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The penalty method for generalized mixed variational-hemivariational inequality problems

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ABSTRACT. It is well known that many popular variational, quasi-variational, hemivariational inequalities and variational inclusions involving constraints in a Banach space to convert a fixed point problems for finding the solution of such problems. This paper is to infuse a sequence of penalized problems without constraints and we show under the few reasonable assumptions to the Kuratowski upper limit with respect to the weak topology of the sets of solutions to penalized problems is nonempty. As an application, we explore two complicated partial differential systems of elliptic mixed boundary value problem involving a nonlinear nonhomogeneous differential operator with an obstacle effect, and a nonlinear elastic contact problem in mechanics with unilateral constraints.

1. INTRODUCTION

The theory of variational inequality problem was first initiated by Hartman and Stampacchia in 1966, *see* [10] for modelling problems arising from mechanics. To study the regularity problem for partial differential equations, Kinderlehrer and Stampacchia [11] studied a generalization of the Lax-Milgram theorem and called all problems of such type the variational inequality problems. The variational inequality problem is also known to have numerous implications in diverse areas such as, physics, science and technology, economics, optimal control theory, mathematical programming and others numerous fields. This theory provides a simple, natural and unified framework for a general treatment of various mathematical problems like the minimization problems, network equilibrium problems, complementarity problems.
Panagiotopoulos demonstrated the hemivariational inequalities as the variational formulation of important classes of unilateral and inequality problems in mechanical sciences, *see* [22, 23, 20]. The notion of hemivariational inequality is a generalization of variational inequality for a case where the function involved in nonconvex and nonsmooth. The hemivariational inequalities is based on the concept of Clarke's generalized gradient for locally Lipschitz functions, *see* [5, 6, 7, 9, 16, 17, 21, 24].

In the past few years, several types of variational and hemivariational inequalities have been developed and the study of variational-hemivariational inequalities has emerged today as a new, noble, innovative and interesting branch of applied and industrial mathematics, *see* [2, 18, 19, 25, 26, 27, 28, 29, 30, 31, 33, 34, 35, 36].
The problem of generalized mixed variational-hemivariational inequality problem demonstrated in this work as follows:
Let $(\mathcal{V}, \|\cdot\|_{\mathcal{V}})$ and $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ be reflexive and separable Banach spaces, and $\mho$ be a nonempty closed convex subset of $\mathcal{V}$. $\mathcal{V}^*$ denotes the dual space of $\mathcal{V}$ and $\langle\cdot,\cdot\rangle$ be the duality pairing between $\mathcal{V}^*$ and $\mathcal{V}$. Given the mappings $\mathscr{N} : \mathcal{V}\times\mathcal{V} \to \mathcal{V}^*$, $J : \mathbb{X} \to \mathbb{R}$, φ :

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$\mathcal{V} \to \mathbb{R} \cup \{+\infty\}$, and an element $f \in \mathcal{V}^*$, we consider the following generalized mixed variational-hemivariational inequality problem with constraints for finding $x \in \mho$ such that

$$\langle \mathscr{N}(x,x) - f, y - x\rangle + \varphi(y) - \varphi(x) + J^0(\varsigma x; \varsigma y - \varsigma x) \geq 0, \ \forall y \in \mho, \tag{1.1}$$

where $J^0(\varsigma, x; y)$ stands for the generalized Clarke directional derivative of $J(\varsigma, \cdot)$ at a point $x \in \mathbb{X}$ in the direction $y \in \mathbb{X}$.
The aim of this paper is to provide an existence result for (1.1) by using the assertion of relaxed monotonicity of $\mathscr{N}$ with respect to the first variable and relaxed Lipschitz continuity of the second variable. Latter on, we intend to discuss penalized problems without constraints as follows:

(i) the Kuratowski upper limit with respect to the weak topology of the sets of solutions to penalized problems, $\rho - \limsup_{n\to\infty} \mathbb{P}_n$ is nonempty and

$$\rho - \limsup_{n\to\infty} \mathbb{P}_n \subset \mathbb{P}$$

where $\mathbb{P}_n$ and $\mathbb{P}$ denotes the sets of solutions to penalized problem and (1.1), *respectively*.

(ii) $\rho - \limsup_{n\to\infty} \mathbb{P}_n = \varrho - \limsup_{n\to\infty} \mathbb{P}_n \subset \mathbb{P}$, if $\mathscr{N}$ satisfies (P_+)-property.

(iii) If $\mathscr{N}$ satisfies (P_+)-property then for each $x \in \varrho - \limsup_{n\to\infty} \mathbb{P}_n$ and sequence $\{\tilde{x}_n\}$ with $\tilde{x}_n \in \arg\min_{x_n \in \mathbb{P}_n} \|x_n - x\|_{\mathcal{V}}$ for each $n \in \mathbb{N}$, there exists a subsequence of $\{\tilde{x}_n\}$ converging strongly to x.

The last objective is to give some application, the first problem is an elliptic mixed boundary value problem involving a nonlinear nonhomogeneous differential operator with an obstacle effect which originates from the semipermeability phenomena. The second is an elastic contact problem in which the constitutive law is described by a convex subdifferential inclusion, while the contact boundary conditions involve non monotone constraints and a reformulated by a generalized Signorini contact condition governed by a Clarke subdifferential of a locally Lipschitz function.

2. Preliminaries

Throughout the text " $\rightharpoonup$ " and " $\to$ " stand for the weak and the strong convergence, *respectively*.

Definition 2.1. A single-valued mapping $\mathscr{A} : \mathcal{V} \to \mathcal{V}^*$ has (P_+)-property, if any sequence $\{x_n\}$ weakly convergent to x with

$$\limsup_{n\to\infty} \langle \mathscr{A}(x_n), x_n - x\rangle \leq 0$$

strongly converges to x.

Definition 2.2. Let $(\mathbb{X}, \zeta)$ be a Hausdorff topological space and $\{\mathscr{A}_n\} \subset 2^{\mathbb{X}}$ for $n \geq 1$. We define

$$\zeta - \lim_{n\to\infty} \inf \mathscr{A}_n = \left\{ x \in \mathbb{X} \,| x = \zeta - \lim_{n\to\infty} x_n, \ x_n \in \mathscr{A}_n \ \forall\, n \geq 1 \right\},$$

and

$$\zeta - \lim_{n\to\infty} \sup \mathscr{A}_n = \left\{ x \in \mathbb{X} \,| x = \zeta - \lim_{\kappa\to\infty} x_{n_\kappa}, \ x_{n_\kappa} \in \mathscr{A}_{n_\kappa}, \ n_1 < n_2 < \cdots < n_{n_\kappa} < \cdots \right\}.$$

The set $\zeta - \lim_{n\to\infty} \inf \mathscr{A}_n$ is called the ζ-Kuratowski lower limit of the sets $\mathscr{A}_n$ and $\zeta - \lim_{n\to\infty} \sup \mathscr{A}_n$ is called the ζ-Kuratowski upper limit of the sets $\mathscr{A}_n$.
We note that if $\mathscr{A} = \zeta - \lim_{n\to\infty} \inf \mathscr{A}_n = \zeta - \lim_{n\to\infty} \sup \mathscr{A}_n$, then $\mathscr{A}$ is called ζ-Kuratowski limit of the sets $\mathscr{A}_n$.

Definition 2.3. [5] A function $J : \mathcal{V} \to \mathbb{R}$ is called locally Lipschitz at $x \in \mathcal{V}$ if there exist a neighborhood $C(x)$ of x in $\mathcal{V}$ and a constant $\ell_x > 0$ such that

$$|J(z) - J(y)| \leq \ell_x \|z - y\|_{\mathcal{V}}, \ \forall\, z, y \in C(x).$$

Definition 2.4. [5] Given a locally Lipschitz function $J : \mathcal{V} \to \mathbb{R}$, we denote by $J^0(x; y)$ the generalized directional derivative of J at $x \in \mathcal{V}$ in the direction $y \in \mathcal{V}$ defined by

$$J^0(x; y) = \limsup_{\lambda \to 0^+, z \to x} \frac{J(z + \lambda y) - J(z)}{\lambda}.$$

The generalized gradient of $J : \mathcal{V} \to \mathbb{R}$ at $x \in \mathcal{V}$ is defined by

$$\partial J(x) = \{\xi \in \mathcal{V}^* | J^0(x; y) \geq \langle \xi, y \rangle, \ \forall\, y \in \mathcal{V}\}.$$

Proposition 2.1. [16] *Let $J : \mathcal{V} \to \mathbb{R}$ be the locally Lipschitz function, then*

(i) *for every $x \in \mathcal{V}$, the function $\mathcal{V} \ni y \to J^0(x; y) \in \mathbb{R}$ is positively homogeneous and subadditive, i.e.,*

$$J^0(x; \varsigma y) = \varsigma J^0(x; y), \ \forall\, \varsigma \geq 0, y \in \mathcal{V}$$

and

$$J^0(x; y_1 + y_2) \leq J^0(x; y_1) + J^0(x; y_2), \ \forall y_1, y_2 \in \mathcal{V}.$$

(ii) *for every $y \in \mathcal{V}$, it holds*

$$J^0(x; y) = \max\{\langle \xi, y \rangle \mid \xi \in \partial J(x)\}.$$

(iii) *the function $\mathcal{V} \times \mathcal{V} \ni (x, y) \to J^0(x; y) \in \mathbb{R}$ is upper semicontinuous.*

(iv) *the graph of generalized gradient $\partial J : \mathcal{V} \to 2^{\mathcal{V}^*}$ is closed in $\mathcal{V} \times (\rho^* - \mathcal{V}^*)$ topology, i.e., if $\{x_n\} \subset \mathcal{V}$ and $\{\xi_n\} \subset \mathcal{V}^*$ are sequences such that*

$$\xi_n \in \partial J(x_n)$$

and

$$x_n \to x, \ \xi_n \to \xi$$

weakly in $\mathcal{V}^*$, then*

$$\xi \in \partial J(x),$$

where $(\rho^* - \mathcal{V}^*)$ denotes the space $\mathcal{V}$ equipped with weak topology.*

Definition 2.5. [32] Let $\mathbb{X}$ be a Hausdorff topological vector space and let $\mho \subseteq \mathbb{X}$. The application $\mathscr{G} : \mho \rightrightarrows \mathbb{X}$ is called a KKM application if for every finite number of elements $x_1, x_2, \cdots, x_n \in \mho$ one has

$$co\{x_1, x_2, \cdots, x_n\} \subseteq \bigcup_{i=1}^{n} \mathscr{G}(x_i).$$

Lemma 2.1. [32] *Let $\mathbb{X}$ be a Hausdorff topological vector space, $\mho \subseteq \mathbb{X}$ and . The application $\mathscr{G} : \mho \rightrightarrows \mathbb{X}$ be a KKM application. If $\mathscr{G}(x)$ is closed for every $x \in \mho$, and there exists $x_0 \in \mho$ such that $\mathscr{G}(x_0)$ is compact, then*

$$\bigcap_{x \in \mho} \mathscr{G}(x) \neq \emptyset.$$

Proposition 2.2. [3] *Assume that $\varphi : \mathbb{X} \to (-\infty, +\infty]$ is convex, lower semi continuous and $\varphi \not\equiv +\infty$. Then $\varphi^* \not\equiv +\infty$, and in particular, φ is bounded below by an affine continuous function.*

Definition 2.6. [1] Let $\mathscr{N} : \mathcal{V} \times \mathcal{V} \to \mathcal{V}^*$ be a mapping. Then $\mathscr{N}$ is said to be

(i) monotone, if

$$\langle \mathscr{N}(y, y) - \mathscr{N}(x, x), y - x \rangle \geq 0, \ \forall x, y \in \mathcal{V}.$$

(ii) strongly monotone, if there exist $c > 0$ and $r > 1$ such that for all $x, y \in \mathcal{V}$,

$$\langle \mathscr{N}(y,y) - \mathscr{N}(x,x), y - x\rangle \geq c\|x-y\|^r.$$

(iii) relaxed monotone with respect to first variable of $\mathscr{N}$ if there exist $c > 0$ and $r > 1$ such that for all $x, y \in \mathcal{V}$,

$$\langle \mathscr{N}(y,y) - \mathscr{N}(x,x), y - x\rangle \geq -c\|x-y\|^r.$$

(iv) relaxed Lipschitz continuous with respect to the second variable if there exist $\beta \geq 0$ and $r > 1$ such that for all $x, y \in \mathscr{N}$,

$$\langle \mathscr{N}(y,y) - \mathscr{N}(x,x), y - x\rangle \leq -\beta\|x-y\|^r.$$

3. Main results

The goal of this section is to establish an existence and convergence result for a generalized penalty method utilized to the generalized mixed variational-hemivariational inequality problem (1.1). First of all we define the following assertions.

H(φ): $\varphi : \mathcal{V} \to \mathbb{R} \cup \{+\infty\}$ is a proper, convex and lower-semicontinuous function.

H(ς): $\varsigma : \mathcal{V} \to \mathbb{X}$ is a linear and continuous operator.

H(J): $J : \mathbb{X} \to \mathbb{R}$ is a locally Lipschitz function.

H($\mathscr{T}$): $\mathscr{T} : \mathcal{V} \to \mathbb{R}$ is a function such that

$$\mathscr{T}(0_{\mathcal{V}}) \leq 0$$

and

(i) $\limsup_{t\to 0^+} \dfrac{\mathscr{T}(ty)}{t} \geq 0, \forall y \in \mathcal{V}$.

