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On the crossing number of the join of the wheel on six vertices with a path

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ABSTRACT. The crossing number $\mathrm{cr}(G)$ of a graph G is the minimum number of edge crossings over all drawings of G in the plane. The main aim of the paper is to give the crossing number of join product $W_5 + P_n$ for the wheel W_5 on six vertices, where P_n is the path on n vertices. Staš and Valiska conjectured that the crossing number of $W_m + P_n$ is equal to $Z(m+1)Z(n) + (Z(m)-1)\lfloor \frac{n}{2} \rfloor + n + 1$, for all $m \geq 3$, $n \geq 2$, where Zarankiewicz's number is defined as $Z(n) = \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$ for $n \geq 1$. Recently, this conjecture was proved for $W_3 + P_n$ by Klešč and Schrötter, and for $W_4 + P_n$ by Staš and Valiska. We establish the validity of this conjecture for $W_5 + P_n$. The conjecture also holds due to some isomorphisms for $W_m + P_2$, $W_m + P_3$ by Klešč, and for $W_m + P_4$ by Staš for all $m \geq 3$.

1. INTRODUCTION

The *crossing number* $\mathrm{cr}(G)$ of a simple graph G with the vertex set $V(G)$ and the edge set $E(G)$ is the minimum possible number of edge crossings in a drawing of G in the plane. (For the definition of a *drawing* see Klešč [10].) It is easy to see that a drawing with the minimum number of crossings (an optimal drawing) is always a *good* drawing, meaning that no edge crosses itself, no two edges cross more than once, and no two edges incident with the same vertex cross. The join product of two graphs G_i and G_j, denoted by $G_i + G_j$, is obtained from vertex-disjoint copies of G_i and G_j by adding all edges between $V(G_i)$ and $V(G_j)$. For $|V(G_i)| = m$ and $|V(G_j)| = n$, the edge set of $G_i + G_j$ is the union of disjoint edge sets of the graphs G_i, G_j, and the complete bipartite graph $K_{m,n}$.

Let D be a good drawing of the graph G. We denote the number of crossings in D by $\mathrm{cr}_D(G)$. Let G_i and G_j be edge-disjoint subgraphs of G. We denote the number of crossings between edges of G_i and edges of G_j by $\mathrm{cr}_D(G_i, G_j)$, and the number of crossings among edges of G_i in D by $\mathrm{cr}_D(G_i)$. For any three mutually edge-disjoint subgraphs G_i, G_j, and G_k of G by [10], the following equations hold:

$$\mathrm{cr}_D(G_i \cup G_j) = \mathrm{cr}_D(G_i) + \mathrm{cr}_D(G_j) + \mathrm{cr}_D(G_i, G_j)\,,$$
$$\mathrm{cr}_D(G_i \cup G_j, G_k) = \mathrm{cr}_D(G_i, G_k) + \mathrm{cr}_D(G_j, G_k)\,.$$

The investigation of the crossing number of graphs is a classical and very difficult problem. Garey and Johnson [7] proved that this problem is NP-complete. The exact values of the crossing numbers are known for some families of graphs, see Clancy *et al.* [4]. The purpose of this paper is to extend the known results concerning this topic. Some parts of proofs will be based on Kleitman's result [9] on the crossing numbers for some complete bipartite graphs. He showed that

$$\mathrm{cr}(K_{m,n}) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor, \quad \text{if} \quad m \leq 6.$$

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Using Kleitman's result, the crossing numbers for the join product of paths with all graphs of order four were studied by Klešč [11] and Klešč and Schrötter [17]. The exact values for the crossing numbers of $G + P_n$ for some graphs G on five vertices are determined in [16, 18, 22, 23, 24, 25]. The crossing numbers of the join product $G + P_n$ are known only for a few graphs G of order six, and so the purpose of this article is to extend the known results concerning this topic to new connected graphs, see [2, 5, 6, 10, 13, 20]. Minimal number of crossings in the Cartesian product and in the strong product of paths have been studied by Klešč *et al.* [14] and [15].

Staš and Valiska [25] observed that the optimal drawing for $W_4 + P_n$ can be generalized to drawings of $W_m + P_n$, which lead them to conjecture that the crossing number of $W_m + P_n$ equals $Z(m+1)Z(n) + (Z(m)-1)\lfloor \frac{n}{2} \rfloor + n + 1$ for all $m \geq 3$, $n \geq 2$, where $Z(n) = \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$ is Zarankiewicz's number. Recently, this conjecture was proved for the crossing numbers of join products $W_3 + P_n$ and $W_4 + P_n$ by Klešč and Schrötter [17] and by Staš and Valiska [25], respectively. Results by Klešč [11] and [12] establish the conjecture for $W_m + P_2$ and $W_m + P_3$, and by Staš [23] for $W_m + P_4$.

The main purpose of the current paper is to show that the conjecture is true for $W_5 + P_n$, for all $n \geq 2$.

