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Oscillations of second-order noncanonical advanced difference equations via canonical transformation

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ABSTRACT. This paper introduces a new improved method for obtaining the oscillation of a second-order advanced difference equation of the form

$$\Delta(\eta(n)\Delta\chi(n)) + f(n)\chi(\sigma(n)) = 0$$

where $\eta(n) > 0$, $\sum_{n=n_0}^{\infty} \frac{1}{\eta(n)} < \infty$, $f(n) > 0$, $\sigma(n) \geq n+1$, and $\{\sigma(n)\}$ is a monotonically increasing integer sequence. We derive new oscillation criteria by transforming the studied equation into the canonical form. The obtained results are original and improve on the existing criteria. Examples illustrating the main results are presented at the end of the paper.

1. INTRODUCTION

Consider the second-order advanced difference equation

$$(E) \qquad \Delta(\eta(n)\Delta\chi(n)) + f(n)\chi(\sigma(n)) = 0, \; n \geq n_0,$$

where $n \in \mathbb{N}(n_0) = \{n_0, n_0+1, ...\}$, n_0 is a positive integer.

Throughout the paper, we assume that:

(1) $[(C_1)]$
(2) $\{\eta(n)\}$ and $\{f(n)\}$ are positive real sequences for all $n \in \mathbb{N}(n_0)$;
(3) $\{\sigma(n)\}$ is a monotonically increasing sequence of integers with $\sigma(n) \geq n+1$ for all $n \in \mathbb{N}(n_0)$;
(4) $\sum_{n=n_0}^{\infty} \frac{1}{\eta(n)} < \infty$.

By a solution of (E), we mean a nontrivial sequence $\{\chi(n)\}$ that satisfies (E) for all $n \in \mathbb{N}(n_0)$. A solution $\{\chi(n)\}$ of (E) is said to be *oscillatory* if it is neither eventually negative nor eventually positive. Otherwise, it is called *nonoscillatory*. Equation (E) is called oscillatory if all its solutions are oscillatory.

It is known that (E) is in the canonical form, if

$$(1.1) \qquad \sum_{n=n_0}^{\infty} \frac{1}{\eta(n)} = \infty$$

and is in the noncanonical form, if

$$(1.2) \qquad \sum_{n=n_0}^{\infty} \frac{1}{\eta(n)} < \infty.$$

In recent years, numerous researches analyzed oscillatory behavior of solutions to various classes of differential equations with deviating arguments (see, e.g, the papers [6] and [8]) and partial differential equations (see example given [13] and [15]), where oscillation and/or delay situations take part in models from mathematical biology and physics

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when their formulation include cross-diffusion terms. Recently, several criteria have been reported in the literature on the oscillation of (E) for the delay case, that is, $\sigma(n) \leq n-1$ while at the same time, either (1.1) or (1.2) holds. See, for example, [1, 4, 10, 11, 12, 14, 18] and the references cited therein. However, there are few oscillation results for the equation (E) with (1.1) by using comparison method and/or Riccati type transformation method, see for example [2, 3, 5, 7, 9, 16, 17, 20].

From the above observations, we see that most of the papers are devoted to the canonical type equations. This is because the canonical type equations are much easier to study than the noncanonical type. Therefore, in this paper, first we transform equation (E) into equivalent canonical type equation and then we derive new oscillation criteria for (E) using a new comparison principle for the advanced type equations.

2. Main results

Without loss of generality, we can study the nonoscillatory solutions of (E) by restricting our attention to positive solutions. It is known that the set of positive solutions of (E) has the following structure:

(I) $\chi(n)>0$, $\Delta\chi(n)>0$ and $\Delta(\eta(n)\Delta\chi(n))<0$;
(II) $\chi(n)>0$, $\Delta\chi(n)<0$ and $\Delta(\eta(n)\Delta\chi(n))<0$.

To overcome this, we transform (E) into the canonical form which essentially simplifies the study of (E).