(ii) for all $\{y_n\} \subset \mathcal{V}$ with $y_n \rightharpoonup y \in \mathcal{V}$, we have

$$\mathscr{T}(y) \leq \limsup_{n\to\infty} \mathscr{T}(y_n).$$

H($\mathscr{N}$): $\mathscr{N} : \mathcal{V} \times \mathcal{V} \to \mathcal{V}^*$ is an operator such that

(i) the inequality

$$\liminf_{t\to 0^+} \langle \mathscr{N}(ty + (1-t)x, x), y - x\rangle \leq \langle \mathscr{N}(x,x), y - x\rangle$$

holds for all $x, y \in \mho$, similarly, we have also

$$\liminf_{t\to 0^+} \langle \mathscr{N}(x, ty + (1-t)x), y - x\rangle \leq \langle \mathscr{N}(x,x), y - x\rangle.$$

(ii) the set-valued map

$$x \to \mathscr{N}(x,x) + \varsigma^* \partial J(\varsigma x)$$

is relaxed $\mathscr{N}$-monotone, *i.e.*, the inequality

$$\langle \mathscr{N}(x,x) - \mathscr{N}(y,y), u - v\rangle + \langle \xi_x - \xi_y, \varsigma(x-y)\rangle_{\mathbb{X}^*\times\mathbb{X}} \geq \mathscr{T}(x-y)$$

holds for all $\xi_x \in \partial J(\varsigma x), \xi_y \in \partial J(\varsigma y)$ and all $x, y \in \mathcal{V}$.

(iii) there exists $y_0 \in \mho \cap D(\varphi)$ such that

$$\liminf_{\substack{x\in\mathcal{V},\\ \|x\|_{\mathcal{V}}\to+\infty}} \frac{\langle \mathscr{N}(x,x), x - y_0\rangle + \inf_{\xi_x\in\partial J(\varsigma x)} \langle \xi_x, \varsigma(x-y_0)\rangle_{\mathbb{X}^*\times\mathbb{X}}}{\|x\|_{\mathcal{V}}} = +\infty. \tag{3.2}$$

Remark 3.1. Condition **H($\mathscr{N}$)(iii)** represents a generalized coercivity condition for the map

$$x \to \mathscr{N}(x,x) + \varsigma^* \partial J(\varsigma x).$$

In particular, if $\mathscr{N} : \mathcal{V} \times \mathcal{V} \to \mathcal{V}^*$ is such that

$$\langle \mathscr{N}(x,x), x\rangle \geq r(\|x\|_{\mathcal{V}})\|x\|_{\mathcal{V}}, \forall x \in \mathcal{V},$$

and

$$x \to \partial J(x)$$

satisfies the growth condition

$$\|\partial J(x)\|_{\mathbb{X}^*} \leq \ell(\|x\|_{\mathbb{X}})\|x\|_{\mathbb{X}}, \forall x \in \mathbb{X},$$

where $r : \mathbb{R} \to \mathbb{R}$ and $\ell : \mathbb{R}_+ \to \mathbb{R}_+$ are such that

$$r(s) \to \infty \text{ as } s \to \infty,$$

and

$$\frac{\ell(s\|\varsigma\|)}{r(s)} \to 0 \text{ as } s \to \infty$$

or

$$\ell(s\|\varsigma\|)\|\varsigma\| < r(s), \forall\, s \geq s_0 \text{ with some } s_0 > 0,$$

then the condition **H($\mathscr{N}$)(iii)** holds.

We now provide the following general existence theorem for (1.1).

Theorem 3.1. *Assume that **H($\mathscr{N}$)**, **H(φ)**, **H(ς)**, **H($\mathscr{T}$)** and **H(J)** hold. Then, we have*

(i) *for each $f \in \mathcal{V}^*$, $x \in \mho$ is a solution to* (1.1) *if and only if it solves the following Minty variational-hemivariational inequality problem for finding $x \in \mho$ such that*

$$\langle \mathscr{N}(y,y) - f, y - x\rangle + J^0(\varsigma y; \varsigma y - \varsigma x) + \varphi(y) - \varphi(x) \geq \mathscr{T}(y - x)\ \forall y \in \mho.$$

(ii) *the set of solutions to (1.1), denoted by $\mathbb{P}$, is nonempty, bounded and weakly closed.*
(iii) *if $\mathscr{T} : \mathcal{V} \to \mathbb{R}$ is a convex function, then $\mathbb{P}$ is convex.*
(iv) *if $\mathscr{T}(y) > 0$ for all $y \in \mathcal{V} \setminus \{0_{\mathcal{V}}\}$, then (1.1) has a unique solution.*

Proof. (i) The part of the proof is standard, and is omitted here.

(ii) It is follows the results of [12], Lemma 3.3 and Theorem 3.4 or [13], Lemma 3.1 and Theorem 3.2. Therefore, we omit the specifics.
First, we show the nonemptiness of the solution set $\mathbb{P}$. Assume that $\mho$ is a bounded set and consider a set-valued mapping $\mathcal{G} : \mho \to 2^{\mho}$ defined by

$$\mathcal{G}(y) = \big\{x \in \mho | \langle \mathscr{N}(y,y) - f, y - x\rangle + \inf_{\xi_y \in \partial J(\varsigma y)} \langle \xi_y, \varsigma(y - x)\rangle_{\mathbb{X}^* \times \mathbb{X}} + \varphi(y) - \varphi(x) \geq \mathscr{T}(y - x)\big\}, \forall y \in \mho.$$

From the assumptions of **H(φ)** and **H($\mathscr{T}$)** we see that $\mathcal{G}(y)$ is nonempty and weakly closed for each $y \in \mho$. Next, we distinguish two cases:
(a) $\mathcal{G}$ is a KKM-map, and
(b) $\mathcal{G}$ is not a KKM-map.
If (a) is true, then by the KKM principle, we have

$$\bigcap_{y \in \mho} \mathcal{G}(y) \neq \emptyset.$$

Therefore, invoking assertion (i) we deduce that $\mathbb{P}$ is nonempty.

Furthermore, if (b) true, then we are able to defined a finite set $\{y_1, y_2, \cdots, y_N\} \subset \mho$ and $x_0 \in \mho$ with $x_0 = \sum_{i=1}^{N} t_i y_i, t_i \in [0,1]$ for all $i = 1, 2, \cdots, N$ and $\sum_{i=1}^{N} t_i = 1$, such that

$$x_0 \notin \bigcup_{i=1}^{N} \mathcal{G}(y_i).$$

Hence,

$$\langle \mathscr{N}(y_i, y_i), y_i - x_0\rangle + \inf_{\xi_i \in \partial J(\varsigma y_i)} \langle \xi_i, \varsigma(y_i - x_0)\rangle_{\mathbb{X}^* \times \mathbb{X}} + \varphi(y_i) - \varphi(x_0) < \langle f, y_i - x_0\rangle$$

$$+\mathscr{T}(y_i - x_0), \quad \text{for all } i = 1, 2, \cdots, N. \tag{3.3}$$

We now assert that there exists a neighborhood O of x_0 such that

$$\langle \mathscr{N}(y_i, y_i), y_i - y\rangle + \inf_{\xi_i \in \partial J(\varsigma y_i)} \langle \xi_i, \varsigma(y_i - y)\rangle_{\mathbb{X}^* \times \mathbb{X}} + \varphi(y_i) - \varphi(y) < \langle f, y_i - y\rangle$$

$$+\mathscr{T}(y_i - y), \quad \text{for all } y \in O \cap \mho \text{ and } i = 1, 2, \cdots, N. \tag{3.4}$$

By contrary we suppose there exist $j_0 \in \{1, 2, \cdots, N\}$ and a sequence $\{x_n\}$ with

$$x_n \to x_0 \text{ as } n \to \infty$$

such that

$$\langle \mathscr{N}(y_{j_0}, y_{j_0}), y_{j_0} - x_n\rangle + \langle \xi_{j_0}, \varsigma(y_{j_0} - x_n)\rangle_{\mathbb{X}^* \times \mathbb{X}} + \varphi(y_{j_0}) - \varphi(x_n) \geq \langle f, y_{j_0} - x_n\rangle$$
$$+ \mathscr{T}(y_{j_0} - x_n), \quad \text{for all } \xi_{j_0} \in \partial J(\varsigma y_{j_0}) \text{ and all } n \in \mathbb{N}.$$

Passing to the upper limit, as $n \to \infty$, and using the hypothesis **H($\mathscr{T}$)(ii)**, we obtain

$$\langle \mathscr{N}(y_{j_0}, y_{j_0}), y_{j_0} - x_0\rangle + \langle \xi_{j_0}, \varsigma(y_{j_0} - x_0)\rangle_{\mathbb{X}^* \times \mathbb{X}} + \varphi(y_{j_0}) - \varphi(x_0) \geq \langle f, y_{j_0} - x_0\rangle$$
$$+ \mathscr{T}(y_{j_0} - x_0), \quad \text{for all } \xi_{j_0} \in \partial J(\varsigma y_{j_0}).$$

This results a contradiction with (3.3), therefore the assertion (3.4) is valid.

By virtue of (3.4) and the monotonicity of

$$x \to \mathscr{N}(x, x) + \varsigma^* \partial J(\varsigma x),$$

one has

$$\begin{aligned}
&\langle \mathscr{N}(y, y), y - y_i\rangle + \langle \xi_y, \varsigma(y - y_i)\rangle_{\mathbb{X}^* \times \mathbb{X}} + \varphi(y) - \varphi(y_i) \\
&\geq \langle \mathscr{N}(y_i, y_i), y - y_i\rangle + \langle \xi_i, \varsigma(y - y_i)\rangle_{\mathbb{X}^* \times \mathbb{X}} + \varphi(y) - \varphi(y_i) + \mathscr{T}(y_i - y) \\
&\geq \langle f, y - y_i\rangle - \mathscr{T}(y_i - y) + \mathscr{T}(y_i - y) \\
&= \langle f, y - y_i\rangle \ \forall\, \xi_y \in \partial J(\varsigma y), \xi_i \in \partial J(\varsigma y_i), \text{ all } y \in O \cap \mho \text{ and all } i = 1, 2, \cdots, N.
\end{aligned}$$

By a direct calculation, we have

$$\langle \mathscr{N}(y, y) - f, y - x_0\rangle + \langle \xi_y, \varsigma(y - x_0)\rangle_{\mathbb{X}^* \times \mathbb{X}} + \varphi(y) - \varphi(x_0) \geq 0, \ \forall\, \xi_y \in \partial J(\varsigma y),$$
$$\text{and all } y \in O \cap \mho.$$

Let $z \in \mho$ be arbitrary and $x_0 \in int(O)$, so we are able to find $t \in (0,1)$ small enough such that

$$y_t = tz + (1-t)x_0 \in O \cap \mho.$$

Inserting $y = y_t$ into the above inequality, a simple calculation gives

$$\langle \mathscr{N}(x_0, x_0) - f, y - x_0\rangle + J^0(\varsigma x_0, \varsigma y - \varsigma x_0) + \varphi(y) - \varphi(x_0) \geq 0, \ \forall\, y \in \mho,$$

that is

$$x_0 \in \mathbb{P} \neq \emptyset.$$

Further, we consider the situation that $\mho$ is unbounded. The above analysis indicates that for each $r > 0$, there exists $x_r \in \mho_r$ such that

$$\langle \mathscr{N}(x_r, x_r) - f, y - x_r\rangle + J^0(\varsigma x_r, \varsigma y - \varsigma x_r) + \varphi(y) - \varphi(x_r) \geq 0, \ \forall\, y \in \mho_r \cap \bar{B}(0, r), \tag{3.5}$$

where

$$\bar{B}(0, r) = \{y \in \mathcal{V} \mid \|y\|_{\mathcal{V}} \leq r\}.$$

Moreover, we shows that there exist $r_0 > 0$ and a solution $x^\star$ to the problem (3.5) with $r = r_0$ such that

$$\|x^*\|_{\mathcal{V}} < r_0$$

Let $z \in \mho$ be arbitrary and $t \in (0, 1)$ be small enough. Putting $y = tz + (1-t)x^*$ into (3.5) for $r = r_0$, and then passing to the upper limit, as $t \to 0^+$, we have

$$\langle \mathscr{N}(x^*, x^*) - f, z - x^*\rangle + J^0(\varsigma x^*, \varsigma z - \varsigma x^*) + \varphi(z) - \varphi(x^*) \geq 0.$$

Since $z \in \mho$ is arbitrary and we infer that x^* is a solution to (1).
We now show that $\mathbb{P}$ is bounded. By the coercivity condition **H($\mathscr{N}$)(iii)** and **H(φ)**, we infer

$$\frac{\langle \mathscr{N}(x, x), x - y_0\rangle + \inf\limits_{\xi \in \partial J(\varsigma x)} \langle \xi_x, \varsigma(x - y_0)\rangle_{\mathbb{X}^* \times \mathbb{X}} + \varphi(x) - \varphi(y_0)}{\|x\|_{\mathbb{X}}}$$
$$\geq \frac{\langle \mathscr{N}(x, x), x - y_0\rangle + \inf\limits_{\xi \in \partial J(\varsigma x)} \langle \xi_x, \varsigma(x - y_0)\rangle_{\mathbb{X}^* \times \mathbb{X}} - \alpha_\varphi \|x\|_{\mathcal{V}} - \beta_\varphi - \varphi(y_0)}{\|x\|_{\mathbb{X}}}$$

where $\alpha_\varphi, \beta_\varphi > 0$ are such that

$$\varphi(y) \geq -\alpha_\varphi \|y\|_{\mathcal{V}} - \beta_\varphi, \ \forall\, y \in \mathcal{V}. \tag{3.6}$$

This implies that

$$\liminf_{x \in \mho, \|x\|_{\mathcal{V}} \to \infty} \frac{\langle \mathscr{N}(x, x), x - y_0\rangle + \inf\limits_{\xi \in \partial J(\varsigma x)} \langle \xi_x, \varsigma(x - y_0)\rangle_{\mathbb{X}^* \times \mathbb{X}} + \varphi(x) - \varphi(y_0)}{\|x\|_{\mathbb{X}}} = +\infty. \tag{3.7}$$

We used the coercivity condition **H($\mathscr{N}$)(iii)** and the fact that $y_0 \in D(\varphi)$. From the preceding condition, it is not difficult to demonstrate that for each $f \in \mathcal{V}^*$ fixed, there exists a constant $\gamma_f > 0$ such that

$$\|x\|_{\mathcal{V}} \leq \gamma_f \ \forall\, x \in \mathbb{P}.$$

Hence the set $\mathbb{P}$ is bounded.