Theorem 1.1. $\operatorname{cr}(W_5 + P_n) = 6 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + n + 3 \left\lfloor \frac{n}{2} \right\rfloor + 1$ *for* $n \geq 2$.

Note that the result of Theorem 3.3 has already been claimed by Su and Huang [21] just as a conjecture without any proof. Clancy *et al.* [4] also placed an asterisk on a number of the results in their survey to essentially indicate that the mentioned results appeared in journals which do not have a sufficiently rigorous peer-review process. In the proofs of the paper, we will often use the term "region" also in nonplanar drawings. In this case, crossings are considered to be vertices of the "map".

2. Cyclic Permutations

This article follows definitions and notation for the crossing numbers from Klešč [11]. Let W_5 be the wheel on six vertices. We first consider the join product of W_5 with the discrete graph on n vertices denoted by D_n. The graph $W_5 + D_n$ consists of just one copy of the graph W_5 and of n vertices $t_1, t_2, \dots, t_n$, where each vertex t_i, $i = 1, 2, \dots, n$, is adjacent to every vertex of W_5. Let T^i, $1 \leq i \leq n$, denote the subgraph induced by the six edges incident with the vertex t_i. This means that the graph $T^1 \cup \dots \cup T^n$ is isomorphic to the complete bipartite graph $K_{6,n}$ and

$$W_5 + D_n = W_5 \cup K_{6,n} = W_5 \cup \left(\bigcup_{i=1}^{n} T^i \right). \tag{2.1}$$

The graph $W_5 + P_n$ contains $W_5 + D_n$ as a subgraph. For the subgraphs of the graph $W_5 + P_n$ which are also subgraphs of the graph $W_5 + nK_1$ we use the same notation as above. Let P_n^* denote the path induced by n vertices of $W_5 + P_n$ not belonging to the subgraph W_5. Hence, P_n^* consists of the vertices $t_1, t_2, \dots, t_n$ and of the edges $\{t_i, t_{i+1}\}$ for $i = 1, 2, \dots, n-1$. One can easily see that

$$W_5 + P_n = W_5 \cup K_{6,n} \cup P_n^* = W_5 \cup \left(\bigcup_{i=1}^{n} T^i \right) \cup P_n^*. \tag{2.2}$$

Let D be a good drawing of the graph $W_5 + D_n$. The *rotation* $\operatorname{rot}_D(t_i)$ of a vertex t_i in the drawing D is the cyclic permutation that records the (cyclic) counterclockwise order in which the edges leave t_i, as defined by Hernández-Vélez *et al.* [8] or Woodall [26]. We use the notation (123456) if the counter-clockwise order the edges incident with the vertex t_i is t_iv_1, t_iv_2, t_iv_3, t_iv_4, t_iv_5, and t_iv_6. Recall that a rotation is a cyclic permutation; that is, (123456), (234561), (345612), (456123), (561234), and (612345) denote the same rotation. We separate all subgraphs T^i, $i = 1, 2, \dots, n$, of the graph $W_5 + D_n$ into four mutually-disjoint families of subgraphs depending on how many times the considered T^i crosses the edges of W_5 in D. Let $R_D = \{T^i : \mathrm{cr}_D(W_5, T^i) = 0\}$, $S_D = \{T^i : \mathrm{cr}_D(W_5, T^i) = 1\}$, and $T_D = \{T^i : \mathrm{cr}_D(W_5, T^i) = 2\}$. Every other subgraph T^i crosses the edges of W_5 at least three times in D. For $T^i \in R_D \cup S_D \cup T_D$, let F^i denote the subgraph $W_5 \cup T^i$, $i \in \{1, 2, \dots, n\}$, of $W_5 + D_n$ and let $D(F^i)$ be its good subdrawing induced by D. Clearly, the four families we mentioned are the same in the drawing D of $W_5 + P_n$ and in the subdrawing D' of $W_5 + D_n$ induced by D without the edges of P_n^*.

3. Possible Drawings of W_5 and the Crossing Number of $W_5 + P_n$

Since the graph W_5 consists of one dominating vertex of degree 5 and of five vertices of degree 3 which form the subgraph isomorphic to the cycle C_5 (for brevity, we will write $C_5^\star$), we only need to consider possibilities of crossings between subdrawings of $C_5^\star$ and the five edges incident with the dominating vertex which form the subgraph isomorphic to the star S_5 on six vertices (also for brevity, we will write $S_5^\star$). In the rest of the paper, let $V(W_5) = \{v_1, v_2, \dots, v_6\}$, and let v_6 be the vertex notation of the dominating vertex of degree 5 in all considered good subdrawings of the graph W_5.

