leq k\rho(\varsigma, \xi) \tag{3.4}$$

and for all $(\varsigma, \xi) \in \wp \times \Re$

$$\rho(\wp, \Re) < \rho(\varsigma, \xi) < \rho(\wp, \Re) + \varepsilon \text{ implies } P(\varsigma, \xi) \leq k\rho(\varsigma, \xi) + (1 - k)\rho(\wp, \Re). \tag{3.5}$$

Then $\mathcal{F}$ has a best cyclic periodic point in $\wp \cup \Re$ provided that one of the following holds:

(i) $K_\varepsilon^o \neq \emptyset$ and there exists an $\varsigma_0 \in \wp \cup \Re$ satisfying $\rho(\varsigma_0, \mathcal{F}^{\min K_\varepsilon^o}\varsigma_0) < \rho(\wp, \Re) + \varepsilon$ such that the Picard sequence $\{\varsigma_n\}$ with the initial point ς_0 has a convergent subsequence in $\wp \cup \Re$, where

$$K_\varepsilon^o = \{q \in \mathbb{Z}_o^+ : \rho(\varsigma, \mathcal{F}^q\varsigma) < \rho(\wp, \Re) + \varepsilon \text{ for some } \varsigma \in \wp \cup \Re\}.$$

(ii) $K_\varepsilon^e \neq \emptyset$ and there exists an $\varsigma_0 \in \wp \cup \Re$ satisfying $\rho(\varsigma_0, \mathcal{F}^{\min K_\varepsilon^e}\varsigma_0) < \varepsilon$ such that the Picard sequence $\{\varsigma_n\}$ with the initial point ς_0 has a convergent subsequence in $\wp \cup \Re$, where

$$K_\varepsilon^e = \{q \in \mathbb{Z}_e^+ : \rho(\varsigma, \mathcal{F}^q\varsigma) < \varepsilon \text{ for some } \varsigma \in \wp \cup \Re\}.$$

Proof. Assume (i) holds and let $\min K_\varepsilon^o = m$. Then there exists an $\varsigma_0 \in \wp \cup \Re$ satisfying $\rho(\varsigma_0, \mathcal{F}^m\varsigma_0) < \rho(\wp, \Re) + \varepsilon$. Note that, since $m \in K_\varepsilon^o \subseteq \mathbb{Z}_o^+$, if $\varsigma_0 \in \wp$, then $\mathcal{F}^m\varsigma_0 \in \Re$ and vice versa. Consider the mentioned Picard sequence $\{\varsigma_n\}$. If there exists $n_0 \in \mathbb{N}$ such that $\rho(\varsigma_{n_0}, \mathcal{F}^m\varsigma_{n_0}) = \rho(\wp, \Re)$, then ς_{n_0} is a best cyclic periodic point of $\mathcal{F}$. Now assume

$$\rho(\varsigma_n, \mathcal{F}^m\varsigma_n) > \rho(\wp, \Re) \tag{3.6}$$

for all $n \in \mathbb{N}$. In this case we investigate the following two cases:

Case 1. Let $m = 1$. Then, we have

$$\rho(\wp, \Re) < \rho(\varsigma_0, \varsigma_1) = \rho(\varsigma_0, \mathcal{F}\varsigma_0) < \rho(\wp, \Re) + \varepsilon. \tag{3.7}$$

Taking $\varsigma = \varsigma_0$ and $\xi = \varsigma_1$ in implication (3.5), we get

$$P(\varsigma_0, \varsigma_1) \leq k\rho(\varsigma_0, \varsigma_1) + (1-k)\rho(\wp, \Re)$$

which implies that

$$\min\{\rho(\varsigma_1, \varsigma_2), \rho(\varsigma_0, \varsigma_1)\} \leq k\rho(\varsigma_0, \varsigma_1) + (1-k)\rho(\wp, \Re) \tag{3.8}$$