Next, we show that $\mathbb{P}$ is weakly closed. Let $\{x_n\} \subset \mathbb{P}$ be such that

$$x_n \rightharpoonup x \text{ as } n \to \infty \text{ for some } x \in \mho.$$

Utilizing the $\mathscr{T}$-relaxed monotonicity of the map

$$x \to \mathscr{N}(x, x) + \varsigma^* \partial J(\varsigma x),$$

we have

$$\langle \mathscr{N}(y, y) - f, y - x_n\rangle + \langle \xi_y, \varsigma(y - x_n)\rangle_{\mathbb{X}^* \times \mathbb{X}} + \varphi(y) - \varphi(x_n)$$
$$\geq \mathscr{T}(y - x_n), \ \forall \xi_y \in \partial J(\varsigma y), \ y \in \mho.$$

We reach to the upper limit, as $n \to \infty$, we have

$$\langle \mathscr{N}(y, y) - f, y - x\rangle + J^0(\varsigma y, \varsigma y - \varsigma x) + \varphi(y) - \varphi(x) \geq \mathscr{T}(y - x), \ \forall y \in \mho, \tag{3.8}$$

that is

$$x \in \mathbb{P}.$$

Consequently, the set $\mathbb{P}$ is weakly closed and proof is completed.

(iii) Assume that the function $\mathscr{T}$ is convex. Let $x_1, x_2 \in \mathbb{P}$, $t \in (0,1)$ be arbitrary, and denote $x_t = tx_1 + (1-t)x_2$. Hence,

$$\langle \mathscr{N}(y,y) - f, y - x_i\rangle + \langle \xi_y, \varsigma(y - x_i)\rangle_{\mathbb{X}^*\times\mathbb{X}} + \varphi(y) - \varphi(x_i) \geq \mathscr{T}(y - x_i),$$

$$\forall \xi_y \in \partial J(\varsigma y), y \in \mho.$$

The convexity of $\mathscr{T}$ and φ reveals

$$\begin{aligned}
&\langle \mathscr{N}(y,y) - f, y - x_t\rangle + J^0(\varsigma y, \varsigma y - \varsigma x_t) + \varphi(y) - \varphi(x_t)\\
&\quad \geq \langle \mathscr{N}(y,y) - f, y - x_t\rangle + \langle \xi_y, \varsigma(y - x_t)\rangle_{\mathbb{X}^*\times\mathbb{X}} + \varphi(y) - \varphi(x_t)\\
&\quad \geq t[\langle \mathscr{N}(y,y) - f, y - x_1\rangle + \langle \xi_y, \varsigma(y - x_1)\rangle_{\mathbb{X}^*\times\mathbb{X}} + \varphi(y) - \varphi(x_1)]\\
&\quad + (1-t)[\langle \mathscr{N}(y,y) - f, y - x_2\rangle + \langle \xi_y, \varsigma(y - x_2)\rangle_{\mathbb{X}^*\times\mathbb{X}} + \varphi(y) - \varphi(x_2)]\\
&\quad \geq t\mathscr{T}(y - x_1) + (1-t)\mathscr{T}(y - x_2)\\
&\quad \geq \mathscr{T}(y - x_t),\ \forall \xi_y \in \partial J(\varsigma y),\ \forall y \in \mho.
\end{aligned}$$

Hence $x_t \in \mathbb{P}$. Therefore the set $\mathbb{P}$ is convex.

(iv) Let $x_1, x_2 \in \mho$ be two solutions to (1), then

$$\mathscr{T}(x_1 - x_2) \leq \langle \mathscr{N}(x_1,x_1) - \mathscr{N}(x_2,x_2), x_1 - x_2\rangle + \langle \xi_1 - \xi_2, \varsigma(x_1 - x_2)\rangle_{\mathbb{X}^*\times\mathbb{X}} \leq 0,$$

where $\xi_1 \in \partial J(\varsigma x_1), \xi_2 \in \partial J(\varsigma x_2)$ are such that

$$\begin{aligned}
J^0(\varsigma x_1; \varsigma x_2 - \varsigma x_1) &= \langle \xi_1, \varsigma(x_2 - x_1)\rangle_{\mathbb{X}^*\times\mathbb{X}},\\
J^0(\varsigma x_2; \varsigma x_1 - \varsigma x_2) &= \langle \xi_2, \varsigma(x_1 - x_2)\rangle_{\mathbb{X}^*\times\mathbb{X}}.
\end{aligned}$$

The along with the fact that

$$\mathscr{T}(y) > 0,\ \forall\, y \in \mathcal{V} \setminus \{O_{\mathcal{V}}\}$$

and implies

$$x_1 = x_2,$$

it shows that (1.1) has an unique solution.

□

Now we are in a position to demonstrate the penalty approximation procedure. Let $\lambda_n \in \mathbb{R}_+$ and $\mathcal{P}_n : \mathcal{V} \to \mathcal{V}^*$ be sequences satisfying the following hypotheses.

H(λ_n): $\lambda_n > 0$ for all $n \in \mathbb{N}$, and

$$\lambda_n \to 0 \text{ as } n \to \infty.$$

H($\mathcal{P}_n$): $\mathcal{P}_n : \mathcal{V} \times \mathcal{V} \to \mathcal{V}^*$ is a bounded, demicontinuous and monotone operator for all $n \in \mathbb{N}$ such that

(i) for each $y \in \mho$, there is a sequence $\{y_n\} \subset \mathcal{V}$ with the property

$$\mathcal{P}_n(y_n, y_n) = O_{\mathcal{V}^*},\ \forall n \in \mathbb{N}$$

and

$$y_n \to y \in \mathcal{V} \text{ as } n \to \infty.$$

(ii) there exists an operator $\mathcal{P} : \mathcal{V} \times \mathcal{V} \to \mathcal{V}^*$ such that
   (a) $\mathcal{P}(x,x) = 0_{\mathcal{V}^*}$ implies $x \in \mho$,

(b) for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$ in $\mathcal{V}$ and

$$\limsup_{n\to\infty}\langle \mathcal{P}_n(x_n,x_n), x_n - x\rangle \le 0,$$

we have

$$\liminf_{n\to\infty}\langle \mathcal{P}_n(x_n,x_n), x_n - y\rangle \ge \langle \mathcal{P}(x,x), x-y\rangle,\ \forall y\in\mathcal{V}.$$

Further, the following stronger versions of **H(φ), H($\mathscr{N}$)(iii)** and **H(ς)** will be used.

$\mathbf{H}(\varphi)'$: $\varphi : \mathcal{V}\to\mathbb{R}$ is a convex and lower semicontinuous function.

$\mathbf{H}(\mathscr{N})(i)'$**:** $\mathscr{N} : \mathcal{V}\times\mathcal{V}\to\mathcal{V}^*$ is a continuous operator.

$\mathbf{H}(\mathscr{N})(iii)'$**:** There exists $y_0\in\mho$ such that the coercivity condition holds

$$\liminf_{\substack{y\in B(y_0,1),x\in\mathcal{V},\\ \|x\|_{\mathcal{V}}\to+\infty}} \frac{\langle \mathscr{N}(x,x), x-y\rangle + \inf\limits_{\xi\in\partial J(\varsigma x)}\langle \xi_x, \varsigma(x-y)\rangle_{\mathbb{X}^*\times\mathbb{X}}}{\|x\|_{\mathcal{V}}} = +\infty,\ \forall y_0\in D.$$

$\mathbf{H}(\varsigma)'$**:** $\varsigma : \mathcal{V}\to\mathbb{X}$ is a linear, continuous and compact operator.

We introduce the following sequence of penalized problems associated to (1.1) for finding $x_n\in\mathcal{V}$ such that

$$\langle \mathscr{N}(x_n,x_n) - f, y - x_n\rangle + \frac{1}{\lambda_n}\langle \mathcal{P}_n(x_n,x_n), y-x_n\rangle + J^0(\varsigma x_n;\varsigma y - \varsigma x_n)$$

$$+\varphi(y) - \varphi(x_n) \ge 0,\ \forall y\in\mathcal{V}. \tag{3.9}$$

The second main result of this section is the following.

Theorem 3.2. *Assume that the hypotheses* $\mathbf{H}(\mathscr{N})(i)'$, ***H($\mathscr{N}$)(ii)***, $\mathbf{H}(\mathscr{N})(iii)'$, ***H($\mathscr{T}$)***, ***H(J)***, $\mathbf{H}(\varsigma)'$, $\mathbf{H}(\varphi)'$, ***H($\mathcal{P}_n$)****, and* ***H(λ_n)*** *hold,* $f\in\mathcal{V}^*$*, and* $\mathscr{T}:\mathcal{V}\to\mathbb{R}$ *is bounded. Then, we have*

(i) *for each* $n\in\mathbb{N}$*, the set of solutions to* (3.9)*, denoted by* $\mathbb{P}_n$*, is nonempty, bounded and weakly closed.*

(ii) $\emptyset \ne \rho - \limsup\limits_{n\to\infty}\mathbb{P}_n \subset \mathbb{P}$.

(iii) *if* $\mathscr{N}$ *satisfies* (P_+)*-property, then* $\rho\text{-}\limsup\limits_{n\to\infty}\mathbb{P}_n = \varrho - \limsup\limits_{n\to\infty}\mathbb{P}_n$.

(iv) *if* $\mathscr{N}$ *satisfies* (P_+)*-property, then for each* $x\in\varrho$ - $\limsup\limits_{n\to\infty}\mathbb{P}_n$ *and any sequence* $\{\tilde{x}_n\}$ *with*

$$\tilde{x}_n \in \arg\min_{x_n\in\mathbb{S}_n}\|x_n - x\|_{\mathcal{V}},\ \forall n\in\mathbb{N},$$

there exists a subsequence of $\{\tilde{u}_n\}$ *converging strongly to* x.

(v) *if* (1.1) *has a unique solution* $x^*\in\mho$*, then* (3.9) *has a unique solutions and the whole sequence of solutions* $\{x_n\}$ *of* (3.9) *converges weakly to* x*. Moreover, if, in addition,* $\mathscr{N}$ *satisfies* (P_+)*-property, then the whole sequence* $\{x_n\}$ *solutions to* (3.9) *converges strongly to* x.

Proof. (i) For each given $n\in\mathbb{N}$, we define an operator $\mathscr{N}_n:\mathcal{V}\times\mathcal{V}\to\mathcal{V}^*$ by

$$\mathscr{N}_n(x,x) = \mathscr{N}(x,x) + \frac{1}{\lambda_n}\mathcal{P}_n(x,x),\ \forall x\in\mathcal{V}.$$

The boundedness of $\mathscr{N}$ and $\mathcal{P}_n$ guarantees that $\mathscr{N}_n$ is bounded too. Therefore, from the continuity of $\mathcal{P}_n$ and hypothesis $\mathbf{H}(\mathscr{N})(i)'$ that $\mathscr{N}_n$ satisfies condition **H($\mathscr{N}$)(i)**. In addition, the monotonicity of $\mathcal{P}_n$ and of

$$x\to\mathscr{N}(x,x) + \varsigma^*\partial J(\varsigma x)$$

imply that

$$x\to\mathscr{N}_n(x,x) + \varsigma^*\partial J(\varsigma x)$$

is $\mathscr{T}$-relaxed monotone, *i.e.*, $\mathscr{N}_n$ satisfies **H($\mathscr{N}$)(ii)**. Next, for any given $y_0 \in \mathcal{V}$, the following estimate

$$\begin{aligned}\frac{1}{\lambda_n}\langle \mathcal{P}_n(x,x), x-y_0\rangle &\geq \frac{1}{\lambda_n}\langle \mathcal{P}_n(y_0,y_0), x-y_0\rangle \\ &\geq -\frac{1}{\lambda_n}\|\mathcal{P}_n(y_0,y_0)\|_{\mathcal{V}^*}\ (\|y_0\|_{\mathcal{V}}+\|x\|_{\mathcal{V}})\end{aligned}$$

together with the coercivity condition **H($\mathscr{N}$)(iii)** entail that $\mathscr{N}_n$ fulfils **H($\mathscr{N}$)(iii)**. Therefore, employing Theorem 3.1, we conclude that for each $n \in \mathbb{N}$, the set $\mathbb{P}_n$ of solutions to (3.9) is nonempty, bounded and weakly closed.

(ii) First, we prove that the set $\rho\text{-}\limsup_{n\to\infty} \mathbb{P}_n$ is nonempty. Indeed, we have the following claim.

