CARPATHIAN J. MATH.
Volume **38** (2022), No. 2,
Pages 445 - 457

Online version at https://semnul.com/carpathian/
Print Edition: ISSN 1584 - 2851; Online Edition: ISSN 1843 - 4401
DOI: https://doi.org/10.37193/CJM.2022.02.14

Common endpoints of generalized Suzuki-Kannan-Ćirić type mappings in hyperbolic spaces

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ABSTRACT. In this paper, we introduce the concept of generalized Suzuki-Kannan-Ćirić type mappings in metric spaces and show that it is weaker than the concept of Suzuki-Kannan-Ćirić type mappings but stronger than the concept of semi-nonexpansive mappings. Moreover, we obtain the semiclosed principle and endpoint theorems for the class of generalized Suzuki-Kannan-Ćirić type mappings. The strong and $\Delta-$convergence theorems of the Kuhfitting iteration for this class of mappings are also discussed.

1. INTRODUCTION

Let D be a nonempty subset of a metric space (M, ρ). A mapping g from D into D is a contraction if there exists a constant λ in $[0, 1)$ such that

$$\rho(g(x), g(y)) \leq \lambda\rho(x, y), \quad \text{for all } \ x, y \in D. \tag{1.1}$$

Moreover, if (1.1) holds when $\lambda = 1$, then g is said to be nonexpansive. A point x in D is called a fixed point of g if $x = g(x)$.

The fixed point theory is a powerful tool for finding solutions of problems in the form of equations and inequalities. One of the remarkable results in the metric fixed point theory is the so-called Banach contraction principle [6] which states that every contraction on a complete metric space always has a unique fixed point. The principle has been studied and generalized in many directions, see, e.g., [2, 5, 8, 10, 14, 17, 20, 23, 41, 43] and references therein.

In 2011, Karapınar and Taş [24] combined the ideas of [14], [23] and [44] to introduce the concept of Suzuki-Kannan-Ćirić type mappings and prove the existence of fixed points for such kind of mappings. In 2015, Chang et al. [9] extended the results of [24] to the setting of multi-valued Suzuki-Kannan-Ćirić type mappings.

The concept of endpoints for multi-valued mappings is an important concept which is weaker than the concept of fixed points for single-valued mappings and stronger than the concept of fixed points for multi-valued mappings. In 1986, Corley [15] proved that a maximization with respect to a cone was equivalent to the problem of finding an endpoint of a certain multi-valued mapping. In 1997, Tarafdar and Yuan [47] proved the existence of Pareto optima for multi-valued mappings by using the concept of endpoints. For further applications of the endpoint theory, the reader is referred to [3, 21, 26, 27, 46, 48].

In 2015, Panyanak [38] proved the existence of endpoints for multi-valued nonexpansive mappings in uniformly convex Banach spaces as well as Banach spaces which satisfy the Opial's condition. It was quickly noted by Espínola et al. [18] that the results of Panyanak can be extended to more general classes of Banach spaces. In 2016, Saejung

Received: 24.04.2021. In revised form: 17.10.2021. Accepted: 24.10.2021
2010 *Mathematics Subject Classification.* 47H09, 47H10.
Key words and phrases. Endpoint, Suzuki-Kannan-Ćirić type mapping, semi-nonexpansive mapping, Kuhfitting iteration, uniformly convex hyperbolic space.
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[42] obtained endpoint theorems for some generalized multi-valued nonexpansive mappings in certain classes of Banach spaces. Since then endpoint results for some generalized multi-valued nonexpansive mappings in several classes of metric and Banach spaces have been developed and many papers have appeared (see, e.g., [11, 12, 22, 29, 31, 32, 35, 39, 40]).

In this paper, we introduce the concept of generalized Suzuki-Kannan-Ćirić type for multi-valued mappings and show that it is more general than the concept of Suzuki-Kannan-Ćirić type mappings. We also give sufficient conditions for the existence of endpoints for generalized Suzuki-Kannan-Ćirić type mappings in uniformly convex hyperbolic spaces with monotone moduli of uniform convexity. Moreover, we also prove the strong and $\Delta-$convergence theorems of the Kuhfitting iteration for the class of generalized Suzuki-Kannan-Ćirić type mappings. Our results extend and improve the results of [9, 12, 24, 29, 44] and many others.

2. Preliminaries

Throughout this paper, $\mathbb{N}$ stands for the set of natural numbers and $\mathbb{R}$ stands for the set of real numbers. Let (M, ρ) be a metric space, $\emptyset \neq D \subseteq M$ and $x \in M$. The distance from x to D is defined by

$$\text{dist}(x, D) := \inf\{\rho(x, y) : y \in D\}.$$

The radius of D relative to x is defined by

$$R(x, D) := \sup\{\rho(x, y) : y \in D\}.$$

We denote by $\mathcal{CB}(D)$ the family of nonempty closed bounded subsets of D and by $\mathcal{K}(D)$ the family of nonempty compact subsets of D. The Pompeiu-Hausdorff distance on $\mathcal{CB}(D)$ is defined by

$$H(A, B) := \max\left\{\sup_{a\in A} \text{dist}(a, B), \sup_{b\in B} \text{dist}(b, A)\right\}, \tag{2.2}$$

for all $A, B \in \mathcal{CB}(D)$.

Now, we collect some basic properties of the radius and the Pompeiu-Hausdorff distance.