Throughout this paper, we are going to use the following notation:

$$\begin{aligned}\Pi(n) &= \sum_{s=n}^{\infty}\frac{1}{\eta(n)},\\ b(n) &= \eta(n)\Pi(n)\Pi(n+1),\\ \mu(n) &= \frac{\chi(n)}{\Pi(n)},\end{aligned}$$

and

$$g(n) = \Pi(n+1)f(n)\Pi(\sigma(n)).$$

Lemma 2.1. *Assume that $(C_1)-(C_3)$ hold. Then*

$$\Delta(\eta(n)\Delta\chi(n)) = \frac{1}{\Pi(n+1)}\Delta\left(\eta(n)\Pi(n)\Pi(n+1)\Delta\left(\frac{\chi(n)}{\Pi(n)}\right)\right). \tag{2.3}$$

Proof. By a straightforward computation, we can show that (2.3) holds for any sequence $\chi(n)$. Indeed,

$$\begin{aligned}&\frac{1}{\Pi(n+1)}\Delta\left(\eta(n)\Pi(n)\Pi(n+1)\Delta\left(\frac{\chi(n)}{\Pi(n)}\right)\right)\\ =\;&\frac{1}{\Pi(n+1)}\Delta\left(\eta(n)\Pi(n)\Pi(n+1)\frac{\eta(n)\Pi(n)\Delta\chi(n)+\chi(n)}{\eta(n)\Pi(n)\Pi(n+1)}\right)\\ =\;&\frac{1}{\Pi(n+1)}\Delta\left(\eta(n)\Pi(n)\Delta\chi(n)+\chi(n)\right)\\ =\;&\frac{1}{\Pi(n+1)}\left[\Pi(n+1)\Delta\left(\eta(n)\Delta\chi(n)-\Delta\chi(n)+\Delta\chi(n)\right)\right]\\ =\;&\Delta\left(\eta(n)\Delta\chi(n)\right).\end{aligned}$$

Moreover

$$\sum_{n=n_0}^{\infty} \frac{1}{\eta(n)\Pi(n)\Pi(n+1)} = \lim_{n\to\infty} \frac{1}{\Pi(n)} - \frac{1}{\Pi(n_0)} = \infty,$$

that is, the operator on the right hand side of (2.3) is canonical. The proof of the lemma is complete. □

As a consequence of Lemma 2.1, we see that (E) can be written in the equivalent form

$$\Delta\left(\eta(n)\Pi(n)\Pi(n+1)\Delta\left(\frac{\chi(n)}{\Pi(n)}\right)\right) + \Pi(n+1)f(n)\chi(\sigma(n)) = 0,$$

or

$$\Delta(b(n)\Delta\mu(n)) + g(n)\mu(\sigma(n)) = 0. \qquad (E_1)$$

The following result follows directly from the above discussion.

Theorem 2.1. *The noncanonical difference equation* (E) *possesses a solution* $\{\chi(n)\}$ *if and only if the canonical equation* (E_1) *has the solution* $\mu(n) = \frac{\chi(n)}{\Pi(n)}$.

Corollary 2.1. *The noncanonical difference equation* (E) *has an eventually positive solution if and only if the canonical equation* (E_1) *has an eventually positive solution.*

Corollary 2.1 significantly simplifies the study of (E) since using (E_1), we deal with only a class of positive solutions, namely,

$$\mu(n) > 0,\ \Delta(\mu(n)) > 0,\ \Delta(b(n)\Delta\mu(n)) < 0. \qquad (2.4)$$

This follows from Lemma 2.1 of [5].

To prove our oscillation results, let us assume that

$$b(n) \ge 1 \ \text{ for all } \ n \in \mathbb{N}(n_0). \qquad (2.5)$$

We define

$$R(n;j) = \begin{cases} \sum_{s=n}^{\infty} g(s), & j = 0 \text{ for } n \ge n_0 \\ \sum_{s=n}^{\infty} \frac{R^2(s,j-1)}{R(s,j-1)+b(s)}, & j \ge 1 \text{ for } n \ge n_0 \end{cases} \qquad (2.6)$$

and

$$G(n) = \sum_{s=n_0}^{n-1} \frac{1}{b(s)}.$$

Theorem 2.2. *Assume that* (2.5) *and*

$$G(n)\sum_{s=n}^{\infty} g(s) \ge \beta > \frac{1}{4} \qquad (2.7)$$

eventually. Then all solutions of (E) *are oscillatory.*

Proof. From (2.7) and (2.6) for $j = 0$, we have $R(n,j) \ge \frac{\beta}{G(n)}$. Since $h(x) = \frac{x^2}{d+x}$ is increasing for $x > 0$ and $d > 0$, we have $h(R(n;0)) \ge h(\frac{\beta}{R(n)})$ and therefore

$$\frac{R^2(n;0)}{R(n;0)+b(n)} \ge \frac{\frac{\beta^2}{G^2(n)}}{\frac{\beta}{G(n)} + b(n)} = \frac{\beta^2}{G(n)(\beta + b(n)G(n))}.$$