Claim 1. The set $\cup_{n\in\mathbb{N}}\mathbb{P}_n$ is uniformly bounded in $\mathcal{V}$.
By contradiction, assume that $\cup_{n\in\mathbb{N}}\mathbb{P}_n$ is unbounded. With out any loss of generality, we may assume that there exists a sequence $\{x_n\}$ with $x_n \in \mathbb{P}_n$ such that

$$\|x_n\|_{\mathcal{V}} \to \infty \text{ as } n \to \infty.$$

Hence,

$$\begin{aligned}&\langle \mathscr{N}(x_n,x_n)-f, y-x_n\rangle + \frac{1}{\lambda_n}\langle \mathcal{P}_n(x_n,x_n), y-x_n\rangle + J^0(\varsigma x_n;\varsigma y-\varsigma x_n)\\ &+\varphi(y)-\varphi(x_n)\geq 0, \forall y\in\mathcal{V},\ n\in\mathbb{N}. \qquad (3.10)\end{aligned}$$

Since $y_0 \in \mho$, *see* $\mathbf{H}(\mathscr{N})(iii)'$ and **H($\mathcal{P}_n$)(i)**, let $\{y_n\} \subset \mathcal{V}$ be a sequence such that

$$\mathcal{P}_n(y_n,y_n) = 0_{\mathcal{V}^*}\ \forall\, n\in\mathbb{N} \text{ and } y_n \to y_0 \text{ as } n\to\infty. \qquad (3.11)$$

Inserting $y = y_n$ into (10) gives

$$\begin{aligned}&\langle \mathscr{N}(x_n,x_n)-f, x_n-y_n\rangle + \varphi(x_n)-\varphi(y_n) - J^0(\varsigma x_n;\varsigma y_n-\varsigma x_n)\\ &\qquad\leq \frac{1}{\lambda_n}\langle \mathcal{P}_n(x_n,x_n)-\mathcal{P}_n(y_n,y_n), y_n-x_n\rangle.\end{aligned}$$

The monotonicity of $\mathcal{P}_n$ and the Cauchy inequality reveal

$$\begin{aligned}0 &\geq \langle \mathscr{N}(x_n,x_n)-f, x_n-y_n\rangle + \varphi(x_n)-\varphi(y_n) - J^0(\varsigma x_n;\varsigma y_n-\varsigma x_n)\\ &= \langle \mathscr{N}(x_n,x_n)-f, x_n-y_n\rangle + \varphi(x_n)-\varphi(y_n) - \langle\tilde{\xi}_n, \varsigma(y_n-x_n)\rangle_{\mathbb{X}^*\times\mathbb{X}}\\ &\geq \langle \mathscr{N}(x_n,x_n), x_n-y_n\rangle - \alpha_\varphi\|x_n\|_{\mathcal{V}} - \beta_\varphi - \varphi(y_n)\\ &\quad - \sup_{\xi_n\in\partial J(\varsigma x_n)}\langle \xi_n, \varsigma(y_n-x_n)\rangle_{\mathbb{X}^*\times\mathbb{X}} - \|f\|_{\mathcal{V}^*}\ (\|x_n\|_{\mathcal{V}}+\|y_n\|_{\mathcal{V}}),\end{aligned}$$

where $\alpha_\varphi,\ \beta_\varphi > 0$ are given in (6), and $\tilde{\xi}_n \in \partial J(\varsigma x_n)$ is such that

$$J^0(\varsigma x_n;\varsigma y_n-\varsigma x_n) = \langle \tilde{\xi}_n, \varsigma(y_n-x_n)\rangle_{\mathbb{X}^*\times\mathbb{X}}.$$

Hence, for n large enough we have

$$\begin{aligned}0 &\geq \frac{\langle \mathscr{N}(x_n,x_n)-f, x_n-y_n\rangle + \varphi(x_n)-\varphi(y_n) - J^0(\varsigma x_n;\varsigma y_n-\varsigma x_n)}{\|x_n\|_{\mathcal{V}}}\\ &\geq \frac{\langle \mathscr{N}(x_n,x_n), x_n-y_n\rangle - \inf_{\xi_n\in\partial J(\varsigma x_n)}\langle \xi_n, \varsigma(y_n-x_n)\rangle_{\mathbb{X}^*\times\mathbb{X}} - \beta_\varphi - \varphi(y_n) - \|f\|_{\mathbb{V}^*}\gamma_0}{\|x_n\|_{\mathcal{V}}}\\ &\quad-(\alpha_\varphi+\|f\|_{\mathcal{V}^*}),\end{aligned}$$

where $\gamma_0 = \|y_0\|_{\mathcal{V}} + 1$. The latter combined with the assumption

$$\|x_n\|_{\mathcal{V}} \to \infty \text{ as } n \to \infty,$$

and the coercivity condition $\mathbf{H}(\mathscr{N})(iii)'$ (due to $y_n \to y_0$ as $n \to \infty$) implies

$$0 \geq \frac{\langle \mathscr{N}(x_n, x_n) - f, x_n - y_n\rangle + \varphi(x_n) - \varphi(y_n) - J^0(\varsigma x_n; \varsigma y_n - \varsigma x_n)}{\|x_n\|_{\mathcal{V}}} > 0,$$

which is a contradiction. Hence, the set

$$\bigcup_{n\in\mathbb{N}} \mathbb{P}_n$$

is uniformly bounded in $\mathcal{V}$. Subsequently, for any sequence $\{x_n\} \subset \mathcal{V}$ with $x_n \in \mathbb{P}_n$ for all $n \in \mathbb{N}$, it is implied by Claim 1 that $\{x_n\}$ is bounded in $\mathcal{V}$ as well. We may presume that along a relabeled subsequence one has

$$x_n \rightharpoonup x \text{ as } n \to \infty \text{ for some } x \in \mathcal{V}. \tag{3.12}$$

This ensures that the set

$$\rho - \limsup_{n\to\infty} \mathbb{P}_n$$

is nonempty.

Next, we shall show that $\rho - \limsup_{n\to\infty} \mathbb{P}_n$ is a subset of $\mathbb{P}$. Let $x \in \rho - \limsup_{n\to\infty} \mathbb{P}_n$ be arbitrary. Without any loss of generality, we may assume that there exists a sequence $\{x_n\}$ with $x_n \in \mathbb{P}_n$ for all $n \in \mathbb{N}$ such that (3.12) holds. Our goal is to prove that $x \in \mathbb{P}$.

Claim 2. We prove that $x \in \mho$. In fact, for each $n \in \mathbb{N}$, we have

$$\frac{1}{\lambda_n}\langle \mathcal{P}_n(x_n, x_n), x_n - y\rangle \leq \langle \mathscr{N}(x_n, x_n) - f, y - x_n\rangle + \varphi(y) - \varphi(x_n) + J^0(\varsigma x_n; \varsigma y - \varsigma x_n),\ \forall y \in \mathcal{V}.$$

Utilizing the hypothesis **H($\mathscr{N}$)(ii)**, we have

$$\begin{aligned}
&\frac{1}{\lambda_n}\langle \mathcal{P}_n(x_n, x_n), x_n - y\rangle \\
&\leq \langle \mathscr{N}(x_n, x_n) - \mathscr{N}(y, y), y - x_n\rangle + J^0(\varsigma x_n; \varsigma y - \varsigma x_n) + J^0(\varsigma y; \varsigma x_n - \varsigma y) \\
&\quad + \langle \mathscr{N}(y, y) - f, y - x_n\rangle + \varphi(y) - \varphi(x_n) - J^0(\varsigma y; \varsigma x_n - \varsigma y) \\
&\leq \langle \mathscr{N}(y, y) - f, y - x_n\rangle + \varphi(y) - \varphi(x_n) - J^0(\varsigma y; \varsigma x_n - \varsigma y) - \mathscr{T}(y - x_n),\ \forall y \in \mathcal{V},
\end{aligned}$$

as a result that

$$\begin{aligned}
\frac{1}{\lambda_n}\langle \mathcal{P}_n(x_n, x_n), x_n - y\rangle \leq{}& \|\mathscr{N}(y, y) - f\|_{\mathcal{V}^*}\ \|y - x_n\|_{\mathcal{V}} \\
&+ \|\partial J(\varsigma y)\|_{\mathbb{X}^*}\ \|\varsigma\|\|y - x_n\|_{\mathcal{V}} \\
&+ \varphi(y) - \varphi(x_n) - \mathscr{T}(y - x_n),\ \forall y \in \mathcal{V}.
\end{aligned}$$

Since $\mathscr{N}$ is bounded, thus for each $y \in \mathcal{V}$, there exists $\chi(y) > 0$, which relies on y but is independent of n, such that

$$\langle \mathcal{P}_n(x_n, x_n), x_n - y\rangle \leq \lambda_n \chi(y), \tag{3.13}$$

where we have made use of the fact that

$$x_n \rightharpoonup x \text{ as } n \to \infty.$$

Combining (3.13) with the hypothesis $\mathbf{H}(\lambda_n)'$ we get

$$\limsup_{n\to\infty}\langle \mathcal{P}_n(x_n, x_n), x_n - y\rangle \leq 0,\ \forall y \in \mathcal{V}. \tag{3.14}$$

Putting $y = x$ in (3.14) and employ the convergence (3.12) and the condition **H($\mathcal{P}_n$)(ii)(b)** to get

$$\begin{aligned}\langle \mathcal{P}(x,x), x-y\rangle &\le \liminf_{n\to\infty}\langle \mathcal{P}_n(x_n,x_n), x_n-y\rangle\\ &\le \limsup_{n\to\infty}\langle \mathcal{P}_n(x_n,x_n), x_n-y\rangle\\ &\le 0.\end{aligned}$$

Since $y \in \mathcal{V}$ is arbitrary, we draw the conclusion that

$$\langle \mathcal{P}(x,x), y\rangle = 0,\ \forall y\in\mathcal{V}$$

implies that

$$\mathcal{P}(x,x) = 0_{\mathcal{V}^*}$$

and the hypothesis **H($\mathcal{P}_n$)(ii)** establishes $x \in \mho$ and is proved.

Claim 3. We will show that $x \in \mathbb{P}$. Let $y \in \mho$ be fixed. The hypothesis **H($\mathcal{P}_n$)(i)** allows to defined a sequence $\{y_n\} \subset \mathcal{V}$ such that

$$\mathcal{P}_n(y_n,y_n) = 0_{\mathcal{V}^*},\ \text{and } y_n \to y \text{ as } n\to\infty. \tag{3.15}$$

Thus, we have

$$\begin{aligned}\langle \mathscr{N}(y_n,y_n), x_n-y_n\rangle + \mathscr{T}(y_n-x_n) \le \langle f, x_n-y_n\rangle + \frac{1}{\lambda_n}\langle \mathcal{P}_n(x_n,x_n), y_n-x_n\rangle\\ + \varphi(y_n)-\varphi(x_n) + J^0(\varsigma y_n;\varsigma y_n-\varsigma x_n).\end{aligned}$$

By virtue of the monotonicity of $\mathcal{P}_n$, and

$$\mathcal{P}_n(y_n,y_n) = 0_{\mathcal{V}^*},$$

it brings in

$$\begin{aligned}\langle \mathscr{N}(y_n,y_n), x_n-y_n\rangle + \mathscr{T}(y_n-x_n) &\le \langle f, x_n-y_n\rangle\\ &\quad + \frac{1}{\lambda_n}\langle \mathcal{P}_n(x_n,x_n)-\mathcal{P}_n(y_n,y_n), y_n-x_n\rangle\\ &\quad + \langle \xi_n, \varsigma(y_n-x_n)\rangle_{\mathbb{X}^*\times\mathbb{X}} + \varphi(y_n)-\varphi(x_n)\\ &\le \langle f, x_n-y_n\rangle + \langle \xi_n, \varsigma(y_n-x_n)\rangle_{\mathbb{X}^*\times\mathbb{X}} + \varphi(y_n)-\varphi(x_n),\end{aligned}\tag{3.16}$$

where $\xi_n \in \partial J(\varsigma y_n)$ is such that

$$J^0(\varsigma y_n;\varsigma y_n-\varsigma x_n) = \langle \xi_n, \varsigma(y_n-x_n)\rangle_{\mathbb{X}^*\times\mathbb{X}}.$$

Since $y_n \to y$ as $n\to\infty$ and the map

$$x \to \partial J(x)$$

is locally bounded, we arrive at the conclusion that the sequence $\{\xi_n\} \subset \mathbb{X}^*$ is bounded too. With no loss of generality, we may assume that

$$\xi_n \rightharpoonup \xi\ \in \mathbb{X}^* \text{ as } n\to\infty \text{ for some } \xi\in\mathbb{X}^*.$$

Invoking Proposition 2.1 (iv) and the convergence $y_n \to y$, as $n\to\infty$, we obtain $\xi \in \partial J(\varsigma y)$. The continuity and convexity of φ, **H** $(\varphi)'$ and the compactness of ς, we have

$$\langle \xi_n, \varsigma(y_n-x_n)\rangle_{\mathbb{X}^*\times\mathbb{X}} \to \langle \xi, \varsigma(y-x)\rangle_{\mathbb{X}^*\times\mathbb{X}}, \tag{3.17}$$

$$\begin{aligned}\limsup_{n\to\infty}(\varphi(y_n)-\varphi(x_n)) &\le \lim_{n\to\infty}\varphi(y_n) - \liminf_{n\to\infty}\varphi(x_n)\\ &\le \varphi(y)-\varphi(x).\end{aligned}\tag{3.18}$$

In addition, the continuity of $\mathscr{N}$ and the condition **H($\mathscr{T}$)(ii)** imply

(3.19) $$\langle \mathscr{N}(y_n, y_n), x_n - y_n\rangle \to \langle \mathscr{N}(y, y), y - x\rangle,$$

(3.20) $$\limsup_{n\to\infty} \mathscr{T}(y_n - x_n) \geq \mathscr{T}(y - x).$$

From (3.17) - (3.20) and the upper limit as $n \to \infty$ in (3.16), we get

$$\begin{aligned}\langle \mathscr{N}(y, y), y - x\rangle + \mathscr{T}(y - x) &\leq \langle f, y - x\rangle + \langle \xi, \varsigma(y - x)\rangle_{\mathbb{X}^*\times\mathbb{X}} + \varphi(y) - \varphi(x)\\ &\leq \langle f, y - x\rangle + J^0(\varsigma y; \varsigma y - \varsigma x) + \varphi(y) - \varphi(x),\end{aligned}$$

where $\xi \in \partial J(\varsigma y)$. Because $y \in \mho$ is arbitrary, it follows from Theorem 3.1 that $x \in \mho$ solves (1), *i.e.*, $x \in \mathbb{P}$. Consequently, we have

$$\rho - \limsup_{n\to\infty} \mathbb{P}_n \subset \mathbb{P}.$$

(iii) Since

$$\varrho - \limsup_{n\to\infty} \mathbb{P}_n \subset \rho - \limsup_{n\to\infty} \mathbb{P}_n,$$

it is adequate to confirm the condition

$$\rho - \limsup_{n\to\infty} \mathbb{P}_n \subset \varrho - \limsup_{n\to\infty} \mathbb{P}_n.$$

Let $x \in \rho - \limsup\limits_{n\to\infty} \mathbb{P}_n$ be arbitrary. Without loss of generality, there exists a sequence $\{x_n\}$ with $x_n \in \mathbb{P}_n$ such that

$$x_n \rightharpoonup x \text{ as } n \to \infty.$$

We claim that $x_n \to x$ as $n \to \infty$. Since $x \in \mho$, so, we are able to find a sequence $\{y_n\} \subset \mathcal{V}$ such that

$$\mathcal{P}_n(y_n, y_n) = 0_{\mathcal{V}^*} \text{ and } y_n \to x \text{ as } n \to \infty.$$