Proposition 2.1. *Let (M, ρ) be a metric space, $x, y \in M$ and $A, B, C \in \mathcal{CB}(M)$. Then the following conclusions hold:*

(i) $R(x, B) = H(\{x\}, B)$.
(ii) $R(x, B) \leq R(x, A) + H(A, B)$.
(iii) $R(x, B) \leq \rho(x, y) + R(y, B)$.
(iv) $H(A, C) \leq H(A, B) + H(B, C)$.

Proof. (i) follows from (2.2) by choosing $A = \{x\}$. For (ii) we let $a \in A$ and $b \in B$. Then $\rho(x, b) \leq \rho(x, a) + \rho(a, b) \leq R(x, A) + \rho(a, b)$. Since $a \in A$ is arbitrary, we get

$$\rho(x, b) \leq R(x, A) + \text{dist}(b, A) \leq R(x, A) + H(B, A).$$

Since $b \in B$ is arbitrary, we have $R(x, B) \leq R(x, A) + H(A, B)$. (iii) follows from (i) and (ii) by choosing $A = \{y\}$. (iv) follows from Theorem 2.1.7 of [45]. □

A mapping S from D into $\mathcal{CB}(M)$ is called a multi-valued mapping. In particular, if Sx is a singleton for all x in D, then S is called a single-valued mapping. A point x in D is called a fixed point of S if $x \in Sx$. Moreover, if $Sx = \{x\}$, then x is called an endpoint of S. We denote by $F(S)$; the set of all fixed points of S, and by $E(S)$; the set of all endpoints of S. It is clear that $E(S) \subseteq F(S)$ for every multi-valued mapping S. Notice also that the following statements hold:

(i) $x \in F(S)$ if and only if $\text{dist}(x, Sx) = 0$.
(ii) $x \in E(S)$ if and only if $R(x, Sx) = 0$.

A sequence $\{x_n\}$ in D is called an approximate endpoint sequence of S [4] if

$$\lim_{n\to\infty} R(x_n, Sx_n) = 0.$$

Moreover, if $\{S_i : i \in I\}$ is a family of multi-valued mappings from D into $\mathcal{CB}(M)$, then $\{x_n\}$ is called an approximate common endpoint sequence of $\{S_i : i \in I\}$ [1] if $\lim_{n\to\infty} R(x_n, S_i x_n) = 0$ for all $i \in I$.

Definition 2.1. A mapping $S : D \to \mathcal{CB}(M)$ is said to be

(i) Suzuki-Kannan-Ćirić type (SKC-type in short) if each $x, y \in D$, the condition $\frac{1}{2}\text{dist}(x, Sx) \le \rho(x, y)$ implies $H(Sx, Sy) \le N_S(x, y)$, where

$$N_S(x,y) := \max\Big\{\rho(x,y), \frac{1}{2}\{\text{dist}(x,Sx) + \text{dist}(y,Sy)\}, \frac{1}{2}\{\text{dist}(x,Sy) + \text{dist}(y,Sx)\}\Big\};$$

(ii) quasi-nonexpansive if $F(S) \neq \emptyset$ and

$$H(Sx, Sp) \le \rho(x,p) \ \text{ for all } x \in D \text{ and } p \in F(S);$$

(iii) semi-nonexpansive if $E(S) \neq \emptyset$ and

$$H(Sx, Sq) \le \rho(x,q) \ \text{ for all } x \in D \text{ and } q \in E(S).$$

It is known from [9] that if S is SKC-type and $F(S) \neq \emptyset$, then S is quasi-nonexpansive. Also notice that if S is quasi-nonexpansive and $E(S) \neq \emptyset$, then S is semi-nonexpansive, see [37]. Moreover, by using the proof of Lemma 1.12 in [9], we can obtain the following result.

Lemma 2.1. *Let D be a nonempty subset of a metric space (M, ρ) and $S : D \to \mathcal{CB}(M)$ an SKC-type mapping. Let $x, y \in D$ and $u_x \in Sx$. Then the following conclusions hold:*

(i) $H(Sx, Su_x) \le \rho(x, u_x)$.
(ii) *Either* $\frac{1}{2}\mathit{dist}(x, Sx) \le \rho(x, y)$ *or* $\frac{1}{2}\mathit{dist}(u_x, Su_x) \le \rho(y, u_x)$.
(iii) *Either* $H(Sx, Sy) \le N_S(x, y)$ *or* $H(Sy, Su_x) \le N_S(y, u_x)$.

As a consequence of Lemma 2.1, we obtain the following corollary.

Corollary 2.1. *Let D be a nonempty subset of a metric space (M, ρ) and $S : D \to \mathcal{CB}(M)$ an SKC-type mapping. Let $x, y \in D$ and $u_x \in Sx$. Then the following conclusions hold:*

(i) $H(Sx, Su_x) \le R(x, Sx)$.
(ii) *Either* $H(Sx, Sy) \le L_S(x, y)$ *or* $H(Sy, Su_x) \le L_S(y, u_x)$, *where*

$$L_S(x,y) := \max\Big\{\rho(x,y), \frac{1}{2}\{R(x,Sx) + R(y,Sy)\}, \frac{1}{2}\{R(x,Sy) + R(y,Sx)\}\Big\}.$$

The following result can be viewed as a counterpart of Lemma 1.13 in [9].

Proposition 2.2. *Let D be a nonempty subset of a metric space (M, ρ) and $S : D \to \mathcal{CB}(M)$ an SKC-type mapping. If $x, y \in D$, then*

$$R(x, Sy) \le 7R(x, Sx) + \rho(x, y). \tag{2.3}$$

Proof. By Corollary 2.1, for any $u_x \in Sx$, we have either $H(Sx, Sy) \le L_S(x, y)$ or $H(Sy, Su_x) \le L_S(y, u_x)$.