In the above inequality, we have used that $d = b(n)$. From (2.6) for $j = 1$, and using the above inequality, we obtain

$$R(n;1) = \sum_{s=n}^{\infty} \frac{R^2(s;0)}{R(s;0)+b(s)} + R(n;0) \ge \sum_{s=n}^{\infty} \frac{\beta^2 b(s)\Delta G(s)}{G(s)(\beta + b(s)G(s))} + \frac{\beta}{G(n)}.$$

To derive the above inequality, we have used that $b(n)\Delta G(n) = 1$.

Since $\frac{\beta}{b(n)} \leq \beta$ for all $n \geq n_0$, it is obvious that

$$R(n;1) \geq \sum_{s=n}^{\infty} \frac{\beta^2 \Delta G(s)}{G(s)(\beta + G(s))} + \frac{\beta}{G(n)}.$$

On the other hand,

$$\sum_{s=n}^{\infty} \frac{\beta^2 \Delta G(s)}{G(s)(\beta + G(s))} \geq \int_{G(s)}^{\infty} \frac{\beta^2 ds}{s(\beta + s)} = \beta \ln\left(\frac{\beta + G(n)}{G(n)}\right).$$

Hence

$$R(n;1) \geq \frac{\beta}{G(n)} \geq +\beta \ln\left(\frac{\beta + G(n)}{G(n)}\right) = \frac{\beta_1}{G(n)}$$

where $\beta_1 = \beta + G(n)\beta \ln\left(\frac{\beta + G(n)}{G(n)}\right) > \beta$.

In general

$$R(n;j) = \sum_{s=n}^{\infty} \frac{\beta_{j-1}^2 \Delta G(s)}{G(s)(\beta_{j-1} + G(s))} + \frac{\beta}{G(n)} \geq \beta_{j-1} \ln\left(\frac{\beta_{j-1} + G(n)}{G(n)}\right) + \frac{\beta}{G(n)} = \frac{\beta_j}{G(n)}$$

where

$$\beta_j = \beta + G(n)\beta_{j-1} \ln\left(\frac{\beta_{j-1} + G(n)}{G(n)}\right). \tag{2.8}$$

It is easy to see that $\beta_j < \beta_{j+1}$ with $\beta = \beta_0$ for $j = 0, 1, 2, ...$ Now we claim that $\lim_{j\to\infty} \beta_j = \infty$. If not, let $M = \lim_{j\to\infty} \beta_j < \infty$.

Then from (2.8), we obtain

$$M = \beta + G(n)M \ln\left(\frac{M + G(n)}{G(n)}\right) = \beta + M\left(M - \frac{M^2}{2G(s)} + ...\right).$$

Letting $n \to \infty$, we have $G(n) \to \infty$ and the last equation yields

$$M = \beta + M^2. \tag{2.9}$$

But equation (2.9) has no real solution if $\beta > \frac{1}{4}$. Hence $\lim_{j\to\infty} \beta_j = \infty$. Then, we see that $\lim_{j\to\infty} R(n;j) = \infty$. Thus from Theorem 1.11.5 of [1] it follows that

$$\Delta(b(n)\Delta\mu(n)) + g(n)\mu(n+1) = 0, n \geq n_0 \tag{2.10}$$

is oscillatory. Now using Theorem 3.5 of [20], we see that (E_1) oscillates and hence (E) is oscillatory. The proof of the theorem is complete. □

Remark 2.1. The above condition is independent of the advanced argument $\{\sigma(n)\}$ and thus, is more appropriate for (2.10).

In our next theorem, we assume the opposite condition of (2.7), namely

$$G(n)\sum_{s=n}^{\infty} g(s) \geq \beta \text{ but } \beta \leq \frac{1}{4}$$

holds.