Let us first note that if D is a good drawing of $W_5 + P_n$ with the empty set $R_D \cup S_D$, then $t = |T_D| < \lceil \frac{n}{2} \rceil$ implies at least $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + 3\lfloor \frac{n}{2} \rfloor + 1$ crossings in D provided by

$$\mathrm{cr}_D(W_5 + P_n) \geq \mathrm{cr}_D(K_{6,n}) + \mathrm{cr}_D(W_5, K_{6,n}) \geq 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2t + 3(n-t) =$$

$$6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 3n - t \geq 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 3n + 1 - \left\lceil \frac{n}{2} \right\rceil \geq 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + n + 3\left\lfloor \frac{n}{2} \right\rfloor + 1.$$

Lemma 3.1. *Let $G \in \{D_n | n \geq 1\} \cup \{P_n | n \geq 2\}$. In any optimal drawing of the join product $W_5 + G$, the edges of $C_5^\star$ do not cross each other. Moreover, the subdrawing of W_5 induced by D, in which no two edges of $C_5^\star$ are crossed by any edge of $S_5^\star$ and with a possibility of obtaining a subgraph T^i whose edges cross the edges of W_5 at most twice, is isomorphic to one of the eight drawings depicted in Fig. 1.*

Proof. The proof for $W_5 + D_n$ has already been presented by Berežný and Staš [3]. In the case of $W_5 + P_n$, the proof uses the same idea, that is, we can always redraw a crossing of two edges of $C_5^\star$ to get a new drawing of $C_5^\star$ (with vertices in a different order) with less number of edge crossings.

Let D be a good drawing of $W_5 + D_n$ with no crossing among edges of $C_5^\star$, and let there be a possibility of obtaining a subgraph T^i by which the edges of W_5 are crossed at most twice. Without lost of generality, let us denote by $v_1v_2v_3v_4v_5v_1$ the vertex notation of the cycle $C_5^\star$. Because any edge of $S_5^\star$ can cross at most one edge of $C_5^\star$, only three main cases need to be considered:

If no edge of the cycle $C_5^\star$ is crossed by any edge of $S_5^\star$, we obtain the planar subdrawing of W_5 shown in Fig. 1(a). For any $i = 1, \dots, 5$, if the edge v_iv_6 crosses some edge of $C_5^\star$ and all three regions of $D(C_5^\star \cup v_iv_6)$ contain at least three vertices of the graph W_5 on its boundary, then we obtain a subdrawing of W_5 isomorphic to the drawing shown in Fig. 1(c). Now, without lost of generality, let the edge v_3v_4 be crossed by the edge v_2v_6. The edge v_3v_6 must be without any crossing in $D(W_5)$, and the edge v_1v_6 either crosses one of the edges v_4v_5, v_3v_4 or does not cross the edges of $C_5^\star$. If v_1v_6 crosses v_4v_5, we obtain the subdrawing of W_5 shown in Fig. 1(e). If v_1v_6 crosses v_3v_4 and also v_4v_6 crosses v_1v_5, we obtain the subdrawing of W_5 shown in Fig. 1(g). If v_1v_6 crosses v_3v_4 and also v_4v_6 is without any crossing in $D(W_5)$, then the edge v_5v_6 either does not cross any edge of $C_5^\star$ or cross the edge v_3v_4 and we obtain the subdrawings of W_5 shown in Fig. 1(f) and (h), respectively. Finally, if the edge v_1v_6 does not cross any edge of the cycle $C_5^\star$, then the edge v_5v_6 either also does not cross any edge of $C_5^\star$ or cross the edge v_3v_4 and we obtain the last two possible subdrawings of W_5 shown in Fig. 1(b) and (d), respectively.

□

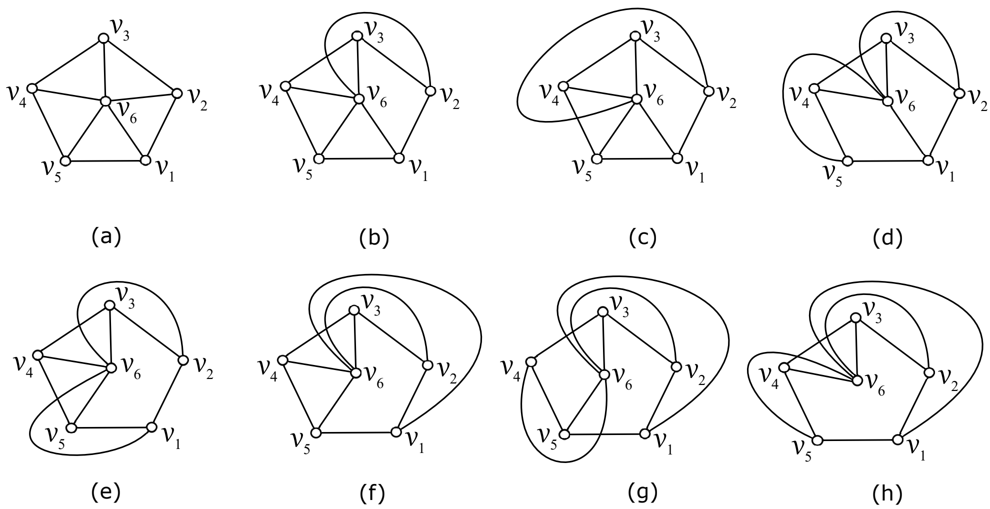

FIGURE 1. Eight possible non isomorphic drawings of the graph W_5 with no crossing among edges of $C_5^\star$, where no two edges of $C_5^\star$ are crossed by any edge of $S_5^\star$, and also with a possibility of obtaining a subgraph $T^i \in R_D \cup S_D \cup T_D$.