Case 1. $H(Sx, Sy) \le L_S(x, y)$.

(1.1) If $L_S(x,y) = \rho(x,y)$, then by Proposition 2.1, we get

$$R(x,Sy) \le R(x,Sx) + H(Sx,Sy) \le R(x,Sx) + \rho(x,y).$$

(1.2) If $L_S(x,y) = \frac{1}{2}\{R(x,Sx) + R(y,Sy)\}$, then

$$\begin{aligned} R(x,Sy) &\le R(x,Sx) + H(Sx,Sy) \\ &\le R(x,Sx) + \frac{1}{2}\{R(x,Sx) + R(y,Sy)\} \\ &\le R(x,Sx) + \frac{1}{2}\{R(x,Sx) + \rho(y,x) + R(x,Sy)\}. \end{aligned}$$

This implies $R(x,Sy) \le 3R(x,Sx) + \rho(x,y)$.

(1.3) If $L_S(x,y) = \frac{1}{2}\{R(x,Sy) + R(y,Sx)\}$, then

$$\begin{aligned} R(x,Sy) &\le R(x,Sx) + H(Sx,Sy) \\ &\le R(x,Sx) + \frac{1}{2}\{R(x,Sy) + R(y,Sx)\} \\ &\le R(x,Sx) + \frac{1}{2}\{R(x,Sy) + \rho(y,x) + R(x,Sx)\}. \end{aligned}$$

This implies $R(x,Sy) \le 3R(x,Sx) + \rho(x,y)$.

Case 2. $H(Sy,Su_x) \le L_S(y,u_x)$.

(2.1) If $L_S(y,u_x) = \rho(y,u_x)$, then by Proposition 2.1 and Corollary 2.1, we have

$$\begin{aligned} R(x,Sy) &\le R(x,Sx) + H(Sx,Su_x) + H(Su_x,Sy) \\ &\le R(x,Sx) + R(x,Sx) + \rho(y,u_x) \\ &\le 2R(x,Sx) + \rho(y,x) + \rho(x,u_x) \\ &\le 3R(x,Sx) + \rho(x,y). \end{aligned}$$

(2.2) If $L_S(y,u_x) = \frac{1}{2}\{R(y,Sy) + R(u_x,Su_x)\}$, then by Proposition 2.1 and Corollary 2.1, we have

$$\begin{aligned} R(x,Sy) &\le R(x,Sx) + H(Sx,Su_x) + H(Su_x,Sy) \\ &\le R(x,Sx) + R(x,Sx) + \frac{1}{2}\{R(y,Sy) + R(u_x,Su_x)\} \\ &\le 2R(x,Sx) + \frac{1}{2}\{\rho(y,x) + R(x,Sy)\} \\ &\quad + \frac{1}{2}\{\rho(u_x,x) + R(x,Sx) + H(Sx,Su_x)\} \\ &\le 2R(x,Sx) + \frac{1}{2}\{\rho(x,y) + R(x,Sy)\} + \frac{3}{2}R(x,Sx). \end{aligned}$$

This implies $R(x,Sy) \le 7R(x,Sx) + \rho(x,y)$.

(2.3) If $L_S(y,u_x) = \frac{1}{2}\{R(y,Su_x) + R(u_x,Sy)\}$, then by Proposition 2.1 and Corollary 2.1, we have

$$\begin{aligned} R(x,Sy) &\le R(x,Sx) + H(Sx,Su_x) + H(Su_x,Sy) \\ &\le R(x,Sx) + R(x,Sx) + \frac{1}{2}\{R(y,Su_x) + R(u_x,Sy)\} \\ &\le 2R(x,Sx) + \frac{1}{2}\{\rho(y,x) + R(x,Sx) + H(Sx,Su_x)\} \\ &\quad + \frac{1}{2}\{\rho(u_x,x) + R(x,Sy)\} \\ &\le 2R(x,Sx) + \frac{1}{2}\{\rho(x,y) + 2R(x,Sx)\} + \frac{1}{2}\{R(x,Sx) + R(x,Sy)\}. \end{aligned}$$

This implies $R(x, Sy) \leq 7R(x, Sx) + \rho(x, y)$. Hence, the proof is completed. □

The previous result leads us to introduce the concept of generalized SKC-type mappings as the following definition.

Definition 2.2. Let D be a nonempty subset of a metric space (M, ρ). A multi-valued mapping $S : D \to \mathcal{CB}(M)$ is said to be generalized SKC-type if there exists $\mu \geq 0$ such that

$$R(x, Sy) \leq \mu R(x, Sx) + \rho(x, y), \text{ for all } x, y \in D. \tag{2.4}$$

Now, we establish the relationships between SKC-type, generalized SKC-type, and semi-nonexpansive mappings.

Proposition 2.3. *Let $S : D \to \mathcal{CB}(M)$ be a multi-valued mapping. Then the following statements hold:*

(i) *If S is SKC-type, then S is generalized SKC-type.*

(ii) *If S is generalized SKC-type and $E(S) \neq \emptyset$, then S is semi-nonexpansive.*

Proof. (i) follows from Proposition 2.2 with $\mu = 7$. For (ii) we let $q \in E(S)$ and $x \in D$. It follows from (2.4) and Proposition 2.1 that

$$H(Sq, Sx) = H(\{q\}, Sx) = R(q, Sx) \leq \mu R(q, Sq) + \rho(q, x) = \rho(q, x).$$

Hence, S is semi-nonexpansive. □

The following examples show that the converses of (i) and (ii) in Proposition 2.3 are not true. Notice also that Example 2.2 below is a modification of Example 2 in [19].

