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On instability in the theory of dipolar bodies with two-temperatures

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ABSTRACT. In this paper we approach a generalized thermoelasticity theory based on a heat conduction equation in bodies with dipolar structure, the heat conduction depends on two distinct temperatures, the thermodynamic temperature and the conductive temperature. In our considerations the difference between two temperatures is highlighted by the heat supply. For the mixed initial boundary value problem defined in this context, we prove the uniqueness of a solution corresponding some specific initial and boundary conditions. Also, if the initial energy is negative or null, we prove that the solutions of the mixed problem are exponentially unstable.

1. INTRODUCTION

It is known that the studies dedicated to classical thermoelasticity used a heat conduction equation which are based on the classical Fourier law. As a consequence, the heat flux vector is depending on the gradient of temperatures and, as a consequence, the thermal signals will propagate with an infinite speed. But this contradicts the causality principle. To avoid this contradiction, a series of new theories of thermoelasticity have emerged that propose different alternatives to the classical heat conduction equation. This is how various models appeared, of which the best known in the literature are Green and Lindsay [7], Lord and Shulman [17], Green and Naghdi [8]-[12], More-Gibson-Thompson [24]. In all these models, the thermal waves propagate with finite speeds and all results from thes generalized theorie are more general and physically more realistic than in the classical theory. In our study we have a temperature rate dependent on two temperatures, by changing the relation between the two temperatures, namely, the thermodynamic temperature and the conductive temperature.

There are many studies that take into account the two temperatures, of which we mention [3], [4], [26] and [17]. Other generalizations of the heat conduction equation can be found in many articles, of which we list [1], [19], [22]. Our uniqueness result is obtained by assuming the initial energy is not strictly positive. Other uniqueness results are based on the assumption that the elastic tensor is a positively defined one. But there are concrete thermoelastic situations in which the positive definition of the elastic tensor cannot be guaranteed. And our result on exponential instability is also obtained on the assumption that the initial energy is not strictly positive. We must emphasize that our mixed problem is considered both in the theory in which it is considered dependent on the rate of both temperature, and the theory depends on the rate of thermodynamic temperature, but not on the rate of the conductive temperature. However, the calculations are quite similar in both situations, which is why the demonstrations are made in detail only in the case of dependence on the rate of thermodynamic temperature.

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We must also say what is the motivation that we took into account the effect due to the dipolar structure. In opinion of many researchers, it is known that this effect makes an important contribution to the general deformations of the media. It is enough to refer to media that have a granular structure, for instance, polymers, human bones or graphite. Also, other concrete usefulness of this effect are for the various materials with pores or composite materials which are reinforced with chopped fibers. From the large number of studies dedicated to media with dipole structure we have selected a few: [2], [5], [6], [13], [18]-[21], [23], [27].

2. The mixed initial-boundary value problem

Consider that the thermoelastic body with dipolar structure occupies the three-dimensional domain Ω from the Euclidian space R^3. The closure of Ω is denoted by $\bar{\Omega}$ and we have $\bar{\Omega} = \Omega \cup \partial\Omega$, where $\partial\Omega$ is the border of the domain Ω and is considered regular enough to allow the application of the divergence theorem. The outward unit normal to $\partial\Omega$ has the components marked with n_i. The vector and tensors fields are denoted by letters in boldface. The notation v_i is used for the components of a vector field $\mathbf{v}$, the notation u_{ij} is used for the components of a tensor field $\mathbf{u}$ of second order, and so on. For the material time derivative we will use a superposed dot. By convention, the subscripts are understood to range over integers $(1, 2, 3)$. The summation rule regarding repeated subscripts is also implied. For the partial differentiation of a function f regarding the spatial variables x_j we will use the notation $f_{,j}$, to simplify the writing. When there are no possibilities of confusion, the time variable and/or the spatial variables of a function may not be highlighted. A fixed system of Cartesian axes $Ox_i,\ i = 1, 2, 3$ will be used to refer the motion of the thermoelastic body.