If $\rho(\varsigma_0, \varsigma_1) \leq \rho(\varsigma_1, \varsigma_2)$, then we have

$$\rho(\varsigma_0, \varsigma_1) \leq k\rho(\varsigma_0, \varsigma_1) + (1-k)\rho(\wp, \Re)$$

and so,

$$\rho(\varsigma_0, \varsigma_1) \leq \rho(\wp, \Re),$$

which contradicts (3.7). Hence we have $\rho(\varsigma_0, \varsigma_1) > \rho(\varsigma_1, \varsigma_2)$ and so, from (3.8), we have

$$\begin{aligned} \rho(\varsigma_1, \varsigma_2) &\leq k\rho(\varsigma_0, \varsigma_1) + (1-k)\rho(\wp, \Re) \\ &< k(\rho(\wp, \Re) + \varepsilon) + (1-k)\rho(\wp, \Re) \\ &= k\varepsilon + \rho(\wp, \Re) \\ &< \rho(\wp, \Re) + \varepsilon. \end{aligned}$$

In a similar way, we obtain

$$\rho(\varsigma_n, \varsigma_{n+1}) \leq k\rho(\varsigma_{n-1}, \varsigma_n) + (1-k)\rho(\wp, \Re) \tag{3.9}$$

for all $n \in \mathbb{N}$. Thus, doing as in the proof of Theorem 2.2, we can show that $\mathcal{F}$ has a best proximity point which is also a best cyclic periodic point of $\mathcal{F}$.

Case 2. Now, assume $m > 1$. That is,

$$\rho(\varsigma, \mathcal{F}\varsigma) \geq \rho(\wp, \Re) + \varepsilon \tag{3.10}$$

for all $\varsigma \in \wp \cup \Re$. In this case, since

$$\rho(\wp, \Re) < \rho(\varsigma_0, \mathcal{F}^m\varsigma_0) = \rho(\varsigma_0, \varsigma_m) < \rho(\wp, \Re) + \varepsilon,$$

taking $\varsigma = \varsigma_0$ and $\xi = \varsigma_m$ in implication (3.5), we have

$$P(\varsigma_0, \varsigma_m) \leq k\rho(\varsigma_0, \varsigma_m) + (1-k)\rho(\wp, \Re)$$

which implies that

$$\min\{\rho(\varsigma_1, \varsigma_{m+1}), \rho(\varsigma_0, \mathcal{F}\varsigma_0), \rho(\varsigma_m, \mathcal{F}\varsigma_m)\} \leq k\rho(\varsigma_0, \varsigma_m) + (1-k)\rho(\wp, \Re).$$

From (3.10), we get

$$\rho(\varsigma_1, \varsigma_{m+1}) \leq k\rho(\varsigma_0, \varsigma_m) + (1-k)\rho(\wp, \Re)$$

and since $\rho(\varsigma_0, \varsigma_m) < \rho(\wp, \Re) + \varepsilon$, we have

$$\begin{aligned} \rho(\varsigma_1, \varsigma_{m+1}) &< k\left(\rho(\wp, \Re) + \varepsilon\right) + (1-k)\rho(\wp, \Re) \\ &= k\varepsilon + \rho(\wp, \Re) \\ &< \rho(\wp, \Re) + \varepsilon. \end{aligned}$$

Then, taking into account (3.6), we have

$$\rho(\wp, \Re) < \rho(\varsigma_1, \varsigma_{m+1}) < \rho(\wp, \Re) + \varepsilon$$

and so taking $\varsigma = \varsigma_1$ and $\xi = \varsigma_{m+1}$ in implication (3.5), we obtain

$$P(\varsigma_1, \varsigma_{m+1}) \leq k\rho(\varsigma_1, \varsigma_{m+1}) + (1-k)\rho(\wp, \Re)$$

which implies that

$$\min\{\rho(\varsigma_2,\varsigma_{m+2}),\rho(\varsigma_1,\mathcal{F}\varsigma_1),\rho(\varsigma_{m+1},\mathcal{F}\varsigma_{m+1})\}\leq k\rho(\varsigma_1,\varsigma_{m+1})+(1-k)\rho(\wp,\Re).$$