Example 2.1. Let $M = \mathbb{R}$, $D = [0, 2]$ and $S : D \to \mathcal{CB}(M)$ be defined by

$$Sx = \begin{cases} [0, \frac{x}{2}] & \text{if } x \neq 2; \\ \{1\} & \text{if } x = 2. \end{cases}$$

It is known from [13] that S is not SKC-type. Now, we show that S is generalized SKC-type. Let $x, y \in D$.

Case 1. If $x = y = 2$, then $R(x, Sy) = R(x, Sx) = R(x, Sx) + |x - y|$.

Case 2. If $x = 2$ and $y \in [0, 2)$, then

$$R(x, Sy) = 2 \leq 2 + (2 - y) = 2R(x, Sx) + |x - y|.$$

Case 3. If $x \in [0, 2)$ and $y = 2$, then

$$R(x, Sy) = |x - 1| \leq 2 = x + (2 - x) = R(x, Sx) + |x - y|.$$

Case 4. If $x, y \in [0, 2)$ and $x \geq y$, then $R(x, Sy) = x \leq R(x, Sx) + |x - y|$. On the other hand, if $x, y \in [0, 2)$ and $x < y$, then

$$R(x, Sy) = \max\{x, \frac{y}{2} - x\} \leq y = x + (y - x) = R(x, Sx) + |x - y|.$$

Therefore, S is a generalized SKC-type mapping with $\mu = 2$.

Example 2.2. Let $M = \mathbb{R}$, $D = [0, 1]$ and $S : D \to \mathcal{CB}(M)$ be defined by

$$Sx = \begin{cases} \left[\left|x(1 - x)\sin(\frac{1}{x})\right|, \left|\frac{x}{1+x}\sin(\frac{1}{x})\right|\right] & \text{if } x \neq 0; \\ \{0\} & \text{if } x = 0. \end{cases}$$

Then $E(S) = \{0\}$. For $x \in (0, 1]$, we have

$$H(Sx, S0) = \left|\frac{x}{1 + x}\sin(\frac{1}{x})\right| \leq \left|\frac{x}{1 + x}\right| \leq |x - 0|. \tag{2.5}$$

This implies that S is semi-nonexpansive. For each $n \in \mathbb{N}$, we set $x_n := \frac{1}{2\pi n+\pi/2}$ and $y_n := \frac{1}{2\pi n}$. Then $Sx_n = \left[x_n(1-x_n), \frac{x_n}{1+x_n}\right]$, $Sy_n = \{0\}$ and $R(x_n, Sy_n) = x_n$. Notice from (2.5) that $R(x_n, Sx_n) = x_n - x_n(1-x_n) = x_n^2$. Thus,

$$\begin{aligned} \frac{R(x_n, Sy_n) - |x_n - y_n|}{R(x_n, Sx_n)} &= \frac{x_n - (y_n - x_n)}{x_n^2} \\ &= \frac{2x_n - y_n}{x_n^2} \\ &= \Big(\frac{2}{2\pi n+\pi/2} - \frac{1}{2\pi n}\Big)(2\pi n+\pi/2)^2 \\ &= \frac{(4\pi n - 2\pi n - \pi/2)(2\pi n+\pi/2)}{2\pi n} \\ &= \frac{(2\pi n)^2 - (\pi/2)^2}{2\pi n} \to \infty. \end{aligned}$$

This implies that S is not generalized SKC-type.

The concept of uniformly convex hyperbolic spaces is introduced by Leuştean [33].

Definition 2.3. A hyperbolic space is a metric space (M, ρ) together with a function W from $M \times M \times [0,1]$ into M such that for $x, y, z, u \in M$ and $s, t \in [0,1]$, we have

(W1) $\rho(z, W(x,y,s)) \leq (1-s)\rho(z,x) + s\rho(z,y)$;
(W2) $\rho(W(x,y,s), W(x,y,t)) = |s-t|\rho(x,y)$;
(W3) $W(x,y,s) = W(y,x,1-s)$;
(W4) $\rho(W(x,z,s), W(y,u,s)) \leq (1-s)\rho(x,y) + s\rho(z,u)$.

To be convenient, from now on, we will use the notation $(1-s)x \oplus sy$ instead of $W(x,y,s)$. A nonempty subset D of M is said to be convex if $(1-s)x \oplus sy \in D$ for all $x, y \in D$ and $s \in [0,1]$. The hyperbolic space (M,ρ) is said to be uniformly convex if each $r \in (0,\infty)$ and $\varepsilon \in (0,2]$, there exists $\delta \in (0,1]$ such that

$$\rho\Big(\frac{1}{2}x \oplus \frac{1}{2}y, z\Big) \leq (1-\delta)r,$$

for all $x, y, z \in M$ with $\rho(x,z) \leq r$, $\rho(y,z) \leq r$ and $\rho(x,y) \geq r\varepsilon$.

In this case, we call δ a modulus of uniform convexity. In particular, if δ is a nonincreasing function of r for every fixed ε, then we call it a monotone modulus of uniform convexity. It is well-known that every uniformly convex Banach space is a uniformly convex hyperbolic space. Also notice that every CAT(0) space is a uniformly convex hyperbolic space, see, e.g., [33]. From now on, M stands for a complete uniformly convex hyperbolic space with a monotone modulus of uniform convexity. The following fact can be found in [25].