In order to characterize the behavior of our media we use the set of variables $(v_i, \phi_{ij}, \varphi, \vartheta)$, where we denoted by v_i the components of the vector of displacement , by ϕ_{ij} the components of the microdeformation tensor, by φ the conductive temperature and by ϑ the thermodynamic temperature measured from the constant absolute temperature ϑ_0 of the body in its reference state.

By using the internal variables (v_i, ϕ_{ij}) we can introduce the kinematic characteristics of the body, that is, the strain tensors, through the following geometrical equations:

$$e_{ij} = \frac{1}{2}\left(v_{j,i} + v_{i,j}\right),\ \epsilon_{ij} = v_{j,i} - \phi_{ji},\ \ \gamma_{ijk} = \phi_{ij,k}. \tag{2.1}$$

As usual, the notation t_{ij} is used for the components of the tensor of stress, τ_{ij} for the components of the tensor of microstress and σ_{ijk} for the components of the tensor of stress moment, all over Ω.

For a homogeneous thermoelastic body, which have in its reference state a center of symmetry at each point, but is otherwise non-isotropic, we can define the stress tensors by means of the following constitutive equations:

$$\begin{aligned}
t_{ij} &= A_{ijkl}e_{kl} + E_{ijkl}\varepsilon_{kl} + F_{ijklm}\gamma_{klm} - \alpha_{ij}\left(\vartheta + a\dot{\vartheta}\right),\\
\tau_{ij} &= E_{klij}e_{kl} + B_{ijkl}\varepsilon_{kl} + G_{ijklm}\gamma_{klm} - \beta_{ij}\left(\vartheta + a\dot{\vartheta}\right),\\
\sigma_{ijk} &= F_{ijklm}e_{lm} + G_{ijklm}\varepsilon_{lm} + C_{ijklmn}\gamma_{lmn} - \delta_{ijk}\left(\vartheta + a\dot{\vartheta}\right),\\
\rho S &= \alpha_{ij}e_{ij} + \beta_{ij}\varepsilon_{ij} + \delta_{ijk}\gamma_{ijk} + d\vartheta + h\dot{\vartheta},\\
q_i &= \kappa_{ij}\varphi_{,j}.
\end{aligned} \tag{2.2}$$

In the absence of body force, of body moment and of heat supply fields, The field of basic equations for the two-temperature thermoelasticity of dipolar bodies are:

- the motion equations:

$$\rho \ddot{v}_i = \left(t_{ji} + \tau_{ji}\right)_{,j}, \tag{2.3}$$

$$I_{jk}\ddot{\phi}_{ik} = \sigma_{kij,k} + \tau_{ji}; \tag{2.4}$$

- the energy equation:

$$\rho \dot{S} = q_{i,i}; \tag{2.5}$$

- two type of the two-temperatures equations:

$$\varphi - c\left(\kappa_{ij}\varphi_{,i}\right)_{,j} = \vartheta + a\dot{\vartheta}; \tag{2.6}$$

$$a\dot{\varphi} + \varphi - c\left(\kappa_{ij}\varphi_{,i}\right)_{,j} = \vartheta + a\dot{\vartheta}. \tag{2.7}$$

It is easy to see that the equation (2.6) will be considered in the theory dependent on rate of conductive temperature and on rate of thermodynamic temperature, where (2.1) and (2.2) are the equations of motion and (2.3) is the equation of energy.
In the above equations we have used the following notations: ρ is the reference constant mass density, S is the specific entropy per unit mass, q_i are the components of heat flux vector.
The above coefficients A_{ijkl}, B_{ijrs}, ..., C_{ijklmn}, α_{ij}, ..., κ_{ij}, c, d, h and a are used to describe the material structure and they satisfy the following symmetry relations:

$$\begin{gathered} A_{ijkl} = A_{klij} = A_{jikl},\ B_{ijkl} = B_{klij},\ C_{ijklmn} = C_{lmnijk}, \\ E_{ijkl} = E_{jikl},\ F_{ijklm} = F_{jiklm},\ \alpha_{ij} = \alpha_{ji},\ \kappa_{ij} = \kappa_{ji}. \end{gathered} \tag{2.8}$$