Lemma 3.2. *Let D be a good drawing of $W_5 + P_n$, $n \geq 2$. If the edges of $C_5^\star$ are crossed at least $\lceil \frac{n}{2} \rceil + 1$ times, then there are at least $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + 3\lfloor \frac{n}{2} \rfloor + 1$ crossings in D.*

Proof. As the wheel W_5 consists of two edge-disjoint subgraphs $C_5^\star$ and $S_5^\star$, let us consider that $\mathrm{cr}_D(C_5^\star) + \mathrm{cr}_D(C_5^\star, S_5^\star + P_n) \geq \lceil \frac{n}{2} \rceil + 1$ is fulfilling in the good drawing D of $W_5 + P_n$. The star $S_5^\star$ is isomorphic to the complete bipartite graph $K_{1,5}$ and the exact value for the crossing number of $K_{1,5} + P_n$ as a direct corollary of the crossing number of $K_{1,5} + D_n$ is given by Mei and Huang [19], that is, $\mathrm{cr}(K_{1,5,n}) = \mathrm{cr}(K_{1,5} + P_n) = 6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 4\lfloor \frac{n}{2} \rfloor$. This enforces that the edges of $S_5^\star + P_n$ must be crossed at least $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 4\lfloor \frac{n}{2} \rfloor$ times in D. Consequently, we have

$$\mathrm{cr}_D(W_5 + P_n) = \mathrm{cr}_D(S_5^\star + P_n) + \mathrm{cr}_D(C_5^\star) + \mathrm{cr}_D(C_5^\star, S_5^\star + P_n)$$
$$\geq 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 4\left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{n}{2} \right\rceil + 1 = 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + n + 3\left\lfloor \frac{n}{2} \right\rfloor + 1.$$

□

Corollary 3.1. *Let D be a good drawing of $W_5 + P_n$, $n \geq 2$, with no crossing among edges of $C_5^\star$, and let $|T_D| \geq \lceil \frac{n}{2} \rceil$. If any subgraph $T^i \in T_D$ crosses some edge of the cycle $C_5^\star$, then there are at least $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + 3\lfloor \frac{n}{2} \rfloor + 1$ crossings in D.*

Proof. In the planar drawing of W_5 (shown in Fig. 1(a)), there is no possibility to obtain a subdrawing of $W_5 \cup T^i$ for a subgraph $T^i \in T_D$. All cases of good drawings of $W_5 + P_n$ with at least one crossing among edges of the graph W_5 enforce $\mathrm{cr}_D(C_5^\star, S_5^\star) \geq 1$. So, if $|T_D| \geq \lceil \frac{n}{2} \rceil$ and any $T^i \in T_D$ crosses some edge of $C_5^\star$, then the edges of $C_5^\star$ are crossed at least $\lceil \frac{n}{2} \rceil + 1$ times, and therefore by Lemma 3.2, there are at least $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + 3\lfloor \frac{n}{2} \rfloor + 1$ crossings in D. □

Corollary 3.2. *Let D be a good drawing of $W_5 + P_n$, $n \geq 2$, with no crossing among edges of $C_5^\star$, and let all vertices t_i of the path P_n^* be placed in the same region of the considered good subdrawing of W_5. If at least two edges of $C_5^\star$ are crossed by the same edge of $S_5^\star$, then there are at least $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + 3\lfloor \frac{n}{2} \rfloor + 1$ crossings in D.*

Proof. Let D be a good drawing of $W_5 + P_n$ with no crossing among edges of $C_5^\star$. If any subgraph T^i crosses the edges of W_5 at least twice, then $|T_D| \geq \lceil \frac{n}{2} \rceil$ or there are at least $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + 3\lfloor \frac{n}{2} \rfloor + 1$ crossings in D.

Let us turn to the good drawing D of the graph $W_5 + P_n$ with the assumption that all vertices of P_n^* are placed in the same region of the considered good subdrawing of W_5, and some edge of $S_5^\star$ crosses at least two different edges of $C_5^\star$. For this purpose, we suppose the drawing with the vertex notation of W_5 in such a way as shown in Fig. 2(a).