Lemma 2.2. *Let $p \in M$ and $\{\alpha_n\}$ be a sequence in $[a,b]$ for some $a, b \in (0,1)$. Let $\{x_n\}$ and $\{y_n\}$ be sequences in M such that $\limsup_{n\to\infty} \rho(x_n, p) \leq c$, $\limsup_{n\to\infty} \rho(y_n, p) \leq c$, and $\lim_{n\to\infty} \rho((1-\alpha_n)x_n \oplus \alpha_n y_n, p) = c$ for some $c \geq 0$. Then $\lim_{n\to\infty} \rho(x_n, y_n) = 0$.*

Let D be a nonempty subset of M and $\{x_n\}$ be a bounded sequence in M. The asymptotic radius of $\{x_n\}$ relative to D is defined by

$$r(D, \{x_n\}) := \inf\big\{\limsup_{n\to\infty} \rho(x_n, x) : x \in D\big\}.$$

The asymptotic center of $\{x_n\}$ relative to D is defined by

$$A(D, \{x_n\}) := \big\{x \in D : \limsup_{n\to\infty} \rho(x_n, x) = r(D, \{x_n\})\big\}.$$

It is known from [34] that if D is a nonempty closed convex subset of M, then $A(D,\{x_n\})$ consists of exactly one point. Now, we give the concept of $\Delta-$convergence and collect some of its basic properties.

Definition 2.4. Let D be a nonempty closed convex subset of M and $x \in D$. Let $\{x_n\}$ be a bounded sequence in M. We will say that $\{x_n\}$ $\Delta-$converges to x if $A(D,\{u_n\}) = \{x\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case we write $x_n \xrightarrow{\Delta} x$ and call x the $\Delta-$limit of $\{x_n\}$.

It is known from [28] that every bounded sequence in X has a $\Delta-$convergent subsequence. The following fact is a consequence of Lemma 2.8 in [16].

Lemma 2.3. *Let D be a nonempty closed convex subset of M and $\{x_n\}$ a bounded sequence in M. If $A(D,\{x_n\}) = \{x\}$ and $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(D,\{u_n\}) = \{u\}$ and the sequence $\{\rho(x_n,u)\}$ converges, then $x = u$.*

Definition 2.5. Let D be a nonempty closed subset of M and $S : D \to \mathcal{CB}(M)$. Let I_D be the identity mapping on D. We say that $I_D - S$ is strongly semiclosed if for any sequence $\{x_n\}$ in D, the conditions $x_n \to x$ and $R(x_n, Sx_n) \to 0$ imply $Sx = \{x\}$. Moreover, if D is closed and convex, then $I_D - S$ is said to be semiclosed if for any sequence $\{x_n\}$ in D such that $x_n \xrightarrow{\Delta} x$ and $R(x_n, Sx_n) \to 0$, one has $Sx = \{x\}$.

Obviously, if $I_D - S$ is semiclosed, then it is strongly semiclosed. Moreover, by using Lemma 2.3 along with the proof of Lemma 3.3 in [31], we can obtain the following result.

Lemma 2.4. *Let D be a nonempty closed convex subset of M and $S : D \to \mathcal{CB}(D)$ a mapping such that $I_D - S$ is semiclosed. If $\{x_n\}$ is a bounded sequence in D such that $\lim_{n\to\infty} R(x_n, Sx_n) = 0$ and $\{\rho(x_n, v)\}$ converges for all $v \in E(S)$, then $\omega_w(x_n) \subseteq E(S)$. Here $\omega_w(x_n) := \bigcup A(D,\{u_n\})$ where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. Moreover, $\omega_w(x_n)$ consists of exactly one point.*

The following fact is also needed.

Lemma 2.5. *Let D be a nonempty closed subset of M and $S : D \to \mathcal{CB}(D)$ a multi-valued mapping. If $I_D - S$ is strongly semiclosed, then $E(S)$ is closed.*

Proof. Let $\{x_n\}$ be a sequence in $E(S)$ such that $\lim_{n\to\infty} x_n = x$. Then $R(x_n, Sx_n) = 0$ for all $n \in \mathbb{N}$. It follows from the strong semiclosedness of $I_D - S$ that $Sx = \{x\}$, and hence $x \in E(S)$. This shows that $E(S)$ is closed. □

3. Endpoint theorems

This section is begun by proving the semiclosed principle for generalized SKC-type mappings in uniformly convex hyperbolic spaces. Notice that it is an extension of Lemma 3.1 in [12].

Theorem 3.1. *Let D be a nonempty closed convex subset of M and $S : D \to \mathcal{CB}(D)$ a generalized SKC-type mapping with $\mu \geq 0$ then $I_D - S$ is semiclosed.*

Proof. Let $\{x_n\}$ be a sequence in D such that $x_n \xrightarrow{\Delta} x$ and $R(x_n, Sx_n) \to 0$. Let $v \in Sx$. By (2.4) we have

$$\rho(x_n, v) \leq R(x_n, Sx) \leq \mu R(x_n, Sx_n) + \rho(x_n, x).$$

This implies that $\limsup_{n\to\infty} \rho(x_n, v) \leq \limsup_{n\to\infty} \rho(x_n, x)$ and so $v \in A(D,\{x_n\}) = \{x\}$. Thus, $v = x$ for all $v \in Sx$. This shows that $Sx = \{x\}$ and hence the proof is complete. □

Now, we prove a common endpoint theorem.