Theorem 2.3. *Assume that* (2.5) *holds. Let* $\{\mu(n)\}$ *be a positive solution of* (E_1) *and*

$$G(n)\sum_{s=n}^{\infty} g(s) \geq \beta > 0, \tag{2.11}$$

eventually. Then there is an integer N *such that for* $n \geq N$, $\{\frac{\mu(n)}{G^{\beta}(n)}\}$ *is monotonically increasing.*

Proof. The proof is similar to that of Theorem 2.3 of [5] and hence, it is omitted. □

Remark 2.2. Theorem 2.3 gives a new monotonic property for the positive solutions of (E_1), ensuring that $\{\mu(n)\}$ is increasing and $\{\frac{\mu(n)}{G^\beta(n)}\}$ is also increasing. This property leads to improve oscillation criteria for second-order noncanonical advanced difference equations.

Next, we state a new comparison theorem for difference equations with advanced arguments.

Theorem 2.4. *Assume that* (2.5) *and* (2.11) *hold. If the difference equation*

$$\Delta(b(n)\Delta\mu(n)) + \left(\frac{G(\sigma(n))}{G(n+1)}\right)^\beta g(n)\mu(n+1) = 0 \tag{2.12}$$

is oscillatory, then (E) *is oscillatory.*

Proof. Assume, for the sake of contradiction, that (E) has an eventually positive solution. Then, from Corollary 2.1, it follows that (E_1) has an eventually positive solution $\{\mu(n)\}$. Now equation (E_1) can be written as

$$\Delta(b(n)\Delta\mu(n)) + \left(\frac{G(\sigma(n))}{G(n+1)}\right)^\beta g(n)\mu(\sigma(n)) = 0.$$

Since $\frac{\mu(n)}{G^\beta(n)}$ is nondecreasing, and $\sigma(n) \geq n+1$, we have

$$\frac{\mu(\sigma(n))}{G^\beta(\sigma(n))} \geq \frac{\mu(n+1)}{G^\beta(n+1)}.$$

Using this relation in the last equation, we see that $\{\mu(n)\}$ is a positive solution of the following inequality

$$\Delta(b(n)\Delta\mu(n)) + \left(\frac{G(\sigma(n))}{G(n+1)}\right)^\beta g(n)\mu(n+1) \leq 0.$$

But by the result in [19], the corresponding difference equation (2.12) also has a positive solution, which leads to a contradiction. The proof of the theorem is complete. □

Remark 2.3. The above theorem ensures that any oscillation criterion obtained for (2.12), leads to an oscillation criterion for (E).

Theorem 2.5. *Let* (2.5) *and* (2.11) *hold. Assume that there is a constant* β_1 *such that*

$$G(n)\sum_{s=n}^{\infty}\left(\frac{G(\sigma(s))}{G(s+1)}\right)^\beta g(s) \geq \beta_1 > \frac{1}{4}, \tag{2.13}$$

eventually. Then all solutions of (E) *are oscillatory.*

Proof. Condition (2.13) guarantees that (2.12) oscillates which in turn implies that (E_1) oscillates. But by Corollary 2.1, equation (E) oscillates. This completes the proof. □

If condition (2.13) fails, that is, $\beta_1 \leq \frac{1}{4}$, then we derive the following new criteria, using the constant β_1.

Theorem 2.6. *Let* (2.5) *and* (2.11) *hold. Assume that* $\{\mu(n)\}$ *is a positive solution of* (E_1) *and*

$$G(n)\sum_{s=n}^{\infty}\left(\frac{G(\sigma(s))}{G(s)}\right)^\beta g(s) \geq \beta_1 > 0 \tag{2.14}$$

eventually. Then there is an integer N such that for $n \geq N$, $\{\frac{\mu(n)}{G^{\beta_1}(n)}\}$ is monotonically nondecreasing.

Proof. The proof is similar to that of Theorem 2.6 of [5] and hence the details are omitted. □

Theorem 2.7. *Let* (2.5), (2.11) *and* (2.14) *hold. If the difference equation*

$$\Delta(b(n)\Delta\mu(n)) + \left(\frac{G(\sigma(n))}{G(n+1)}\right)^{\beta_1} g(n)\mu(n+1) = 0 \tag{2.15}$$

is oscillatory, then all solutions of (E) are oscillatory.