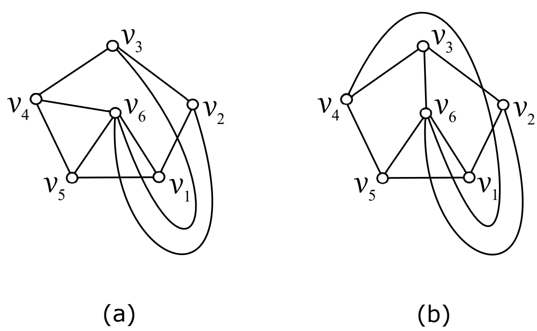

FIGURE 2. Two good drawings of the graph W_5 in which at least two edges of $C_5^\star$ are crossed by the same edge of $S_5^\star$.

Because no region is incident to at least five vertices in the subdrawing $D(W_5)$, there is no possibility to obtain a subdrawing of $W_5 \cup T^i$ for a $T^i \in R_D \cup S_D$. As $r = 0$ and $s = 0$, we can assume that $|T_D| \geq \lceil \frac{n}{2} \rceil$, otherwise, we obtain the considered number of crossings in D. If all vertices of the path P_n^* are placed in the outer region with four vertices v_2, v_3, v_4, and v_5 of the graph W_5 on its boundary, then the edges $t_i v_6$ cross some edge of the cycle $C_5^\star$, and therefore by Corollary 3.1, there are at least $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + 3\lfloor \frac{n}{2} \rfloor + 1$ crossings in D again. Now, let all vertices of P_n^* be placed in the inner region with four vertices v_1, v_3, v_4, and v_6 of W_5 on its boundary. The graph W_5 contains the cycle $v_3v_4v_6v_3$ as a subgraph by which the vertices v_2, v_5 and t_i are separated in $D(W_5)$, that is, each subgraph T^i crosses the edges of the 3-cycle $v_3v_4v_6v_3$ at least twice in $D(W_5 \cup T^i)$. If the edges of $C_5^\star$ are crossed by all subgraphs T^i at most $\lceil \frac{n}{2} \rceil - \mathrm{cr}_D(W_5)$ times due to Lemma 3.2, then there are at least $2n - (\lceil \frac{n}{2} \rceil - 3) + 2$ crossings on the edges v_3v_6 and v_4v_6 in D. Let us denote by H the subgraph of W_5 with the vertex set $V(W_5)$, and the edge set $E(W_5) \setminus \{v_3v_6, v_4v_6\}$. Since the exact value for the crossing number of the join product $H \setminus \{v_1v_6, v_3v_4\} + D_n$ is given in [1], i.e., $\mathrm{cr}(H \setminus \{v_1v_6, v_3v_4\} + D_n) = 6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2\lfloor \frac{n}{2} \rfloor$, the edges of $H + P_n$ are crossed at least $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2\lfloor \frac{n}{2} \rfloor$ times in D. Thus, the edges of $W_5 + P_n$ are crossed at least $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2\lfloor \frac{n}{2} \rfloor + 2n - (\lceil \frac{n}{2} \rceil - 3) + 2$ times in D.

This result completes the proof for the considered subdrawing of W_5 in D given in Fig. 2(a), and the proof proceeds in the similar way also for the remaining good subdrawings of W_5 in which one edge of $S_5^\star$ crosses at least two different edges of the cycle $C_5^\star$. In several cases (shown in Fig. 2(b)), it is sufficient to use only Corollary 3.1. $\square$

According to Lemma 3.1, the edges of the cycle $C_5^\star$ do not cross each other in any optimal drawing D of the join product $W_5 + P_n$. Using Corollary 3.2, we will consider only eight possible non isomorphic drawings of W_5 as shown in Fig. 1, in which there is no crossing among edges of $C_5^\star$, no two edges of $C_5^\star$ are crossed by any edge of $S_5^\star$, and there is a possibility of obtaining a subgraph T^i by which the edges of W_5 are crossed at most twice. In the proof of Theorem 3.3, several parts are also based on the following Theorem 3.2.

Theorem 3.2 ([3], Theorem 3.1). $\mathrm{cr}(W_5 + D_n) = 6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + 3\lfloor \frac{n}{2} \rfloor$ *for* $n \geq 1$.

Even though we can compute the exact values of crossing numbers of two graphs $W_5 + P_2$ and $W_5 + P_3$ using algorithm located on the website `http://crossings.uos.de/`, due to the simplicity of these proofs, we prove the following Lemma 3.3.

Lemma 3.3. $\mathrm{cr}(W_5 + P_2) = 6$ *and* $\mathrm{cr}(W_5 + P_3) = 13$.