Theorem 3.2. *Let D be a nonempty closed convex subset of M and $\{S_i : i \in I\}$ a family of generalized SKC-type mappings from D into $\mathcal{CB}(D)$. If $\{S_i : i \in I\}$ has a bounded approximate common endpoint sequence in D, then it has a common endpoint in D.*

Proof. Let $\{x_n\}$ be a bounded approximate common endpoint sequence of $\{S_i : i \in I\}$. As we have observed, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \xrightarrow{\Delta} x \in D$. It follows from Theorem 3.1 that $S_i x = \{x\}$ for all $i \in I$. Thus, x is a common endpoint of $\{S_i : i \in I\}$. □

As a consequence of Theorem 3.2, we obtain the following result.

Corollary 3.2. *Let D be a nonempty closed convex subset of M and $S : D \to \mathcal{CB}(D)$ a generalized SKC-type mapping. If S has a bounded approximate endpoint sequence in D, then S has an endpoint in D.*

The following result can be viewed as an extension of Theorem 3.2 in [32].

Theorem 3.3. *Let D be a nonempty closed convex subset of M and $\{S_i : i \in I\}$ a family of generalized SKC-type mappings from D into $\mathcal{CB}(D)$. Suppose there exist two disjoint subsets A and B of I such that $A \cup B = I$. Also, suppose each $i \in A$, S_i has a bounded approximate endpoint sequence in $\cap_{j\in B} E(S_j)$ then $\{S_i : i \in I\}$ has a common endpoint in D.*

Proof. Fix $i \in A$ and let $\{x_n\}$ be a bounded approximate endpoint sequence of S_i in $\cap_{j\in B} E(S_j)$. Without loss of generality, we may assume that $x_n \xrightarrow{\Delta} x \in D$. According to Theorem 3.1, $x \in E(S_i)$; fixing $j \in B$ and letting $w \in S_j x$, since S_j is generalized SKC-type so there exists $\mu_j \geq 0$ such that

$$\rho(x_n, w) \leq R(x_n, S_j x) \leq \mu_j R(x_n, S_j x_n) + \rho(x_n, x) \text{ for all } n \in \mathbb{N}.$$

This implies $\limsup_{n\to\infty} \rho(x_n, w) \leq \limsup_{n\to\infty} \rho(x_n, x)$ and hence $w \in A(D, \{x_n\}) = \{x\}$. Thus, $w = x$ for all $w \in S_j x$ and therefore $S_j x = \{x\}$. This shows that x is a common endpoint of $\{S_i : i \in I\}$. □

Corollary 3.3. *Let D be a nonempty closed convex subset of M and $S, T : D \to \mathcal{CB}(D)$ be generalized SKC-type mappings. Suppose that S has a bounded approximate endpoint sequence in $E(T)$. Then S and T has a common endpoint in D.*

4. CONVERGENCE THEOREMS

In this section, we prove strong and $\Delta-$convergence theorems of the Kuhfitting iteration [30] for finding a common endpoint of generalized SKC-type mappings. Let D be a nonempty convex subset of M and $S_i : D \to \mathcal{K}(D)$ $(i = 1, 2, ..., m)$ be a finite family of multi-valued mappings. For each $i \in \{1, 2, ..., m\}$, let $\{\alpha_{n,i}\}$ be a sequence in $[0, 1]$. The sequence of Kuhfitting iteration is defined by given $x_1 \in D$ and for $n \in \mathbb{N}$, we let

$$(4.6) \quad \begin{cases} y_{n,1} = (1 - \alpha_{n,1})x_n \oplus \alpha_{n,1} z_{n,1} \\ y_{n,2} = (1 - \alpha_{n,2})x_n \oplus \alpha_{n,2} z_{n,2} \\ \vdots \\ y_{n,m-1} = (1 - \alpha_{n,m-1})x_n \oplus \alpha_{n,m-1} z_{n,m-1} \\ x_{n+1} = (1 - \alpha_{n,m})x_n \oplus \alpha_{n,m} z_{n,m}, \end{cases}$$

where $z_{n,1} \in S_1 x_n$ such that $\rho(x_n, z_{n,1}) = R(x_n, S_1 x_n)$ and $z_{n,i} \in S_i y_{n,i-1}$ such that $\rho(x_n, z_{n,i}) = R(x_n, S_i y_{n,i-1})$ for $i \in \{2, 3, ..., m\}$.

A sequence $\{x_n\}$ in M is said to be Fejér monotone with respect to D [7] if $\rho(x_{n+1}, p) \leq \rho(x_n, p)$ for all $p \in D$ and $n \in \mathbb{N}$. The following result shows that the sequence of Kuhfitting iteration is Fejér monotone with respect to the common endpoint set of generalized SKC-type mappings.

Lemma 4.6. *Let D be a nonempty convex subset of M and $S_i : D \to \mathcal{K}(D)$ $(i = 1, 2, ..., m)$ be a finite family of generalized SKC-type mappings such that $E := \cap_{i=1}^m E(S_i) \neq \emptyset$. Let $\{x_n\}$ be the sequence of Kuhfitting iteration defined by (4.6). Then $\{x_n\}$ is Fejér monotone with respect to E.*

Proof. Let $p \in E$. By Proposition 2.3, S_i is semi-nonexpansive for all $i \in \{1, 2, ..., m\}$. For each $n \in \mathbb{N}$ and $i \in \{1, 2, ..., m-1\}$, we have

$$\begin{aligned} \rho(y_{n,i}, p) &\leq (1-\alpha_{n,i})\rho(x_n, p) + \alpha_{n,i}\rho(z_{n,i}, p) \\ &\leq (1-\alpha_{n,i})\rho(x_n, p) + \alpha_{n,i}H(S_i x_n, S_i p) \\ &\leq \rho(x_n, p). \end{aligned} \tag{4.7}$$

This implies that

$$\begin{aligned} \rho(x_{n+1}, p) &\leq (1-\alpha_{n,m})\rho(x_n, p) + \alpha_{n,m}\rho(z_{n,m}, p) \\ &\leq (1-\alpha_{n,m})\rho(x_n, p) + \alpha_{n,m}H(S_m y_{n,m-1}, S_m p) \\ &\leq (1-\alpha_{n,m})\rho(x_n, p) + \alpha_{n,m}\rho(y_{n,m-1}, p) \\ &\leq \rho(x_n, p). \end{aligned} \tag{4.8}$$

Thus, $\{x_n\}$ is Fejér monotone with respect to E. □

Now, we prove $\Delta-$convergence theorem.