From (3.10), we get

$$\rho(\varsigma_2,\varsigma_{m+2})\leq k\rho(\varsigma_1,\varsigma_{m+1})+(1-k)\rho(\wp,\Re)$$

and since $\rho(\varsigma_1,\varsigma_{m+1})<\rho(\wp,\Re)+\varepsilon$, we have

$$\begin{aligned}\rho(\varsigma_2,\varsigma_{m+2}) &< k(\rho(\wp,\Re)+\varepsilon)+(1-k)\rho(\wp,\Re)\\ &= k\varepsilon+\rho(\wp,\Re)\\ &< \rho(\wp,\Re)+\varepsilon.\end{aligned}$$

Continuing this process, we obtain

$$\rho(\varsigma_n,\varsigma_{m+n})\leq k\rho(\varsigma_{n-1},\varsigma_{m+n-1})+(1-k)\rho(\wp,\Re)$$

for all $n\in\mathbb{N}$. Thus, we have

$$\begin{aligned}\rho(\varsigma_n,\varsigma_{m+n}) &\leq k\rho(\varsigma_{n-1},\varsigma_{m+n-1})+(1-k)\rho(\wp,\Re)\\ &\leq k^2\rho(\varsigma_{n-2},\varsigma_{m+n-2})+(1+k)(1-k)\rho(\wp,\Re)\\ &\quad\vdots\\ &\leq k^n\rho(\varsigma_0,\varsigma_m)+(1-k)\left(1+k+k^2+\cdots k^{n-1}\right)\rho(\wp,\Re)\\ &= k^n\rho(\varsigma_0,\varsigma_m)+(1-k)\left(\frac{1-k^n}{1-k}\right)\rho(\wp,\Re)\\ &= k^n\rho(\varsigma_0,\varsigma_m)+(1-k^n)\,\rho(\wp,\Re).\end{aligned}$$

Therefore, we get

$$\rho(\varsigma_n,\varsigma_{m+n})\to\rho(\wp,\Re) \tag{3.11}$$

as $n\to+\infty$. From the condition (i), there exists a subsequence $\{\varsigma_{n_k}\}$ of the sequence $\{\varsigma_n\}$ such that $\varsigma_{n_k}\to\varsigma^*\in\wp\cup\Re$. On the other hand, since $\mathcal{F}$ is orbitally continuous, then $\mathcal{F}^m$ is also orbitally continuous. Thus, we have

$$\varsigma_{m+n_k}=\mathcal{F}^m\varsigma_{n_k}\to\mathcal{F}^m\varsigma^* \text{ as } k\to+\infty.$$

Hence, from (3.11), we obtain

$$\rho(\varsigma^*,\mathcal{F}^m\varsigma^*)=\rho(\wp,\Re).$$

Thus, $\mathcal{F}$ has a best cyclic periodic point in $\wp\cup\Re$.

Now assume (ii) holds and let $\min K_\varepsilon^e=m$. Then there exists an $\varsigma_0\in\wp\cup\Re$ satisfying $\rho(\varsigma_0,\mathcal{F}^m\varsigma_0)=\rho(\varsigma_0,\varsigma_m)<\varepsilon$. Note that, since $m\in K_\varepsilon^e\subseteq\mathbb{Z}_e^+$, if $\varsigma_0\in\wp$ (resp. $\varsigma_0\in\Re$), then $\mathcal{F}^m\varsigma_0\in\wp$ (resp. $\mathcal{F}^m\varsigma_0\in\Re$). Consider the mentioned Picard sequence $\{\varsigma_n\}$. If there exists $n_0\in\mathbb{N}$ such that $\rho(\varsigma_{n_0},\mathcal{F}^m\varsigma_{n_0})=0$, then ς_{n_0} is a best cyclic periodic point of $\mathcal{F}$. Now assume

$$\rho(\varsigma_n,\mathcal{F}^m\varsigma_n)>0 \tag{3.12}$$

for all $n\in\mathbb{N}$. Also we can assume

$$\rho(\varsigma_n,\mathcal{F}\varsigma_n)\geq\rho(\wp,\Re)+\varepsilon \tag{3.13}$$

for all $n\in\mathbb{N}$. Otherwise, since the condition (i) holds, the proof is completed. Now taking $\varsigma=\varsigma_0$ and $\xi=\varsigma_m$ in implication (3.4), we have

$$P(\varsigma_0,\varsigma_m)\leq k\rho(\varsigma_0,\varsigma_m)$$

which implies that

$$\min\{\rho(\varsigma_1, \varsigma_{m+1}), \rho(\varsigma_0, \mathcal{F}\varsigma_0), \rho(\varsigma_m, \mathcal{F}\varsigma_m)\} \leq k\rho(\varsigma_0, \varsigma_m).$$