Also, we have

$$\langle \mathscr{N}(x_n, x_n), x_n - y_n\rangle \leq \langle f, x_n - y_n\rangle + \varphi(y_n) - \varphi(x_n) + J^0(\varsigma x_n; \varsigma y_n - \varsigma x_n).$$

Passing to the upper limit as $n \to \infty$ in the aforemention inequality, and using the following result

$$\begin{aligned}&\limsup_{n\to\infty}\langle \mathscr{N}(x_n, x_n), x_n - y_n\rangle\\ &= \limsup_{n\to\infty}\langle \mathscr{N}(x_n, x_n), x_n - x + x - y_n\rangle\\ &\leq \limsup_{n\to\infty}\langle \mathscr{N}(x_n, x_n), x_n - x\rangle + \limsup_{n\to\infty}\langle \mathscr{N}(x_n, x_n), x - y_n\rangle\\ &= \limsup_{n\to\infty}\langle \mathscr{N}(x_n, x_n), x_n - x\rangle\\ &\leq 0.\end{aligned}$$

The latter coupled with the convergence $x_n \rightharpoonup x$ as $n \to \infty$ and the fact that $\mathscr{N}$ satisfies (P_+)-property implies $x_n \to x$ as $n \to \infty$ imply that $x \in \varrho - \limsup_{n\to\infty} \mathbb{P}_n$. Therefore, it holds

$$\varrho - \limsup_{n\to\infty} \mathbb{P}_n = \rho - \limsup_{n\to\infty} \mathbb{P}_n.$$

(iv) For any $x \in \varrho - \limsup_{n\to\infty} \mathbb{P}_n$, without any loss of generality, we may assume that there exists a sequence $\{x_n\} \subset \mathcal{V}$ with $x_n \in \mathbb{P}_n$ such that $x_n \to x$ as $n \to \infty$. Recall that for each $n \in \mathbb{N}$, the set $\mathbb{P}_n$ of solutions to (3.9) is nonempty and closed. Consider the sequence $\{\tilde{x}_n\} \subset \mathcal{V}$ such that

$$\tilde{x}_n \in \arg\min_{y_n \in \mathbb{P}_n} \|y_n - x\|_{\mathcal{V}}, \ \text{ for each } n \in \mathbb{N}.$$

As a consequence of Claim 1 that the sequence $\{\tilde{x}_n\}$ is bounded. Therefore we may assume a subsequence, not relabeled, that

$$\tilde{x}_n \rightharpoonup \tilde{x} \text{ as } n \to \infty \text{ for some } \tilde{x} \in \mathcal{V}.$$

Using the same logic as in the proof of Claim 2, we obtain that $\tilde{x} \in \mho$. Let $y \in \mho$. Then, there exists a sequence $\{y_n\} \subset \mathcal{V}$ such that (3.15) is available, and

$$\begin{aligned}\langle \mathscr{N}(y_n, y_n), \tilde{x}_n - y_n\rangle + \mathscr{T}(y_n - \tilde{x}_n) \le& \langle f, \tilde{x}_n - y_n\rangle + \varphi(y_n) - \varphi(\tilde{x}_n)\\ &+ J^0(\varsigma y_n; \varsigma y_n - \varsigma\tilde{x}_n).\end{aligned}$$

Taking upper limit as $n \to \infty$, in the above inequality, we get

$$\langle \mathscr{N}(y, y), \tilde{x} - y\rangle + \mathscr{T}(y - \tilde{x}) \le \langle f, \tilde{x} - y\rangle + \varphi(y) - \varphi(\tilde{x}) + J^0(\varsigma y; \varsigma y - \varsigma\tilde{x}).$$

Since $y \in \mho$ is arbitrary, by Theorem 3.1(i), we conclude that $\tilde{x} \in \mathbb{P}$. From assertion (iii), it follows that $\tilde{x} \in \varrho - \limsup_{n\to\infty} \mathbb{P}_n \subset \mathbb{P}$. Next, we shall demonstrate that $x = \tilde{x}$. By the virtue of definition of $\tilde{x}_n$ and $x_n \in \mathbb{P}_n$, we have

$$\begin{aligned}\|\tilde{x}_n - x\|_{\mathcal{V}} &= d(x, \mathbb{P}_n)\\ &\le \|x - x_n\|_{\mathcal{V}}.\end{aligned}$$

Since $x_n \to x$ as $n \to \infty$, the above results show

$$\|\tilde{x}_n - x\|_{\mathcal{V}} \to 0 \text{ as } n \to \infty.$$

This together with the convergence $\tilde{x}_n \to \tilde{x}$ as $n \to \infty$ implies $x = \tilde{x}$. Consequently, for any sequence $\{\tilde{x}_n\} \subset \mathcal{V}$ with $\tilde{x}_n \in \arg\min_{y_n \in \mathbb{P}_n} \|y_n - x\|_{\mathcal{V}}$ for each $n \in \mathbb{N}$, there exists a subsequence $\{\tilde{x}_{n_k}\}$ of $\{\tilde{x}_n\}$ such that $\tilde{x}_{n_k} \to \tilde{x}$ as $k \to \infty$.

(v) Assume that $\mathbb{P} = \{x^*\}$. It goes without saying that for each $n \in \mathbb{N}$, (3.9) has a unique solution $x_n \in \mathcal{V}$. Since $\emptyset = \rho - \limsup_{n\to\infty} \mathbb{P}_n \subset \mathbb{P}$, we deduce that there exists a subsequence of $\{x_n\}$ converging weakly to x^*. We now justify that the whole sequence $\{x_n\}$ converges weakly to x^*. In fact, we can see that each subsequence of $\{x_n\}$ converges weakly to the same limit in $\mathcal{V}$ that coincides with the unique solution of (1.1). The latter combined with the boundedness of $\{x_n\}$ entails that the whole sequence $\{x_n\}$ converges weakly to x^*. Moreover, if $\mathscr{N}$ satisfies (P_+)-property, then we can apply the similar arguments as in the proof of assertion (iii) to obtain that the whole sequence $\{x_n\}$ converges strongly to x^*.

□

Let $\mathcal{P} : \mathcal{V} \times \mathcal{V} \to \mathcal{V}^*$ be a penalty operator of $\mho$. Consider the following problem for finding $x_n \in \mathcal{V}$ such that

$$\langle \mathscr{N}(x_n, x_n) - f, y - x_n\rangle + \frac{1}{\lambda_n}\langle \mathcal{P}(x_n, x_n), y - x_n\rangle + J^0(\varsigma x_n; \varsigma y - \varsigma x_n)$$

$$+\varphi(y) - \varphi(x_n) \ge 0, \ \forall y \in \mathcal{V}. \tag{3.21}$$

Theorem 3.3. *Assume that the hypotheses* **H($\mathscr{N}$)(i)-(iii)**, **H($\mathscr{T}$)**, **H(J)**, **H(ς)′**, **H(φ)**, *and* **H(λ_n)** *are satisfied, $f \in \mathcal{V}^*$, $\mathscr{T} : \mathcal{V}^* \to \mathbb{R}$ is bounded, and $\mathcal{P} : \mathcal{V} \times \mathcal{V} \to \mathcal{V}^*$ is a penalty operator. Then*

(i) *for each $n \in \mathbb{N}$, the set of solutions to* (3.21)*, denoted by $\tilde{\mathbb{P}}_n$, is nonempty, bounded and weakly closed.*

(ii) $\emptyset \neq \rho - \limsup_{n\to\infty} \tilde{\mathbb{P}}_n \subset \mathbb{P}$.

(iii) *if $\mathscr{N}$ satisfies* (P_+)*-property, then*

$$\rho - \limsup_{n\to\infty} \tilde{\mathbb{P}}_n = \varrho - \limsup_{n\to\infty} \tilde{\mathbb{P}}_n.$$

(iv) *if $\mathscr{N}$ satisfies* (P_+)*-property, then for each $x \in \varrho - \limsup_{n\to\infty} \tilde{\mathbb{P}}_n$ and any sequence $\{\tilde{x}_n\}$ with $\tilde{x}_n \in \arg\min_{x_n \in \tilde{\mathbb{P}}_n} \|x_n - x\|_{\mathcal{V}}$ for each $n \in \mathbb{N}$, there exists a subsequence of $\{\tilde{x}_n\}$ converging strongly to x.*

(v) *if (1) has a unique solution $x^* \in \mho$, then* (3.21) *has a unique solution, and the whole sequence $\{x_n\}$ of solutions to* (3.21) *converges weakly to x. Moreover, if, in addition, $\mathscr{N}$ satisfies* (P_+)*-property, then the whole sequence $\{x_n\}$ of solutions to* (3.21) *converges strongly to x.*

4. Nonlinear nonhomogeneous mixed boundary value problems

This is devoted with the applicability of the theoretical systems of nonlinear nonhomogeneous mixed boundary value problem (*see*, more detail [8]) involving a differential operator with an obstacle effect which comes from the modeling of semipermeability. Let Ω be a bounded domain in $\mathbb{R}^{\mathbb{N}}(N \geq 2)$ and $\daleth = \partial\Omega$ is a boundary of class $\mathcal{C}^2$. Assume that $\daleth$ is divided into three disjoint measurable parts $\daleth_1$, $\daleth_2$ and $\daleth_3$ with meas$(\daleth_1) > 0$. Let $\boldsymbol{\nu}$ be the outward unit normal to the boundary $\daleth$. Consider the following nonlinear nonhomogeneous mixed boundary value problem with constraints for finding a function $x : \bar{\Omega} \to \mathbb{R}$ such that

$$-div(a(\mathbf{u}, \nabla x(\mathbf{u}))) + \partial_c\phi(\mathbf{u}, x(\mathbf{u})) \ni f_0(\mathbf{u}) \qquad \text{in } \Omega \tag{4.22}$$

where $\partial_c\phi$ is a convex subdifferential of the function $\phi : \Omega \times \mathbb{R} \to \mathbb{R}$ with respect to second variable. The function x denotes the electric potential or temperature, the function $a = a(\mathbf{u}, \nabla x)$ is the dielectric coefficient, magnetic permeability or thermal conductivity, and $f_0 = f_0(\mathbf{u})$ is a given source term. The material which occupies Ω is non-isotropic and heterogeneous, and therefore the effectively depends on $\mathbf{u}$.

$$x(\mathbf{u}) \leq \Phi(\mathbf{u}) \qquad \text{in } \Omega, \tag{4.23}$$

represents an additional unilateral constraint for the solution.

$$x(\mathbf{u}) = 0 \qquad \text{in } \daleth_1, \tag{4.24}$$

$$\frac{\partial x(\mathbf{u})}{\partial \mathbf{n}_a} = (a(\mathbf{u}, \nabla x(\mathbf{u})), \boldsymbol{\nu})_{\mathbb{R}^{\mathbb{N}}} = f_2(\mathbf{u}) \qquad \text{on } \daleth_2, \tag{4.25}$$

$$\frac{\partial x(\mathbf{u})}{\partial \mathbf{n}_a} = -g(\mathbf{u}) sgn(x(\mathbf{u})) \qquad \text{on } \daleth_3 \tag{4.26}$$

where *sgn* is a function defined by

$$sgn(t) = \begin{cases} [-1, 1] & \text{if } t = 0 \\ 1 & \text{it } t > 0 \\ -1 & \text{otherwise.} \end{cases} \tag{4.27}$$

The mathematical model (4.22)-(4.26) is motivated by the study of semi permeability phenomena which may appear in the interior and on the boundary of the body Ω. Now, we are in position to study the weak solutions of (4.22)-(4.26) to its variational formulation for the following function space

$$\mathcal{V} = \{y \in W^{1,p}(\Omega) | \varsigma y(\mathbf{u}) = 0 \text{ for a.e.} \mathbf{u} \in \daleth_1\},$$

where $W^{1,p}(\Omega)$, $2 \leq p < \infty$, is the well-known Sobolev space with the usual norm

$$\|x\|_{W^{1,p}(\Omega)} = \|x\|_{L^p(\Omega)} + \|\nabla x\|_{L^p(\Omega;\mathbb{R}^{\mathbb{N}})},$$

and $\varsigma : W^{1,p}(\Omega) \to L^p(\daleth)$ stands for the trace operator which is known to be linear, bounded, and compact. Also, we need the constraint set defined by

$$\mho = \{y \in \mathcal{V} | y(\mathbf{u}) \leq \phi(\mathbf{u}) \text{ for a.e.} \mathbf{u} \in \Omega\}.$$

Further, we impose the assumptions for the data of (4.22)-(4.26).

H(ϕ) $\phi : \Omega \times \mathbb{R} \to \mathbb{R}$ is such that

(i) for each $s \in \mathbb{R}$, $\mathbf{u} \to \phi(\mathbf{u}, s)$ is measurable on Ω, and there exists $e \in L^p(\Omega)$ such that

$$\mathbf{u} \to \phi(\mathbf{u}, e(\mathbf{u})) \in L^1(\Omega),$$

(ii) for a.e. $\mathbf{u} \in \Omega$, $s \to \phi(\mathbf{u}, s)$ is convex and lower semicontinuous.