Theorem 4.4. *Let D be a nonempty closed convex subset of M and $S_i : D \to \mathcal{K}(D)$ $(i = 1, 2, ..., m)$ be a finite family of generalized SKC-type mappings such that $E := \cap_{i=1}^m E(S_i) \neq \emptyset$. Let $\{\alpha_{n,i}\} \subset [a, b] \subset (0, 1)$ $(i = 1, 2, ..., m)$, and $\{x_n\}$ be the sequence of Kuhfitting iteration defined by (4.6). Then $\{x_n\}$ $\Delta-$converges to a common endpoint of $\{S_1, S_2, ..., S_m\}$.*

Proof. For each $i \in \{1, 2, ..., m\}$, there exists $\mu_i \geq 0$ such that

$$R(x, S_i y) \leq \mu_i R(x, S_i x) + \rho(x, y) \text{ for all } x, y \in D. \tag{4.9}$$

We will show that

$$\lim_{n\to\infty} R(x_n, S_i x_n) = 0 \text{ for all } i \in \{1, 2, ..., m\}. \tag{4.10}$$

Fix $p \in E$. By Lemma 4.6, $\lim\limits_{n\to\infty} \rho(x_n, p) = c$ for some $c \geq 0$. If $c = 0$, then for each $i \in \{1, 2, ..., m\}$ we have

$$\begin{aligned} R(x_n, S_i x_n) &\leq \rho(x_n, p) + R(p, S_i x_n) \\ &= \rho(x_n, p) + H(S_i p, S_i x_n) \\ &\leq 2\rho(x_n, p) \to 0 \text{ as } n \to \infty. \end{aligned}$$

If $c > 0$, then by (4.7) we have

$$\limsup_{n\to\infty} \rho(y_{n,i}, p) \leq c \text{ for all } i \in \{1, 2, ..., m-1\}. \tag{4.11}$$

We note that $\rho(z_{n,1}, p) = \text{dist}(z_{n,1}, S_1 p) \leq H(S_1 x_n, S_1 p) \leq \rho(x_n, p)$ and for each $i \in \{2, 3, ..., m\}$, we have

$$\rho(z_{n,i}, p) = \text{dist}(z_{n,i}, S_i p) \leq H(S_i y_{n,i-1}, S_i p) \leq \rho(y_{n,i-1}, p).$$

It follows that $\limsup_{n\to\infty} \rho(z_{n,i}, p) \le c$ for all $i \in \{1, 2, ..., m\}$. Since $\lim_{n\to\infty} \rho(x_{n+1}, p) = \lim_{n\to\infty} \rho((1-\alpha_{n,m})x_n \oplus \alpha_{n,m} z_{n,m}, p) = c$, by Lemma 2.2 we have

$$\lim_{n\to\infty} \rho(x_n, z_{n,m}) = 0. \tag{4.12}$$

On the other hand, it follows from (4.8) that

$$\begin{aligned} \rho(x_n, p) &\le \frac{\rho(x_n, p) - \rho(x_{n+1}, p)}{\alpha_{n,m}} + \rho(y_{n,m-1}, p) \\ &\le \frac{\rho(x_n, p) - \rho(x_{n+1}, p)}{a} + \rho(y_{n,m-1}, p), \end{aligned}$$

which implies $c \le \liminf_{n\to\infty} \rho(y_{n,m-1}, p)$. This, together with (4.11), implies that $\lim_{n\to\infty} \rho(y_{n,m-1}, p) = c$. Also, by Lemma 2.2 we have $\lim_{n\to\infty} \rho(x_n, z_{n,m-1}) = 0$. Since $\rho(y_{n,m-1}, p) \le (1-\alpha_{n,m-1})\rho(x_n, p) + \alpha_{n,m-1}\rho(y_{n,m-2}, p)$, we have

$$\begin{aligned} \rho(x_n, p) &\le \frac{\rho(x_n, p) - \rho(y_{n,m-1}, p)}{\alpha_{n,m-1}} + \rho(y_{n,m-2}, p) \\ &\le \frac{\rho(x_n, p) - \rho(y_{n,m-1}, p)}{a} + \rho(y_{n,m-2}, p), \end{aligned}$$

which implies $c \le \liminf_{n\to\infty} \rho(y_{n,m-2}, p)$. This, together with (4.11), implies that $\lim_{n\to\infty} \rho(y_{n,m-2}, p) = c$. By Lemma 2.2, we have $\lim_{n\to\infty} \rho(x_n, z_{n,m-2}) = 0$. Similarly, we can show that for each $i \in \{1, 2, ..., m-3\}$,

$$\lim_{n\to\infty} \rho(y_{n,i}, p) = c \;\text{ and }\; \lim_{n\to\infty} \rho(x_n, z_{n,i}) = 0. \tag{4.13}$$