From (3.13), we get

$$\rho(\varsigma_1, \varsigma_{m+1}) \leq k\rho(\varsigma_0, \varsigma_m)$$

and since $\rho(\varsigma_0, \varsigma_m) < \varepsilon$, we have

$$\rho(\varsigma_1, \varsigma_{m+1}) < k\varepsilon < \varepsilon.$$

Then, taking into account (3.12), we have

$$0 < \rho(\varsigma_1, \varsigma_{m+1}) < \varepsilon$$

and so taking $\varsigma = \varsigma_1$ and $\xi = \varsigma_{m+1}$ in implication (3.4), we obtain

$$P(\varsigma_1, \varsigma_{m+1}) \leq k\rho(\varsigma_1, \varsigma_{m+1})$$

which implies that

$$\min\{\rho(\varsigma_2, \varsigma_{m+2}), \rho(\varsigma_1, \mathcal{F}\varsigma_1), \rho(\varsigma_{m+1}, \mathcal{F}\varsigma_{m+1})\} \leq k\rho(\varsigma_1, \varsigma_{m+1}).$$

From (3.13), we get

$$\rho(\varsigma_2, \varsigma_{m+2}) \leq k\rho(\varsigma_1, \varsigma_{m+1})$$

and since $\rho(\varsigma_1, \varsigma_{m+1}) < \varepsilon$, we have

$$\rho(\varsigma_2, \varsigma_{m+2}) < k\varepsilon < \varepsilon.$$

Continuing this process, we obtain

$$\rho(\varsigma_n, \varsigma_{m+n}) \leq k\rho(\varsigma_{n-1}, \varsigma_{m+n-1}) \leq \cdots \leq k^n\rho(\varsigma_{0,}\varsigma_m)$$

for all $n \in \mathbb{N}$. Therefore, we get

$$\rho(\varsigma_n, \varsigma_{m+n}) \to 0 \tag{3.14}$$

as $n \to +\infty$. From the condition (ii), there exists a subsequence $\{\varsigma_{n_k}\}$ of the sequence $\{\varsigma_n\}$ such that $\varsigma_{n_k} \to \varsigma^* \in \wp \cup \Re$. On the other hand, since $\mathcal{F}$ is orbitally continuous, then $\mathcal{F}^m$ is also orbitally continuous. Thus, we have

$$\varsigma_{m+n_k} = \mathcal{F}^m\varsigma_{n_k} \to \mathcal{F}^m\varsigma^* \text{ as } k \to +\infty.$$

Hence, from (3.14) we obtain

$$\rho(\varsigma^*, \mathcal{F}^m\varsigma^*) = 0.$$

Thus, $\mathcal{F}$ has a best cyclic periodic point in $\wp \cup \Re$. □

Example 3.2. Let $\mathbb{R}^2$ endowed with the taxicab metric ρ. Let's consider the sets

$$\wp = \{(0, \varsigma) : \varsigma \in \mathbb{Z}\backslash\{0\}\}$$

and

$$\Re = \{(1, \varsigma) : \varsigma \in \mathbb{Z}\backslash\{0\}\}.$$

Then, $\rho(\wp, \Re) = 1$. Define a mapping $\mathcal{F} : \wp \cup \Re \to \wp \cup \Re$ by

$$\mathcal{F}\varsigma = \begin{cases} (1, -t) & , \quad \varsigma = (0, t), t \neq -1, 2 \\ (0, -t) & , \quad \varsigma = (1, t), t \neq -1, 2 \\ (1, 2) & , \quad \varsigma = (0, -1) \\ (1, 1) & , \quad \varsigma = (0, 2) \\ (0, 2) & , \quad \varsigma = (1, -1) \\ (0, 1) & , \quad \varsigma = (1, 2) \end{cases}.$$