Proof. The graphs $W_5 + P_2$ and $W_5 + P_3$ are isomorphic to the join product of the cycle C_5 with the cycle C_3 and with the graph $K_4 \setminus e$ obtained by removing one edge from the complete graph K_4, respectively. In [11] and [12] were proved that $\mathrm{cr}(C_5 + C_3) = 6$ and $\mathrm{cr}(C_5 + K_4 \setminus e) = 13$, respectively, and so $\mathrm{cr}(W_5 + P_2) = 6$ and $\mathrm{cr}(W_5 + P_3) = 13$. $\square$

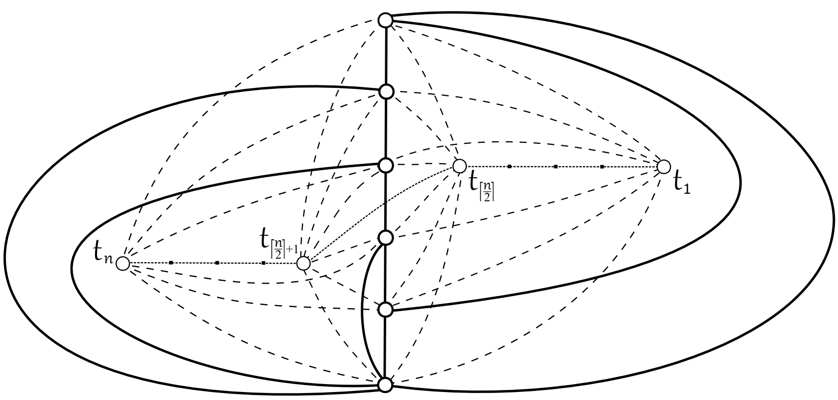

FIGURE 3. The good drawing of $W_5 + P_n$ with $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + 3\lfloor \frac{n}{2} \rfloor + 1$ crossings.

Theorem 3.3. $\mathrm{cr}(W_5 + P_n) = 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + n + 3\left\lfloor \frac{n}{2} \right\rfloor + 1$ *for* $n \geq 2$.

Proof. In Fig. 3, the edges of $K_{6,n}$ cross each other $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$ times, each subgraph T^i, $i = 1, \ldots, \lceil \frac{n}{2} \rceil$ on the right side crosses the edges of $C_5^\star$ exactly once and each subgraph T^i, $i = \lceil \frac{n}{2} \rceil + 1, \ldots, n$ on the left side crosses the edges of $S_5^\star$ exactly four times. The path P_n^* crosses W_5 once, and so $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + 3\lfloor \frac{n}{2} \rfloor + 1$ crossings appear among the edges of the graph $W_5 + P_n$ in this drawing. Thus, $\mathrm{cr}(W_5 + P_n) \leq 6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + 3\lfloor \frac{n}{2} \rfloor + 1$. By Lemma 3.3, the result is true for $n = 2$ and $n = 3$. We prove the reverse inequality by induction on n. Suppose now that, for some $n \geq 4$, there is a drawing D with

$$\mathrm{cr}_D(W_5 + P_n) < 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + n + 3\left\lfloor \frac{n}{2} \right\rfloor + 1, \tag{3.3}$$

and that

$$\text{(3.4)} \qquad \mathrm{cr}(W_5 + P_m) = 6\Big\lfloor\frac{m}{2}\Big\rfloor\Big\lfloor\frac{m-1}{2}\Big\rfloor + m + 3\Big\lfloor\frac{m}{2}\Big\rfloor + 1 \text{ for any integer } 2 \leq m < n.$$

As the graph $W_5 + D_n$ is a subgraph of the graph $W_5 + P_n$, by Theorem 3.2, the edges of $W_5 + P_n$ are crossed exactly $6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + n + 3\left\lfloor\frac{n}{2}\right\rfloor$ times, and therefore, no edge of the path P_n^* is crossed in D. This also enforces that all vertices t_i of the path P_n^* must be placed in the same region of the considered good subdrawing of W_5. Moreover, if $r = |R_D|$, $s = |S_D|$, and $t = |T_D|$, the assumption (3.3) together with the well-known fact $\mathrm{cr}(K_{6,n}) = 6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor$ imply that, in D, if $r = 0$ and $s = 0$, then there are at least $\left\lceil\frac{n}{2}\right\rceil$ subgraphs T^i by which the edges of W_5 are crossed exactly twice. More precisely:

$$\mathrm{cr}_D(W_5) + \mathrm{cr}_D(W_5, K_{6,n}) \leq n + 3\left\lfloor\frac{n}{2}\right\rfloor,$$

i.e.,

$$\text{(3.5)} \qquad \mathrm{cr}_D(W_5) + 0r + 1s + 2t + 3(n - r - s - t) \leq n + 3\left\lfloor\frac{n}{2}\right\rfloor.$$

This forces that $3r + 2s + t \geq \left\lceil\frac{n}{2}\right\rceil + \mathrm{cr}_D(W_5)$, and therefore, $t \geq \left\lceil\frac{n}{2}\right\rceil + \mathrm{cr}_D(W_5)$ if both sets R_D and S_D are empty. By Lemma 3.1, we can also suppose that there is no crossing among edges of $C_5^\star$ in all contemplated subdrawings of the graph W_5. Now, we will deal with the possibilities of obtaining a subgraph $T^i \in R_D \cup S_D \cup T_D$ in the drawing D and we show that in all cases a contradiction with the assumption (3.3) is obtained. Recall that the drawing D is also satisfying the restriction of placement all vertices t_i in the same region of the considered subdrawing of W_5.