H(0): $f_0 \in L^p(\Omega)$, $f_2 \in L^p(\daleth_2)$, $\Phi \in \mathcal{V}$ with $\Phi(\mathbf{w}) \geq c_\phi > 0$ for *a.e.* $\mathbf{w} \in \Omega$, $g \in L^\infty(\daleth_3)$, $g(\mathbf{u}) \geq 0$ for *a.e.* $\mathbf{u} \in \daleth_3$ and $g \not\equiv 0$. Assume that x is smooth function on Ω which solves problem (4.22)-(4.26) and let $y \in \mho$. Multiplying (4.22) by $y - x$ and applying Green's Theorem, we have

$$\int_\Omega (a(\mathbf{u}, \nabla x(\mathbf{u})), \nabla y(\mathbf{u}) - \nabla x(\mathbf{u}))_{\mathbb{R}^{\mathbb{N}}}\, d\mathbf{u} = \int_\Omega (f_0(\mathbf{u}) - \xi(\mathbf{u}))(y(\mathbf{u}) - x(\boldsymbol{\nu}))d\mathbf{u} + \int_\daleth \frac{\partial x(\mathbf{u})}{\partial \mathbf{n}_a}(y(\mathbf{u}) - x(\mathbf{u}))d\daleth,$$

where $\xi(\mathbf{u}) \in \partial_c\phi(\mathbf{u}, x(\mathbf{u}))$ for *a.e.* $\mathbf{u} \in \Omega$ is such that

$$-div\,(a(\mathbf{u}, \nabla x(\mathbf{x}))) + \xi(\mathbf{u}) = f_0(\mathbf{u}) \text{ for a.e. } \mathbf{u} \in \Omega.$$

Now by using the Riesz representation theorem to find a function $f \in \mathcal{V}^*$ such that

$$\langle f, v\rangle = \int_\Omega f_0(\mathbf{u})y(\mathbf{u})d\mathbf{u} + \int_{\daleth_2} f_2(\mathbf{u})\varsigma y(\mathbf{u})d\daleth,\ \forall y \in \mathcal{V}. \tag{4.28}$$

Furthermore, using the equality

$$\int_\daleth \frac{\partial x(\mathbf{u})}{\partial \mathbf{n}_a}(y(\mathbf{u}) - x(\mathbf{u}))d\daleth = \int_{\daleth_1} \frac{\partial x(\mathbf{u})}{\partial \mathbf{n}_a}(y(\mathbf{u}) - x(\mathbf{u}))d\daleth + \int_{\daleth_2} \frac{\partial x(\mathbf{u})}{\partial \mathbf{n}_a}(y(\mathbf{u}) - x(\mathbf{u}))d\daleth + \int_{\daleth_3} \frac{\partial x(\mathbf{u})}{\partial \mathbf{n}_a}(y(\mathbf{u}) - x(\mathbf{u}))d\daleth$$

and boundary conditions (4.24)-(4.25), we get

$$\int_\Omega (a(\mathbf{u}, \nabla x(\mathbf{u})), \nabla y(\mathbf{u}) - \nabla x(\mathbf{u}))_{\mathbb{R}^{\mathbb{N}}}\, d\mathbf{u} + \int_\Omega \xi(\mathbf{u})(y(\mathbf{u}) - x(\boldsymbol{\nu}))d\mathbf{u} = \langle f, y - x\rangle + \int_{\daleth_3} \frac{\partial x(\mathbf{u})}{\partial \mathbf{n}_a}(y(\mathbf{u}) - x(\mathbf{u}))d\daleth.$$

From (4.26)-(4.27) and the definition of convex subgradient that

$$\int_\Omega (a(\mathbf{u}, \nabla x(\mathbf{u})), \nabla y(\mathbf{u}) - \nabla x(\mathbf{u}))_{\mathbb{R}^{\mathbb{N}}}\, d\mathbf{u} + \int_\Omega \phi(\mathbf{u}, y(\mathbf{u})) - \phi(\mathbf{u}, x(\mathbf{y}))d\mathbf{u} + \int_{\daleth_3} g(\mathbf{u})(|y(\mathbf{u})| - |x(\mathbf{u})|)d\daleth \geq \langle f, y - x\rangle.$$

Hence we get the following variational formulation of (4.22)-(4.26) for finding $x \in \mho$ such that

$$\int_\Omega (a(\mathbf{u}, \nabla x(\mathbf{u})), \nabla y(\mathbf{u}) - \nabla x(\mathbf{u}))_{\mathbb{R}^\mathbb{N}}\, d\mathbf{u} + \int_\Omega \phi(\mathbf{u}, y(\mathbf{u})) - \phi(\mathbf{u}, x(\mathbf{u}))d\mathbf{u}$$

$$+ \int_{\daleth_3} g(\mathbf{u})(|y(\mathbf{u})| - |x(\mathbf{u})|)d\daleth \geq \langle f, y - X\rangle,\ \forall y \in \mho. \tag{4.29}$$

Theorem 4.4. *Assume that* ***H(a)****,* ***H(ϕ)*** *and* ***H(0)*** *hold. Then (4.29) has a unique solution $x \in \mho$.*

Proof. Consider a function $\varphi : \mathcal{V} \to \mathbb{R}$ defined by

$$\varphi(x) = \int_\Omega \phi(\mathbf{u}, x(\mathbf{u}))d\mathbf{u} + \int_{\daleth_3} g(\mathbf{u})|x(\mathbf{u})|d\daleth\, \forall x \in \mathcal{V}.$$

Under this notation, we can be rewritten (4.29) for finding $x \in \mho$ such that

$$\langle \mathscr{N}(x, x) - f, y - x\rangle + \varphi(y) - \varphi(x) \geq 0,\ \forall y \in \mho. \tag{4.30}$$

From hypotheses **H(0)** and **H(ϕ)**, we see that the function φ is convex and continuous and conditions **H($\mathscr{N}$)(i)-(ii)** hold with $\mathscr{T} = J = 0$. Since the trace operator $\varsigma : W^{1,p}(\Omega) \to L^p(\daleth)$ is linear, bounded and compact then we show that the coercivity condition **H($\mathscr{N}$)(iii)** is satisfied. Therefore the following holds:

$$\begin{aligned}\langle \mathscr{N}(x, x), x\rangle &= \int_\Omega (a(\mathbf{u}, \nabla x(\mathbf{u})), \nabla x(\mathbf{u}))_{\mathbb{R}^\mathbb{N}} d\mathbf{u}\\ &\geq \frac{a_3}{p-1}\int_\Omega \|\nabla x(\mathbf{u})\|^p d\mathbf{u}\\ &= \frac{a_3}{p-1}\|u\|^p_{\mathcal{V}}\ \forall x \in \mathcal{V},\end{aligned}$$

where we used the Poincar inequality [8]. This implies that the coercivity condition **H($\mathscr{N}$)(iii)** holds with $J = 0$. Then we conclude that (4.29) has an unique solution. ☐

Let $\{\Phi_n\}$ be a sequence such that

$$\Phi_n \to \Phi\ \in \mathcal{V} \text{ as } n \to \infty. \tag{4.31}$$

From assumption **H(0)**, we assume that

$$\Phi_n(\mathbf{w}) \geq 0 \text{ for a.e. } \mathbf{w} \in \Omega, \text{ and all } n \in \mathbb{N}.$$

For some $\lambda_n > 0$, the penalized problem associated with (4.22)-(4.26) to find a function $x_n : \bar{\Omega}_n \to \mathbb{R}$ such that

$$\begin{aligned} -div\,(a(\mathbf{u}, \nabla x_n(\mathbf{u}))) + \partial_c\phi(\mathbf{u}, x_n(\mathbf{u})) + \frac{1}{\lambda_n}(x_n(\mathbf{u}) - \Phi_n(\mathbf{u}))^+ \ni f_0(\mathbf{u}) \quad & \text{in } \Omega,\\ x_n(\mathbf{u}) = 0 \quad & \text{on } \daleth_1,\\ \frac{\partial x_n(\mathbf{u})}{\partial \mathbf{n}_a} = (a(\mathbf{u}, \nabla x_n(\mathbf{u})), \boldsymbol{\nu})_{\mathbb{R}^\mathbb{N}} = f_2(\mathbf{u}) \quad & \text{on } \daleth_2,\end{aligned}$$

$$\frac{\partial x_n(\mathbf{u})}{\partial \mathbf{n}_a} = -g(\mathbf{u})sgn(x_n(\mathbf{u}))\text{on } \daleth_3. \tag{4.32}$$

Here, $r^+ = \max\{0, r\}$ stands for the positive part of $r \in \mathbb{R}$.
Consider the operator $\mathcal{P}_n : \mathcal{V} \times \mathcal{V} \to \mathcal{V}^*$ given by

$$\langle \mathcal{P}_n(x, x), y\rangle = \int_\Omega (x(\mathbf{u}) - \Phi_n(\mathbf{u}))^+ y(\mathbf{u})d\mathbf{u},\ \forall x, y \in \mathcal{V}, n \in \mathbb{N}. \tag{4.33}$$

Lemma 4.2. *Assume that the sequence $\{\Phi_n\}$ satisfies (4.31), then the sequence of operators $\{\mathcal{P}_n\}$ defined by (4.33) satisfies the condition **H($\mathcal{P}_n$)**.*

Proof. From (4.33), it is easy to see that $\mathcal{P}_n$ is a bounded, continuous and monotone operator for each $n \in \mathbb{N}$. For $y \in \mho$, let the sequence $\{y_n\} \subset \mathcal{V}$ be defined by

$$y_n(\mathbf{u}) = \frac{y(\mathbf{u})\Phi_n(\mathbf{u})}{\Phi(\mathbf{u})}, \ \forall \mathbf{u} \in \Omega \text{ and } n \in \mathbb{N}.$$

Since $y \in \mho$, it satisfied

$$y_n(\mathbf{u}) \le \frac{\Phi(\mathbf{u})\Phi_n(\mathbf{u})}{\Phi(\mathbf{u})}, \ \forall\ \mathbf{u} \in \Omega \text{ and } n \in \mathbb{N}.$$

Hence, by (4.33), we have

$$\mathcal{P}_n(y_n, y_n) = 0_{\mathcal{V}^*}, \ \forall n \in \mathbb{N}.$$

Using (4.31) and the Lebesgue convergence theorem, we have

$$\begin{aligned}
\lim_{n\to\infty} \|y_n - y\|_{\mathcal{V}}^p &= \lim_{n\to\infty} \int_\Omega \|\nabla y_n(\mathbf{u}) - \nabla y(\mathbf{u})\|^p d\mathbf{u} \\
&= \lim_{n\to\infty} \int_\Omega \left\| \nabla y(\mathbf{u}) \left(1 - \frac{\Phi_n(\mathbf{u})}{\Phi(\mathbf{u})}\right) + y(\mathbf{u}) \nabla \left(\frac{\Phi_n}{\Phi}\right) \right\|^p d\mathbf{u} \\
&= \int_\Omega \lim_{n\to\infty} \left\| \nabla y(\mathbf{u}) \left(1 - \frac{\Phi_n(\mathbf{u})}{\Phi(\mathbf{u})}\right) + y(\mathbf{u}) \nabla \left(\frac{\Phi_n}{\Phi}\right) \right\|^p d\mathbf{u} \\
&= 0.
\end{aligned}$$

Therefore **H($\mathcal{P}$)(i)** is justified.

Consider the operator $\mathcal{P} : \mathcal{V} \times \mathcal{V} \to \mathcal{V}^*$ given by

$$\langle \mathcal{P}(x, x), y \rangle = \int_\Omega (x(\mathbf{u}) - \Phi(\mathbf{u}))^+ y(\mathbf{u}) d\mathbf{u} \, \forall x, y \in \mathcal{V}.$$

Now we show that

$$\mho = \{x \in \mathcal{V} | \mathcal{P}(x, x) = 0_{\mathcal{V}^*}\},$$

hence **H($\mathcal{P}$)(ii)** holds. Next, we assume that $\{x_n\}$ is a sequence satisfying

$$x_n \rightharpoonup x \in \mathcal{V} \text{ and } \limsup_{n\to\infty} \langle \mathcal{P}_n(x_n, x_n), x_n - x \rangle \le 0, \text{ for some } x \in \mathcal{V}.$$

Since the embedding of $\mathcal{V}$ into $L^p(\Omega)$ is compact then we may assume that

$$x_n(\mathbf{u}) \to x(\mathbf{u}) \text{ and } \Phi_n(\mathbf{u}) \to \Phi(\mathbf{u}) \text{ as } n \to \infty \text{ for a.e. } \mathbf{u} \in \Omega.$$

The boundedness of the operator $\mathcal{P}_n$ and the Lebesgue convergence theorem imply

$$\begin{aligned}
\lim_{n\to\infty} \langle \mathcal{P}_n(x_n, x_n), x_n - y \rangle &= \lim_{n\to\infty} \int_\Omega (x_n(\mathbf{u}) - \Phi_n(\mathbf{u}))^+ (x_n(\mathbf{u}) - y(\mathbf{u})) d\mathbf{u} \\
&= \int_\Omega \lim_{n\to\infty} (x_n(\mathbf{u}) - \Phi_n(\mathbf{u}))^+ (x_n(\mathbf{u}) - y(\mathbf{u})) d\mathbf{u} \\
&= \int_\Omega (x(\mathbf{u}) - \Phi(\mathbf{u}))^+ (x(\mathbf{u}) - y(\mathbf{u})) d\mathbf{u} \\
&= \langle \mathcal{P}(x, x), x - y \rangle, \forall y \in \mathcal{V}.
\end{aligned}$$

This shows that $\mathbf{H}(\mathcal{P}_n)(b)$ is satisfied, and the proof is completed. □

Theorem 4.5. *Assume that **H(a)**, **H(ϕ)**, $\mathbf{H}(0)$ and (4.31) are satisfied. If, in addition, $\{\lambda_n\}$ is such that $\lambda_n > 0$ and $\lambda_n \to 0$ as $n \to \infty$, then*

(i) *(4.32) has a unique solution $x_n \in \mathcal{V}$.*
(ii) *the solution x_n of (4.32) converges strongly to the solution x of (4.22)-(4.26).*

5. A FRICTIONAL ELASTIC CONTACT PROBLEMS

In the section we discuss the frictional elastic contact problems which is described by a convex subdifferential operator, and a generalized Signorini contact condition associated with the Clarke subdifferential term of a locally Lipschitz function. Consider a nonlinear elastic body which occupies a bounded domain Ω in $\mathbb{R}^d$, where $d = 2, 3$, and the boundary $\daleth = \partial\Omega$ of Ω is Lipschitz continuous which is composed of four mutually disjoint parts $\daleth_D$, $\daleth_{D_1}$, $\daleth_{D_2}$ and $\daleth_{D_3}$ such that $\text{meas}(\daleth_D) > 0$. Let $\mathfrak{F}^d$ the space of real symmetric $d \times d$ matrices. We utilize $\mathbb{R}^d$ for inner product, $\mathfrak{F}^d$ for norms and defined by

$$\boldsymbol{\xi} \cdot \boldsymbol{\eta} = \xi_i\, \eta_i,\ \|\boldsymbol{\xi}\| = \sqrt{(\boldsymbol{\xi} \cdot \boldsymbol{\xi})} \text{ for } \boldsymbol{\xi} = (\xi_i),\ \boldsymbol{\eta} = (\eta_i) \in \mathbb{R}^d,$$
$$\boldsymbol{\sigma} \cdot \boldsymbol{\tau} = \xi_{ij}\, \tau_{ij},\ \|\boldsymbol{\sigma}\| = \sqrt{(\boldsymbol{\sigma} \cdot \boldsymbol{\sigma})} \text{ for } \boldsymbol{\sigma} = (\sigma_{ij}),\ \boldsymbol{\tau} = (\tau_{ij}) \in \mathfrak{F}^d,$$

where $i, j, k, l \in \{1, \cdots, d\}$ and the summation convention over repeated indices is used. The normal and tangential components of a vector field $\boldsymbol{\xi}$ on the boundary are defined by $\xi_\nu = \xi \cdot \boldsymbol{\nu}$ and $\boldsymbol{\xi_\tau} = \boldsymbol{\xi} - \xi_\nu \boldsymbol{\nu}$ where $\boldsymbol{\nu}$ is a outward unit normal at the boundary $\daleth$. Also, σ_ν and $\boldsymbol{\sigma}_\tau$ denotes the normal and tangential components of the stress field $\boldsymbol{\sigma}$ on the boundary, that is, $\sigma_\nu = (\sigma\boldsymbol{\nu}) \cdot \boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma\nu} - \sigma_\nu \boldsymbol{\nu}$.