Since $O_{\mathcal{F}}(\varsigma)$ is finite set for all $\varsigma \in \wp \cup \Re$, then $\mathcal{F}$ is an orbitally continuous mapping and the sequence $\{\varsigma_n\}$ has a convergent subsequence for every Picard sequence $\{\varsigma_n\}$. Further, note that for all $(\varsigma, \xi) \in \wp^2 \cup \Re^2$ with $\rho(\varsigma, \xi) > 0$, we have $\rho(\varsigma, \xi) \geq 1$ and for all $(\varsigma, \xi) \in \wp \times \Re$

with $\rho(\varsigma,\xi) > \rho(\wp,\Re)$, we have $\rho(\varsigma,\xi) \geq 2$. Therefore, the implications (3.4) and (3.5) hold for $\varepsilon = \frac{1}{2}$. Moreover, for all $\varsigma \in \wp \cup \Re$ we have $\rho(\varsigma,\mathcal{F}\varsigma) \geq \rho(\wp,\Re) + \frac{1}{2}$ and for $\varsigma_0 = (0,1) \in \wp$, we have $\rho(\varsigma_0,\mathcal{F}^3\varsigma_0) = \rho(\wp,\Re)$. This shows that $\min K^o_{\frac{1}{2}} = 3$. Hence the condition (i) is satisfied. Therefore, all hypothesis of Theorem 3.3 are satisfied, then $\mathcal{F}$ has a best cyclic periodic point in $\wp \cup \Re$. Note that, for $\varsigma_0 = (1,4) \in \Re$ we have $\rho(\varsigma_0,\mathcal{F}^2\varsigma_0) = 0$ and so $\min K^e_{\frac{1}{2}} = 2$. Hence the condition (ii) is also satisfied.

Note that, if we take $\wp = \Re = \mho$ in Theorem 3.3, we obtain the following periodic point result:

Corollary 3.4. *Let $(\mho,\rho)$ be an orbitally complete metric space, $\mathcal{F} : \mho \to \mho$ be an orbitally continuous mapping and $\varepsilon > 0$. Assume that there exists an $\varsigma \in \mho$ such that $\rho(\varsigma,\mathcal{F}^q\varsigma) < \varepsilon$ for some $q \in \mathbb{Z}^+$ and there exists a k in $[0,1)$ such that*

$$0 < \rho(\varsigma,\xi) < \varepsilon \; implies \; \min\{\rho(\mathcal{F}\varsigma,\mathcal{F}\xi),\rho(\varsigma,\mathcal{F}\varsigma),\rho(\xi,\mathcal{F}\xi)\} \leq k\rho(\varsigma,\xi)$$

for all $\varsigma,\xi \in \mho$. Then $\mathcal{F}$ has a periodic point in $\mho$.

Proof. Define

$$K_\varepsilon = \{q \in \mathbb{N} : \rho(\varsigma,\mathcal{F}^q\varsigma) < \varepsilon \text{ for some } \varsigma \in \mho\}.$$

Then from the hypothesis K_ε is nonempty. Let $m = \min K_\varepsilon$, then there exists an $\varsigma_0 \in \mho$ such that $\rho(\varsigma_0,\mathcal{F}^m\varsigma_0) < \varepsilon$. It can be show that the subsequence $\{\varsigma_{nm}\}$ of the Picard sequence $\{\varsigma_n\}$ with the initial point ς_0 is a Cauchy sequence in $\mho$. Hence, by the orbitally completeness of $\mho$, we have $\{\varsigma_{nm}\}$ is convergent. Therefore all conditions of Theorem 3.3 are satisfied and so $\mathcal{F}$ has a best cyclic periodic point, which is actually a periodic point in $\mho$. $\square$

Acknowledgement. The authors are thankful to the referees for making valuable suggestions leading to the better presentations of the paper.