Case 1: $\mathrm{cr}_D(W_5) = 0$. The drawing of W_5 is uniquely determined in such a way as shown in Fig. 1(a). It is obvious that the sets R_D and T_D are empty. As the set S_D cannot be empty, all vertices t_i must be placed in the region of $D(W_5)$ with five vertices of the graph W_5 on its boundary. Since each subgraph T^i crosses some edge of $C_5^\star$ at least once, the edges of the cycle $C_5^\star$ are crossed at least n times. Lemma 3.2 forces a contradiction with the assumption (3.3) in D.

Case 2: $\mathrm{cr}_D(W_5) = 1$. At first, without loss of generality, we suppose the drawing with the vertex notation of W_5 in such a way as shown in Fig. 1(b). Since the sets R_D and S_D are empty, there are at least $\left\lceil\frac{n}{2}\right\rceil + 1$ subgraphs T^i whose edges cross the edges of W_5 exactly twice. It is easy to see that $\mathrm{cr}_D(C_5^\star, T^i) \geq 1$ holds for each such subgraph $T^i \in T_D$, and so Lemma 3.2 again contradicts the assumptions of D.

In addition, without loss of generality, we can choose the vertex notation of the graph W_5 in such a way as shown in Fig. 1(c). Clearly, also the sets R_D and S_D are empty, that is, $t \geq \left\lceil\frac{n}{2}\right\rceil + 1$. Moreover, all vertices t_i must be placed in the region of $D(W_5)$ with four vertices v_1, v_2, v_3, and v_6 of W_5 on its boundary. For a $T^i \in T_D$, there is only one subdrawing of $F^i \setminus \{v_4, v_5\}$ represented by the rotation (1236), which yields that there are four ways to obtain the subdrawing of F^i depending on which two edges of W_5 are crossed by the edges t_iv_4 and t_iv_5. The edges of $C_5^\star$ are not crossed by any $T^i \in T_D$ only if $\mathrm{rot}_D(t_i) = (123465)$. It is not difficult to verify in six possible regions of $D(W_5 \cup T^i)$ that $\mathrm{cr}_D(W_5 \cup T^i, T^k) \geq 6$ is true for any subgraph T^k with $k \neq i$. Thus, by fixing the subgraph $W_5 \cup T^i$, we have

$$\mathrm{cr}_D(W_5 + P_n) \geq 6\Big\lfloor\frac{n-1}{2}\Big\rfloor\Big\lfloor\frac{n-2}{2}\Big\rfloor + 6(n-1) + 3 \geq 6\Big\lfloor\frac{n}{2}\Big\rfloor\Big\lfloor\frac{n-1}{2}\Big\rfloor + n + 3\Big\lfloor\frac{n}{2}\Big\rfloor + 1.$$

If there is no $T^i \in T_D$ with $\mathrm{rot}_D(t_i) = (123465)$, then each subgraph $T^i \in T_D$ crosses some edge of $C_5^\star$ at least once, and so Lemma 3.2 also confirms a contradiction with the assumption (3.3) in D.

Case 3: $\mathrm{cr}_D(W_5) = 2$. At first, without loss of generality, we can choose the vertex notation of the graph W_5 in such a way as shown in Fig. 1(d). As the sets R_D and S_D are also empty, suppose that all vertices t_i are placed in the region of $D(W_5)$ with four vertices v_1, v_2, v_3, and v_6 of W_5 on its boundary according to a certain symmetry of this drawing W_5. Consequently, we can apply the same idea as in the previous subcase regarding the existence or non existence of a subgraph $T^i \in T_D$ which does not cross the edges of $C_5^\star$.

In addition, without loss of generality, we can consider the drawing of W_5 with the vertex notation in such a way as shown in Fig. 1(e). Clearly, all vertices t_i are placed in the region of $D(W_5)$ with five vertices v_1, v_2, v_3, v_5, and v_6 of W_5 on its boundary, and the set R_D is empty but the set S_D can be nonempty. So, two possible subcases may occur:

a) Let S_D be the nonempty set, that is, only the edge v_1v_2 of W_5 is crossed by the edge t_iv_4 of each subgraph $T^i \in S_D$. It is not difficult to verify in six possible regions of $D(W_5 \cup T^i)$ that $\mathrm{cr}_D(W_5 \cup T^i, T^k) \geq 6$ holds for any subgraph T^k, $k \neq i$. By fixing the subgraph $W_5 \cup T^i$, we obtain a contradiction with the assumption (3.3) in D.

b) Let S_D be the empty set, that is, each subgraph T^i crosses the edges of W_5 at least twice. The edges of $C_5^\star$ are not crossed by any $T^i \in T_D$ only if either $\mathrm{rot}_D(t_i) = (123465)$ or $\mathrm{rot}_D(t_i) = (123645)$. In both cases, $\mathrm{cr}_D(W_5 \cup T^i, T^k) \geq 6$ is fulfilling for any subgraph T^k, $k \neq i$, which yields a contradiction by fixing the subgraph $W_5 \cup T^i$. If there is no subgraph $T^i \in T_D$ with $\mathrm{cr}_D(C_5^\star, T^i) = 0$, then the discussed drawing contradicts the assumption of D again by Lemma 3.2.