The classical formulation of the frictional elastic contact problem for finding a displacement field $\mathbf{x} : \Omega \to \mathbb{R}^d$, a stress field $\boldsymbol{\sigma} : \Omega \to \mathfrak{F}^d$ and an interface force $\xi_\nu : \daleth_{D_1} \to \mathbb{R}$ such that

$$div\ \boldsymbol{\sigma} + \mathbf{f}_0 = \mathbf{0} \text{ in } \Omega, \tag{5.34}$$

here $\mathbf{f}_0$ is a density of the body forces,

$$\boldsymbol{\sigma} \in \mathscr{N}(\boldsymbol{\varepsilon}(\mathbf{x}), \boldsymbol{\varepsilon}(\mathbf{x})) + \partial_c\psi(\boldsymbol{\varepsilon}(\mathbf{x})) \text{ in } \Omega, \tag{5.35}$$

here $\mathscr{N}$ is a elasticity operator and $\partial_c\psi$ the convex subdifferential operator of a convex function ψ'

$$\mathbf{x} = \mathbf{0} \text{ on } \daleth_D, \tag{5.36}$$

describe the displacement,

$$\boldsymbol{\sigma\nu} = \mathbf{f}_{D_1} \text{ on } \daleth_{D_1}, \tag{5.37}$$

represent the traction boundary conditions. Assumed the body is fixed on Γ_D and surface traction of density $\mathbf{f}_{D_1}$ act on $\daleth_{D_1}$,

$$\begin{cases} x_\nu \le g, \\ \sigma_\nu + \xi_\nu \le 0, \\ (x_\nu - g)(\sigma_\nu + \xi_\nu) = 0, \quad \text{on } \daleth_{D_2} \\ \xi_\nu \in \partial J_\nu(x_\nu) \\ \boldsymbol{\sigma}_\tau = \mathbf{0} \end{cases} \tag{5.38}$$

$$\begin{cases} -\boldsymbol{\sigma}_\nu = \mathscr{F}, \\ \|\boldsymbol{\sigma}_\tau\| \le \mu|\sigma_\nu|, \qquad\qquad \text{on } \daleth_{D_3} \\ -\boldsymbol{\sigma}_\tau = \mu|\sigma_\nu| \dfrac{\mathbf{x}_\tau}{\|\mathbf{x}_\tau\|},\ \text{if } \mathbf{x}_\tau \ne \mathbf{0}. \end{cases} \tag{5.39}$$

The (5.38) is the model the frictionless contact with a foundation made of a rigid body covered by a layer made of elastic material which shows the penetration of restricted, since $x_\nu \le g$ where g is a thickness of the elastic layer [14] and the normal displacement does not reach the bound g, the contact is described by a multivalued normal compliance condition of the form $\xi_\nu \in \partial j_\nu(x_\nu)$. Therefore ξ_ν can be interpreted as the opposite of the normal stress on the contact surface. On boundary $\urcorner_{D_3}$, the normal stress on the contact boundary is assumed to be given by a function $\mathscr{F}$. By friction law, if at a point $\mathbf{u} \in \urcorner_{D_3}$, the inequality $\|\boldsymbol{\sigma}_\tau(\mathbf{u})\| < \mu(\mathbf{u})\mathscr{F}(\mathbf{u})$ holds, then $\mathbf{x}_\tau(\mathbf{u}) = \mathbf{0}$ and the material point $\mathbf{u}$ is in the so-called stick zone; but if $\|\boldsymbol{\sigma}_\tau(\mathbf{u})\| = \mu(\mathbf{u})\mathscr{F}(\mathbf{u})$, then the point $\mathbf{u}$ is in the so-called slip zone, *see* [[30], pp. 108 and 109].

(A) $\mathscr{N} : \Omega \times \Omega \times \mathfrak{F}^d \to \mathfrak{F}^d$ is such that

(i) the mapping $\mathbf{u} \to \mathscr{N}(\mathbf{u}, \mathbf{u}, \boldsymbol{\varepsilon})$ is measurable on Ω, for any $\boldsymbol{\varepsilon} \in \mathfrak{F}^d$,

(ii) there exist $d_0 > 0$ and $d_1 > 0$ such that

$$\|\mathscr{N}(\mathbf{u}, \mathbf{u}, \boldsymbol{\varepsilon})\|_{\mathfrak{F}^d} \le d_0 + d_1 \|\boldsymbol{\varepsilon}\|_{\mathfrak{F}^d}, \ \forall\, \boldsymbol{\varepsilon} \in \mathfrak{F}^d \text{ and a.e. } \mathbf{u} \in \Omega, \tag{5.40}$$

(iii) the mapping $\boldsymbol{\varepsilon} \to \mathscr{N}(\mathbf{u}, \mathbf{u}, \boldsymbol{\varepsilon})$ is continuous for a.e. $\mathbf{u} \in \Omega$,

(iv) relaxed monotone with respect to the first variable then there exists $\alpha_{\mathscr{N}} > 0$ such that

$$(\mathscr{N}(\mathbf{u}, \mathbf{u}, \boldsymbol{\varepsilon}_1) - \mathscr{N}(\mathbf{u}, \mathbf{u}, \boldsymbol{\varepsilon}_2)) \cdot (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2)) \ge -\alpha_{\mathscr{N}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|^2_{\mathfrak{F}^d}, \ \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathfrak{F}^d, \text{ and } \mathbf{u} \in \Omega, \tag{5.41}$$

(v) relaxed Lipschitz continuous with respect to the second variable then there exists $\beta_{\mathscr{N}} > 0$ such that

$$(\mathscr{N}(\mathbf{u}, \mathbf{u}, \boldsymbol{\varepsilon}_1) - \mathscr{N}(\mathbf{u}, \mathbf{u}, \boldsymbol{\varepsilon}_2)) \cdot (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2)) \le -\beta_{\mathscr{N}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|^2_{\mathfrak{F}^d}, \ \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathfrak{F}^d, \text{ and } \mathbf{u} \in \Omega, \tag{5.42}$$

(B) $\psi : \Omega \times \mathfrak{F}^d \to \mathbb{R}$ is such that

(i) $\psi(\cdot, \boldsymbol{\varepsilon})$ is measurable on Ω for all $\boldsymbol{\varepsilon} \in \mathfrak{F}^d$ and there exists $\mathbf{w} \in L^2(\Omega; \mathfrak{F}^d)$ such that

$$\psi(\cdot, \mathbf{w}(\cdot)) \text{ belongs to } L^1(\Omega),$$

(ii) $\psi(\mathbf{u}, \cdot)$ is convex and lower semicontinuous for a.e. $\mathbf{u} \in \Omega$.

(C) $j_\nu : \urcorner_{D_2} \times \mathbb{R} \to \mathbb{R}$ is such that

(i) $j_\nu(\cdot, r)$ is measurable on $\urcorner_{D_2}$ for all $r \in \mathbb{R}$ and there exists $e \in L^2(\urcorner_{D_2})$ such that $j_\nu(\cdot, e(\cdot)) \in L^1(\urcorner_{D_2})$,

(ii) $j_\nu(\mathbf{u}, \cdot)$ is locally Lipschitz on $\mathbb{R}$ for a.e. $\mathbf{u} \in \urcorner_{D_2}$.

(iii) there are constants $\pi_{0_\nu}, \pi_{1_\nu} \ge 0$ such that

$$|\partial j_\nu(\mathbf{u}, r)| \le \pi_{0_\nu} + \pi_{1_\nu} |r|, \ \text{for a.e.} \mathbf{u} \in \mathbb{R}, \tag{5.43}$$

(iv) there exists $\alpha_{j_\nu} \ge 0$ and all $r_1, r_2 \in \mathbb{R}$ such that

$$j^0_\nu(\mathbf{u}, r_1; r_2 - r_1) + j^0_\nu(\mathbf{u}, r_2; r_1 - r_2) \le \alpha_{j_\nu} |r_1 - r_2|^2, \ \text{for a.e. } \mathbf{u} \in \urcorner_{D_2}. \tag{5.44}$$

The regularity hypotheses where densities of volume forces and surface traction satisfying

$$\mathbf{f}_0 \in L^2(\Omega; \mathbb{R}^d), \ \mathbf{f}_{D_1} \in L^2(\urcorner_{D_1}; \mathbb{R}^d), \tag{5.45}$$

and μ, $\mathscr{F}$, g fulfill the following assumptions

$$\begin{cases} g \ge o, \\ \mathscr{F} \in L^\infty(\urcorner_{D_2}), \ \mathscr{F} \ge 0, & \text{a.e. on } \urcorner_{D_3}, \\ \mu \in L^\infty(\urcorner_{D_3}), \ \mu \ge 0, & \text{a.e. on } \urcorner_{D_3}. \end{cases} \tag{5.46}$$

Now we define the following function spaces as

$$V = \{\mathbf{y} \in H^1(\Omega; \mathbb{R}^d) \mid \mathbf{y} = \mathbf{0} \text{ on } \urcorner_D\},$$
$$H = L^2(\Omega; \mathbb{R}^d),$$
$$\mathcal{H} = L^2(\Omega; \mathfrak{F}^d).$$

Here the trace operator $\varsigma : V \to L^2(\daleth_D;\mathbb{R}^d)$ is linear, bounded and compact. Therefore V is endowed with the Hilbertian structure by the inner product and the corresponding norm

$$\langle \mathbf{x},\mathbf{y}\rangle = \langle \boldsymbol{\varepsilon}(\mathbf{x}),\boldsymbol{\varepsilon}(\mathbf{y})\rangle_{\mathcal{H}},\ \|\mathbf{y}\|_V = \|\boldsymbol{\varepsilon}(\mathbf{y})\|_{\mathcal{H}}\ \forall \mathbf{x},\mathbf{y}\in V.$$

Since set $\mho$ of admissible displacement fields is defined by

$$\mho = \{\mathbf{y}\in V\ |\mathbf{y}_\nu \le g \text{ a.e. on } \daleth_{D_2}\}.$$

By Green formula, we have the following variational formulation of (5.34)-(5.39) for finding a displacement field $\mathbf{x}\in\mho$ such that

$$\langle \mathscr{N}(\boldsymbol{\varepsilon}(\mathbf{x}),\boldsymbol{\varepsilon}(\mathbf{x})),\boldsymbol{\varepsilon}(\mathbf{y})-\boldsymbol{\varepsilon}(\mathbf{x})\rangle_{\mathcal{H}} + \int_\Omega \psi(\boldsymbol{\varepsilon}(\mathbf{x}))d\mathbf{u} + \int_\Omega \psi(\boldsymbol{\varepsilon}(\mathbf{x}))d\mathbf{u} + \int_{\daleth_{D_3}} \mathscr{F}(y_\nu - x_\nu)d\daleth$$
$$+ \int_{\daleth_{D_3}} \mu\mathscr{F}(\|\mathbf{y}_\tau\| - \|\mathbf{x}_\tau\|)d\daleth + \int_{\daleth_{D_2}} j^0_\tau(\mathbf{x}_\tau;\mathbf{y}_\tau - \mathbf{x}_\tau)d\daleth$$
$$\ge \langle \mathbf{f},\mathbf{y}-\mathbf{x}\rangle,\ \forall \mathbf{y}\in\mho, \tag{5.47}$$

where $\mathbf{f}\in\mathcal{V}^*$ is such that

$$\langle \mathbf{f},\mathbf{y}\rangle = \langle \mathbf{f}_0,\mathbf{y}\rangle_H + \langle \mathbf{f}_{D_1},\mathbf{y}\rangle_{L^p(\daleth_{D_1},\mathbb{R}^d)},\ \forall \mathbf{y}\in\mathcal{V}.$$

Theorem 5.6. *Assume that (A)-(C), (5.45)-(5.46) hold. If the inequalities*

$$\alpha_{j_\nu}\|\varsigma\|^2 \le \alpha_{\mathscr{N}} - \beta_{\mathscr{N}} \text{ and } \pi_{1_\nu}\|\varsigma\|^2\sqrt{2} < \alpha_{\mathscr{N}} - \beta_{\mathscr{N}}. \tag{5.48}$$

hold, then the solution set to (5.47), denoted by $\mathbb{P}$, is nonempty, bounded, closed and convex.