REFERENCES

[1] Altun, I.; Aslantas, M.; Sahin, H. Best proximity point results for p-proximal contractions. *Acta Math. Hungar.* **162** (2020), 393–402.

[2] Alsulami, H. H.; Karapınar, E.; Rakočević, V. Ćirić type nonunique fixed point theorems on b-Metric Spaces. *Filomat* **31** (2017), 3147–3156.

[3] Arshad, M.; Shoaib, A.; Abbas, M.; Azam, A. Fixed points of a pair of Kannan type mappings on a closed ball in ordered partial metric spaces. *Miskolc Mathematical Notes* **14** (2013), 769–784.

[4] Banach, S. Sur les opérations dans les ensembles abstraits et leur applications aux équations intégrales. *Fund. Math.* **3** (1922), 133–181.

[5] Basha, S. S., Veeramani, P., *Best approximations and best proximity pairs*, Acta Sci. Math. **63** (1997), 289-300.

[6] Chiming, C.; Lin, C. Best periodic proximity points for cyclic weaker Meir-Keeler contractions. *J. Appl. Math.* **2012** (2012), Article ID: 782389.

[7] Ćirić, L. B. On contraction type mappings. *Math. Balkanica* **1** (1971), 52–57.

[8] Ćirić, L. B. On some maps with a nonunique fixed point. *Institut Mathématique* **17** (1974), 52–58.

[9] Darwish, M. A.; Jleli, M.; O'Regan, D.; Samet, B. On the study of fixed points for a new class of α-admissible mappings. *Mathematics* **7** (2019).

[10] Eldred, A. A.; Veeramani, P. Existence and convergence of best proximity points. *J. Math. Anal. Appl.* **323** (2006), 1001–1006.

[11] Gabeleh, M.; Markin, J. Common best proximity pairs via the concept of complete proximal normal structure. *Ann. Funct. Anal.* **11** (2020), 831–847.

[12] Hussain, N.; Mitrović, Z. D.; Radenović, S. A common fixed point theorem of Fisher in b-metric spaces. *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas.* **113** (2019), 949–956.

[13] Jeong, G. S.; Rhoades, B. E. More maps for which $F(T) = F(T^n)$. *Demonstratio Mathematica.* **40** (2007), 671–680.

[14] Jleli, M.; Samet, B. A new generalization of the Banach contraction principle. *J. Inequal. Appl.* **2014**, 2014:38, 8 pp.

[15] Kirk, W. A.; Srinivasan, P. S.; Veeramani, P. Fixed points for mappings satisfying cyclical contractive conditions. *Fixed Point Theory.* **4** (2003), 79–89.
[16] Nadler, S. B. Multi-valued contraction mappings. *Pacific J. Math.* **30** (1969), 475–488.
[17] Radenovic, S. Classical fixed point results in 0-complete partial metric spaces via cyclic-type extension. *Bull. Allahabad Math. Soc.* **31** (2016), 39–55.
[18] Radenovic, S. Some remarks on mappings satisfying cyclical contractive conditions. *Afrika Matematika.* **27** (2016), 291–295.
[19] Rahimi, H.; Radenovic, S.; Rad, G. S.; Kumam, P. Quadrupled fixed point results in abstract metric spaces. *Comput. Appl. Math.* **33** (2014), 671-685.
[20] Raj, V. S. Best proximity point theorems for non-self mappings. *Fixed Point Theory.* **14** (2013), 447–454.
[21] Sintunavarat, W.; Kumam, P. Coupled best proximity point theorem in metric spaces. *Fixed Point Theory Appl.* **2012** (2012), 93.

DEPARTMENT OF MATHEMATICS
ÇANKIRI KARATEKIN UNIVERSITY
18100, ÇANKIRI, TURKEY
Email address: maslantas@karatekin.edu.tr

DEPARTMENT OF MATHEMATICS
AMASYA UNIVERSITY
05100, AMASYA, TURKEY
Email address: hakan.sahin@amasya.edu.tr

DEPARTMENT OF MATHEMATICS
KIRIKKALE UNIVERSITY
71450 YAHSIHAN, KIRIKKALE, TURKEY
Email address: ishakaltun@yahoo.com