Thus, for each $i \in \{1, 2, ..., m-1\}$, we have

$$\rho(y_{n,i}, x_n) \le \alpha_{n,i}\rho(z_{n,i}, x_n) \to 0 \text{ as } n \to \infty \tag{4.14}$$

and

$$R(x_n, S_i y_{n,i-1}) = \rho(x_n, z_{n,i}) \to 0 \text{ as } n \to \infty. \tag{4.15}$$

From (4.12), we have $\lim_{n\to\infty} R(x_n, S_1 x_n) = 0$. For $i \in \{2, 3, ..., m\}$, by (4.9), (4.14) and (4.15), we have

$$\begin{aligned} R(x_n, S_i x_n) &\le \rho(x_n, y_{n,i-1}) + R(y_{n,i-1}, S_i x_n) \\ &\le \rho(x_n, y_{n,i-1}) + \mu_i R(y_{n,i-1}, S_i y_{n,i-1}) + \rho(y_{n,i-1}, x_n) \\ &\le 2\rho(x_n, y_{n,i-1}) + \mu_i\{\rho(y_{n,i-1}, x_n) + R(x_n, S_i y_{n,i-1})\} \\ &\quad \to 0 \text{ as } n \to \infty. \end{aligned}$$

Hence, (4.10) helds. By Lemma 4.6, $\{\rho(x_n, v)\}$ converges for all $v \in E$. By Lemma 2.4, $\omega_w(x_n)$ consists of exactly one point and is contained in E. This shows that $\{x_n\}$ $\Delta-$converges to an element of E. □

Next, we prove strong convergence theorems. A family of mappings $\{S_1, S_2, ..., S_m\}$ from D into $\mathcal{K}(D)$ is said to satisfy condition (J) [36] if $E := \cap_{i=1}^m E(S_i) \ne \emptyset$ and there exists a nondecreasing function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$, $g(r) > 0$ for $r \in (0, \infty)$ and

$$\max_{1\le i\le m} \{R(x, S_i x)\} \ge g(\text{dist}(x, E)) \;\text{ for all }\; x \in D. \tag{4.16}$$

The following fact can be found in [12].

Lemma 4.7. *Let D be a nonempty closed subset of M and $\{x_n\}$ a Fejér monotone sequence with respect to D. Then $\{x_n\}$ converges strongly to an element of D if and only if $\lim_{n\to\infty} dist(x_n, D) = 0$.*

Theorem 4.5. *Let D be a nonempty closed convex subset of M and $S_i : D \to \mathcal{K}(D)$ $(i = 1, 2, ..., m)$ be a finite family of generalized SKC-type mappings which satisfies condition (J). Let $\{\alpha_{n,i}\} \subset [a, b] \subset (0, 1)$ $(i = 1, 2, ..., m)$, and $\{x_n\}$ be the sequence of Kuhfitting iteration defined by (4.6). Then $\{x_n\}$ converges strongly to a common endpoint of $\{S_1, S_2, ..., S_m\}$.*

Proof. Let $E = \cap_{i=1}^m E(S_i)$. It follows from Lemmas 3.1 and 2.5 that E is closed. By (4.10) and (4.16) we get $\lim_{n\to\infty} g(\text{dist}(x_n, E)) = 0$ and hence $\lim_{n\to\infty} \text{dist}(x_n, E) = 0$. By Lemma 4.6, $\{x_n\}$ is Fejér monotone with respect to E. The conclusion follows from Lemma 4.7. □

A mapping $S : D \to \mathcal{K}(D)$ is said to be semicompact [36] if any sequence $\{x_n\}$ in D with $\lim_{n\to\infty} R(x_n, Sx_n) = 0$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim_{k\to\infty} x_{n_k} = q \in D$.

Theorem 4.6. *Let D be a nonempty convex subset of M and $S_i : D \to \mathcal{K}(D)$ $(i = 1, 2, ..., m)$ be a finite family of generalized SKC-type mappings such that $\cap_{i=1}^m E(S_i) \neq \emptyset$ and S_j is semicompact for some $j \in \{1, 2, ..., m\}$. Let $\{\alpha_{n,i}\} \subset [a, b] \subset (0, 1)$ $(i = 1, 2, ..., m)$, and $\{x_n\}$ be the sequence of Kuhfitting iteration defined by (4.6). Then $\{x_n\}$ converges strongly to a common endpoint of $\{S_1, S_2, ..., S_m\}$.*

Proof. Since S_j is semicompact, by (4.10) there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to q \in D$. For each $i \in \{1, 2, ..., m\}$, there exists $\mu_i \geq 0$ such that

$$R(x_{n_k}, S_i q) \leq \mu_i R(x_{n_k}, S_i x_{n_k}) + \rho(x_{n_k}, q) \text{ for all } k \in \mathbb{N}.$$

This implies that

$$\begin{aligned} R(q, S_i q) &\leq \rho(q, x_{n_k}) + R(x_{n_k}, S_i q) \\ &\leq 2\rho(x_{n_k}, q) + \mu_i R(x_{n_k}, S_i x_{n_k}) \ \to 0 \ \text{ as } \ k \to \infty. \end{aligned}$$

Thus $q \in E(S_i)$ for all $i \in \{1, 2, ..., m\}$. According to Lemma 4.6, $\lim_{n\to\infty} \rho(x_n, q)$ exists and hence q is the strong limit of $\{x_n\}$. □

Acknowledgements. This research was supported by Thailand Science Research and Innovation under the project IRN62W0007 and Chiang Mai University.

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