Proof. Let $\mathbb{X} = L^2(\daleth_{D_2})$ and the functional $J:\mathbb{X}\to\mathbb{R}$ defined by

$$J(\mathbf{z}) = \int_{\daleth_{D_2}} j_\tau(\mathbf{z}_\tau)d\daleth,\ \forall \mathbf{z}\in\mathbb{X}.$$

It follows from hypothesis (C) and locally Lipschitz function of J, the inequalities

$$\begin{cases} J^0(\mathbf{z};\mathbf{w}) \le \int_{\daleth_{D_2}} J^0_\tau(\mathbf{z}_\tau;\mathbf{w}_\tau)d\daleth, \\ \|\partial J(\mathbf{z})\|_{\mathbb{X}^*} \le \pi_{2_\nu} + \sqrt{2}\pi_{1_\nu}\|\mathbf{z}\|_{\mathbb{X}},\ \forall\, \mathbf{z},\mathbf{w}\in\mathbb{X} \text{ with some } \pi_{2_\nu}\ge 0. \end{cases} \tag{5.49}$$

Now, the intermediate problem for (1.1) is to find $\mathbf{x}\in\mho$ such that

$$\langle \mathscr{N}(\mathbf{x},\mathbf{x}) - \mathbf{f},\mathbf{y}-\mathbf{x}\rangle + J^0(\varsigma\mathbf{x};\varsigma\mathbf{y}-\varsigma\mathbf{x}) + \varphi(\mathbf{y}) - \varphi(\mathbf{x}) \ge 0,\ \forall \mathbf{y}\in\mho, \tag{5.50}$$

where $\mathscr{N}:\mathcal{V}\times\mathcal{V}\to\mathcal{V}^*$ and $\varphi:\mathcal{V}\to\mathbb{R}$ are defined by

$$\langle \mathscr{N}(\mathbf{x},\mathbf{x}),\mathbf{y}\rangle = \langle \mathscr{N}(\boldsymbol{\varepsilon}(\mathbf{x}),\boldsymbol{\varepsilon}(\mathbf{x})),\boldsymbol{\varepsilon}(\mathbf{y})\rangle_{\mathcal{H}}$$
$$\varphi(\mathbf{x}) = \int_\Omega \psi(\boldsymbol{\varepsilon}(\mathbf{x}))d\mathbf{u} + \int_{\daleth_{D_3}} \mathscr{F}x_\nu + \int_{\daleth_{D_3}} \mu\|\mathbf{x}_\tau\|d\daleth,\ \forall x,y\in\mathcal{V},$$

Since $\mathbf{x}\in\mathcal{V}$ is a solution of (5.50), then it is a solution of (5.47) as well. Therefore, we show that (5.47) has at least one solution, it is enough to prove that (5.50) is solvable.

Let $\mathcal{K}\subset\mathcal{V}$ be a bounded set and for any $\mathbf{y}_0\in\mathcal{K}$. From the inequality $\pi_{1_\nu}\|\varsigma\|^2\sqrt{2} < \alpha_{\mathscr{N}}$ and $\mathbf{y}_0\in\mathcal{K}$, we have

$$\liminf_{\mathbf{x}\in\mathcal{V},\|\mathbf{x}\|_{\mathcal{V}}\to\infty} \frac{\langle \mathscr{N}(\mathbf{x},\mathbf{x}),\mathbf{x}-\mathbf{y}_0\rangle + \inf_{\boldsymbol{\xi}_x\in\partial J(\varsigma\mathbf{x})}\langle \boldsymbol{\xi}_x,\mathbf{x}-\mathbf{y}_0\rangle_{\mathbb{X}^*\times\mathbb{X}}}{\|\mathbf{x}\|_{\mathcal{V}}} = +\infty.$$

Hence **H($\mathscr{N}$)(iii)** is valid.

Therefore the set of solutions to (5.50) is nonempty, bounded, closed and convex and this show that solution set of (5.47) has the same properties. □

Now, we consider a normal compliance function $p_\nu : \daleth_{D_2} \times \daleth_{D_2} \times \mathbb{R} \to \mathbb{R}_+$ satisfying the following conditions:

(a) for all $r \in \mathbb{R}$, the function $\mathbf{u} \to p_\nu(\mathbf{u}, \mathbf{u}, r)$ is measurable on $\daleth_{D_2}$,

(b) there exists $\mathscr{L}_{p_\nu} > 0,\ \mathscr{L}'_{p_\nu} > 0$ such that

$$|p_\nu(\mathbf{u}, \mathbf{u}, r_1) - p_\nu(\mathbf{u}, \mathbf{u}, r_2)| \leq \mathscr{L}_{p_\nu}|r_1 - r_2| + \mathscr{L}'_{p_\nu}|r_1 - r_2|$$

$$\leq (\mathscr{L}_{p_\nu} + \mathscr{L}'_{p_\nu})|r_1 - r_2|,\ \forall r_1, r_2 \in \mathbb{R} \text{ and } \mathbf{u} \in \daleth_{D_2}, \tag{5.51}$$

(c) for all $r_1, r_2 \in \mathbb{R}$ and $\mathbf{u} \in \daleth_{D_2}$,

$$(p_\nu(\mathbf{u}, \mathbf{u}, r_1) - p_\nu(\mathbf{u}, \mathbf{u}, r_2))(r_1 - r_2) \geq 0, \tag{5.52}$$

(d) for a.e. $\mathbf{u} \in \daleth_{D_2}$, $p_\nu \in (\mathbf{u}, \mathbf{u}, r) = 0$ if and only if

$$r \leq 0.$$

Let $\{g_n\}$, $\{\lambda_n\} \subset \mathbb{R}_+$ be such that

$$g_n > 0,\ \text{for all } n \in \mathbb{N} \text{ and } g_n \to g \text{ as } n \to \infty, \tag{5.53}$$

and

$$\lambda_n > 0,\ \text{for all } n \in \mathbb{N} \text{ and } \lambda_n \to \lambda \text{ as } n \to \infty. \tag{5.54}$$

Then we find a displacement field $\mathbf{x}_n : \Omega \to \mathbb{R}^d$, a stress field $\boldsymbol{\sigma}_n : \Omega \to \mathfrak{F}^d$ and an interface force $\xi_{n\nu} : \daleth_{D_2} \to \mathbb{R}$ such that

$$Div\ \boldsymbol{\sigma}_n + \mathbf{f}_0 = \mathbf{0} \text{ in } \Omega, \tag{5.55}$$

$$\boldsymbol{\sigma}_n \in \mathscr{N}(\boldsymbol{\varepsilon}(\mathbf{x}_n), \boldsymbol{\varepsilon}(\mathbf{x}_n)) + \partial_c\psi(\boldsymbol{\varepsilon}(\mathbf{x}_n)) \text{ in } \Omega, \tag{5.56}$$

$$\mathbf{x}_n = \mathbf{0} \text{ on } \daleth_D, \tag{5.57}$$

$$\boldsymbol{\sigma}_n\boldsymbol{\nu} = \mathbf{f}_N \text{ on } \daleth_{D_1}, \tag{5.58}$$

$$\begin{cases} -\sigma_{n\nu} = \dfrac{1}{\lambda_n} p_\nu(x_{n\nu} - g_n)\xi_{n\nu}, \\ \xi_{n\nu} \in \partial j_\nu(x_{n\nu}), \\ \boldsymbol{\sigma}_{n\tau} = \mathbf{0}, \qquad \text{on } \daleth_{D_2} \end{cases} \tag{5.59}$$

here λ_n be the deformability coefficient and $\dfrac{1}{\lambda_n}$ denotes the surface stiffness coefficient.

$$\begin{cases} -\sigma_{n\nu} = \mathscr{F}, \\ \|\boldsymbol{\sigma}_{n\tau}\| \leq \mu|\sigma_{n\nu}|, \qquad \text{on } \daleth_{D_3} \\ -\boldsymbol{\sigma}_{n\tau} = \mu|\sigma_{n\nu}|\dfrac{\mathbf{x}_{n\tau}}{\|\mathbf{x}_{n\tau}\|},\ \text{if } \mathbf{x}_{n\tau} \neq \mathbf{0}. \end{cases} \tag{5.60}$$

The variational formulation of (5.55)-(5.60) for finding a displacement field $\mathbf{x}_n \in \mathcal{V}$ such that

$$\langle \mathscr{N}(\boldsymbol{\varepsilon}(\mathbf{x}_n), \boldsymbol{\varepsilon}(\mathbf{x}_n)), \boldsymbol{\varepsilon}(\mathbf{y}) - \boldsymbol{\varepsilon}(\mathbf{x}_n)\rangle_{\mathcal{H}} + \int_\Omega \psi(\boldsymbol{\varepsilon}(\mathbf{y}))d\mathbf{u} - \int_\Omega \psi(\boldsymbol{\varepsilon}(\mathbf{x}_n))d\mathbf{u} + \int_{\daleth_{D_3}} \mathscr{F}(y_\nu - x_{n\nu})d\daleth$$

$$+ \int_{\daleth_{D_3}} \mu\mathscr{F}(\|\mathbf{y}\tau\| - \|\mathbf{x}_{n\tau}\|)d\daleth + \frac{1}{\lambda_n}\int_{\daleth_{D_2}} p_\nu(x_{n\nu} - g_n)(y_\nu - x_{n\nu})d\daleth$$

$$+ \int_{\daleth_{D_2}} j^0_\tau(\mathbf{x}_{n\tau}; \mathbf{y}_\tau - \mathbf{x}_{n\tau})d\daleth \geq \langle \mathbf{f}, \mathbf{y} - \mathbf{x}_n\rangle,\ \forall \mathbf{y} \in \mathcal{V}. \tag{5.61}$$

Theorem 5.7. *Assume that (A)-(C), (5.45)-(5.46) and (5.51)-(5.54) are fulfilled. If, in addition, the inequalities (5.48) hold, then*

(i) *for each $n \in \mathbb{N}$, the set of solutions of (5.61), denoted by $\mathbb{P}_n$, is nonempty, bounded, closed and convex.*
(ii) $\emptyset \neq \varrho - \limsup_{n\to\infty} \mathbb{P}_n = \rho - \limsup_{n\to\infty} \mathbb{P}_n \subset \mathbb{P}$.
(iii) *for each $\mathbf{x} \in \varrho - \limsup_{n\to\infty} \mathbb{P}_n$, there exists a subsequence of $\{\tilde{\mathbf{x}}_n\}$ converging strongly to $\mathbf{x}$, where $\tilde{\mathbf{x}}_n = proj_{\mathbb{P}_n}(\mathbf{x})$ for all $n \in \mathbb{N}$.*

Proof. Consider the operator $\mathscr{P}_n : \mathcal{V} \times \mathcal{V} \to \mathcal{V}^*$ defined by

$$\langle \mathscr{P}_n(\mathbf{x},\mathbf{x}), \mathbf{y}\rangle = \int_{\daleth_{D_2}} p_\nu(x_\nu - g_n) y_\nu d\daleth, \ \forall\, \mathbf{x},\mathbf{y} \in \mathcal{V}.$$

From hypotheses (5.51)-(5.52), $\mathscr{P}_n$ is bounded, continuous and monotone for each $n \in \mathbb{N}$. Let $\mathbf{y} \in \mho$ be arbitrary and consider the sequence $\{\mathbf{y}_n\}$ defined by

$$y_n = \begin{cases} \frac{g_n}{g}\mathbf{y} \text{ if } g > 0 \\ \mathbf{y} \text{ if } g = 0. \end{cases}$$

Since $\mathbf{y} \in \mho$, $y_\nu \le g$ on $\daleth_{D_2}$ and $g_n > 0$, then $y_{n\nu} - g_n \le 0$. Combining $p_\nu(\mathbf{u}, s) = 0$ with $s \le 0$, entails that

$$\mathscr{P}_n(\mathbf{y}_n, \mathbf{y}_n) = \mathbf{0}\ \forall n \in \mathbb{N}.$$

Additionally,

$$\mathbf{y}_n \to \mathbf{y} \text{ as } n \to \infty,$$

therefore the condition **H(**$\mathscr{P}_n$**)(i)** holds. Next, we define the operator $\mathscr{P} : \mathcal{V} \times \mathcal{V} \to \mathcal{V}^*$ by

$$\langle \mathscr{P}(\mathbf{x},\mathbf{x}), \mathbf{y}\rangle = \int_{\daleth_{D_2}} p_\nu(x_\nu - g_n) y_\nu d\daleth, \ \forall \mathbf{x},\mathbf{y} \in \mathcal{V}.$$

Using the assumption (5.51)-(5.52), $\mathscr{P}$ satisfies condition **H(**$\mathscr{P}_n$**)(ii)**. Let $\{\mathbf{x}_n\} \subset \mathcal{V}$ be such that

$$\mathbf{x}_n \rightharpoonup \mathbf{x}$$

and

$$\limsup_{n\to\infty} \langle \mathscr{P}_n(\mathbf{x}_n,\mathbf{x}_n), \mathbf{x}_n - \mathbf{x}\rangle \le 0.$$

Hence, the compactness of the embedding of $\mathcal{V}$ into $L^2(\daleth)$, and the continuity of $p_\nu(\mathbf{x},\mathbf{x},\cdot)$ for a.e. $\mathbf{x} \in \daleth_{D_2}$ imply

$$\begin{aligned} \liminf_{n\to\infty} \int_{\daleth_{D_2}} p_\nu(x_{n\nu} - g_n)(x_{n\nu} - y_\nu) d\daleth &\ge \liminf_{n\to\infty} \int_{\daleth_{D_2}} p_\nu(x_{n\nu} - g_n)(x_{n\nu} - x_\nu) d\daleth \\ &\quad + \liminf_{n\to\infty} \int_{\daleth_{D_2}} p_\nu(x_{n\nu} - g_n)(x_\nu - y_\nu) d\daleth \\ &\ge \int_{\daleth_{D_2}} p_\nu(x_\nu - g)(x_\nu - y_\nu) d\daleth \\ &= \langle \mathscr{P}(\mathbf{x},\mathbf{x}), \mathbf{x} - \mathbf{y}\rangle. \end{aligned}$$

This showed that the condition **H(**$\mathscr{P}_n$**)(ii)(b)** holds and proof is completed. □

6. Declaration statements

Availability of data and material: The data sets used and/or analysed during the current study are available from the corresponding author on reasonable request.
Declaration of Competing interest: The authors declare that they have no conflicts of interest to this work. We declare that we do not have any commercial or associative interest that represents a conflict of interest in connection with that work submitted.
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