Finally, without loss of generality, we assume the drawing of W_5 with the vertex notation in such a way as shown in Fig. 1(f). The set R_D is empty, but the set S_D can be nonempty, and so the proof proceeds in a similar way as for the drawing of W_5 in Fig. 1(e). Only if $S_D \neq \emptyset$, there are two possibilities of obtaining a subgraph $T^i \in S_D$ with either $\mathrm{rot}_D(t_i) = (123645)$ or $\mathrm{rot}_D(t_i) = (123654)$.

Case 4: $\mathrm{cr}_D(W_5) = 3$. If we consider the drawing of W_5 as in Fig. 1(g), by applying the same process as for the drawing in Fig. 1(b), we obtain at least $\left\lceil \frac{n}{2} \right\rceil + 1$ subgraphs $T^i \in T_D$ whose edges cross the edges of $C_5^\star$. Otherwise, if we assume the drawing of W_5 as shown in Fig. 1(h), then the same idea as in such a case in [3] regarding the existence or non existence of a subgraph $T^i \in R_D$ could be exploited.

We have shown, in all cases, that there are at least $6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + n + 3\left\lfloor \frac{n}{2} \right\rfloor + 1$ crossings in each good drawing D of the graph $W_5 + P_n$. This completes the proof of Theorem 3.3. □

4. Conclusions

Staš and Valiska were able to postulate that

$$\mathrm{cr}(W_m + P_n) = Z(m+1)Z(n) + (Z(m) - 1)\left\lfloor \frac{n}{2} \right\rfloor + n + 1 \quad \text{for all } m \geq 3, n \geq 2,$$

where $Z(n) = \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$ is Zarankiewicz's number. Recently, this conjecture was proved for the graph $W_3 + P_n$ by Klešč and Schrötter [17], and for the graph $W_4 + P_n$ by Staš and Valiska [25]. Theorem 3.3 also confirms the validity of this conjecture for $W_5 + P_n$.

Theorem 4.4 ([17], Theorem 4.2). $\mathrm{cr}(W_3 + P_n) = 2\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + n + 1$ *for* $n \geq 2$.

Theorem 4.5 ([25], Theorem 3.3). $\mathrm{cr}(W_4 + P_n) = 4\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + n + \left\lfloor \frac{n}{2} \right\rfloor + 1$ *for* $n \geq 2$.

On the other hand, the graphs $W_m + P_2$ and $W_m + P_3$ are isomorphic to the join product of the cycle C_m with the cycle C_3 and with the graph $K_4 \setminus e$ obtained by removing one edge from the complete graph K_4, respectively. The exact values for the crossing numbers of the graphs $C_m + C_n$ are given by Klešč [11], that is, $\mathrm{cr}(C_m + C_n) = Z(m)Z(n) + 2$

for any $m, n \geq 3$ with $\min\{m, n\} \leq 6$. The crossing numbers of $K_4 \setminus e + C_m$ equal to $2\lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor + \lfloor \frac{m}{2} \rfloor + 3$ were established also by Klešč [12]. Further, the graph $W_m + P_4$ is isomorphic to the join product of the cycle C_m with the graph $K_{1,4} + 3e$ obtained by adding three non incident edges with the same vertex to the complete bipartite graph $K_{1,4}$. Using the result of Staš [23], the crossing numbers of the graphs $(K_{1,4} + 3e) + C_m$ are given by $4\lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor + 2\lfloor \frac{m}{2} \rfloor + 3$ for each $m \geq 3$. These facts allow us to determine another results for the join product of the wheels W_m with the path on two, three, and four vertices.

Theorem 4.6 ([25], Theorem 5.1). $\mathrm{cr}(W_m + P_2) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor + 2$ *for* $m \geq 3$.

Theorem 4.7 ([25], Theorem 5.2). $\mathrm{cr}(W_m + P_3) = 2\left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor + \left\lfloor \frac{m}{2} \right\rfloor + 3$ *for* $m \geq 3$.

Theorem 4.8 ([23], Corollary 7). $\mathrm{cr}(W_m + P_4) = 4\left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor + 2\left\lfloor \frac{m}{2} \right\rfloor + 3$ *for* $m \geq 3$.

One can easily verify that these results also confirm the validity of this conjecture for the graphs $W_m + P_2$, $W_m + P_3$, and $W_m + P_4$.

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