Theorem 2.8. *Let* (2.5), (2.11) *and* (2.14) *hold. If there exists a constant β_2 such that*

$$G(n)\sum_{s=n}^{\infty}\left(\frac{G(\sigma(s))}{G(s+1)}\right)^{\beta_1} g(s) \geq \beta_2 > \frac{1}{4}, \tag{2.16}$$

eventually, then all solutions of (E) are oscillatory.

Theorems 2.7 and 2.8 can be proved similarly to Theorem 2.4 and 2.5. Thus, we omit the details of the proofs.

By repeatedly applying the above process, we can obtain progressively improved oscillation criteria. Assume that there is a positive constant λ such that

$$\frac{G(\sigma(n))}{G(n+1)} \geq \lambda > 1 \tag{2.17}$$

eventually. Thus, using (2.5) and (2.11), conditions (2.14) and (2.16) can be rewritten in similar forms as

$$\begin{aligned} \beta_1 &= \lambda^{\beta}\beta > \frac{1}{4} \\ \beta_2 &= \lambda^{\beta_1}\beta > \frac{1}{4} \end{aligned}$$

respectively. By repeating the above process, we can obtain an increasing sequence $\{\beta_j\}_{j=0}^{\infty}$ defined as follows:

$$\begin{aligned} \beta_0 &= \beta \\ \beta_{j+1} &= \lambda^{\beta_j}\beta. \end{aligned} \tag{2.18}$$

Now, we can generalize the oscillation criteria of Theorems 2.5 and 2.8, in the following theorem.

Theorem 2.9. *Let* (2.5), (2.11) *and* (2.17) *hold. If there exists a positive integer ℓ such that $\beta_j \leq \frac{1}{4}$ for $j = 0, 1, ..., \ell - 1$, and*

$$\beta_\ell > \frac{1}{4},$$

then all solutions of (E) are oscillatory.

3. Examples

In this section, we provide three examples to illustrate the main results.

Example 3.1. Consider the second-order advanced Euler type difference equation

$$\Delta(n(n+1)\Delta\chi(n)) + \lambda d\chi(\lambda n) = 0, \; n \geq 1, \tag{3.19}$$

where $d > 0$ and $\lambda \geq 2$ is an integer.

A simple calculation shows that $b(n) = 1$, $G(n) = n-1$, and $g(n) = \frac{d}{n(n+1)}$. Now $d = \beta$, and by Theorem 2.5, equation (3.19) is oscillatory provided that

$$d\lambda^d > \frac{1}{4}.$$

For example, if $d = \frac{1}{5}$, then it is required that $\lambda \geq 4$ or conversely, for $\lambda = 2$, we need $d = \frac{1}{4}$.

Example 3.2. Consider again the difference equation (3.19). For this equation, we see that $\beta_1 = d\lambda^d$.

By Theorem 2.8, we see that (3.19) is oscillatory provided that

$$d\lambda^{\beta_1} > \frac{1}{4}.$$

Since $\beta_1 > d$, Theorem 2.8 implies Theorem 2.5. Now for $d = \frac{1}{5}$ we need $\lambda \geq 3$.

Example 3.3. Consider the second-order advanced type difference equation

$$\Delta(n(n+1)\Delta\chi(n)) + 0.422\chi(2n) = 0, \ n \geq 1. \tag{3.20}$$

In this equation $\beta = \beta_0 = 0.211$ and $\lambda = 2$. By a straightforward calculation, we obtain

$$\beta_1 = 0.244 \ \text{ and } \ \beta_2 = 0.249.$$

Hence, Theorems 2.5 and 2.8 fail for (3.20). But

$$\beta_3 = 0.2509 > 0.25$$

and Theorem 2.9 guarantees the oscillation of (3.20).

Remark 3.4. The results obtained in [5] do not apply to the above examples, since the equations in these are of the Euler type.

4. Conclusion

In this paper, we have obtained new monotonic properties of the nonoscillatory solutions of second-order noncanonical advanced difference equations. Based on these, we have established new oscillation criteria for this form of equations. The results obtained in this paper are original and complement those in [2, 3, 5, 17, 7, 16, 20, 11, 10].

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