We wish to outline that the coefficients a, c, d and h are specific constants of the heat.
The temperature ϑ_0 and density ρ are given strict positive constants. From the entropy production inequality (see Gren and Lindsay [7]) we obtain the following conditions

$$c > 0,\ h > 0,\ da - h \geq 0, \tag{2.9}$$

and, according to the same entropy inequality, we assume that A_{ijkl}, B_{ijkl}, C_{ijklmn} and k_{ij} are positive definite tensors, i.e.

$$\begin{gathered} A_{ijkl}x_{ij}x_{kl} \geq k_1 x_{ij}x_{ij},\ k_1 > 0,\ \forall\ x_{ij} = x_{ji}, \\ B_{ijkl}y_{ij}y_{kl} \geq k_2 y_{ij}y_{ij},\ k_2 > 0,\ \forall\ y_{ij} = y_{ji}, \\ C_{ijklmn}z_{ijk}z_{lmn} \geq k_3 z_{ijk}z_{ijk},\ k_3 > 0,\ \forall\ z_{ijk}, \\ \kappa_{ij}x_i x_j \geq k_4 x_i x_i,\ k_4 > 0,\ \forall\ x_i. \end{gathered} \tag{2.10}$$

Along with the above basic equations (2.3)-(2.7), we consider the following homogeneous boundary conditions of Dirichlet type:

$$v_i = 0,\ \phi_{ij} = 0, \vartheta = 0,\ \varphi = 0 \text{ on } \partial\Omega \times [0, \infty). \tag{2.11}$$

To this system of equations we adjoin the following initial conditions:

$$\begin{gathered} u_i(x,0) = u_i^0(x),\ \dot{u}_i(x,0) = u_i^1(x), \\ \phi_{ij}(x,0) = \phi_{ij}^0(x),\ \dot{\phi}_{ij}(x,0) = \phi_{ij}^1(x), \\ \varphi(x,0) = \varphi^0(x),\ \vartheta(x,0) = \vartheta^0(x),\ \dot{\vartheta}(x,0) = \vartheta^1(x), \end{gathered} \tag{2.12}$$

which are satisfied for any $x \in \Omega$.
Considering the geometric equations and the constitutive equations (2.2), which are introduced in the basic equations (2.3)-(2.5), we are led the following system of partial differential equations:

$$(A_{ijmn}+E_{ijmn})\,v_{n,mj}+(E_{mnij}+B_{ijmn})\,(v_{n,mj}-\phi_{mn,j})+$$
$$+(F_{ijklm}+D_{ijklm})\,\phi_{lm,kj}-(\alpha_{ij}+\beta_{ij})\left(\vartheta_{,j}+a\dot{\vartheta}_{,j}\right)=\rho\ddot{v}_i,$$
$$F_{jklmn}v_{n,mj}+D_{mnjkl}\,(v_{n,mj}-\phi_{mn,j})+C_{kljmnr}\phi_{nr,mj}-\delta_{klj}\left(\vartheta_{,j}+a\dot{\vartheta}_{,j}\right)+ \tag{2.13}$$
$$+E_{klmn}v_{m,n}+B_{klmn}(v_{n,m}-\phi_{mn})+D_{klmnr}\phi_{nr,m}-\beta_{kl}\left(\vartheta+a\dot{\vartheta}\right)=I_{kr}\ddot{\phi}_{lr},$$

which are satisfied for any $(x)\in\Omega\times(0,\infty)$.
By a solution of the mixed initial boundary value problem in the two temperatures thermoelastic theory of dipolar bodies in the cylinder $\Omega\times[0,\infty)$ we mean an ordered array $(v_i,\ \phi_{ij},\ \varphi,\ \vartheta)$ which satisfies the system of equations (2.13), the boundary conditions (2.11) and the initial conditions (2.12).

3. MAIN RESULTS

At the beginning of this section we must specify an energy conservation law in the case we consider the rate of the conductive temperature, that is, we consider the two-temperatures relation (2.7):

$$W_1(t)=W_1(0),\ t\in[0,\infty), \tag{3.14}$$

where

$$W_1(t)=\frac{1}{2}\int_\Omega\Big[\rho\dot{v}_i(t)\dot{v}_i(t)+I_{jk}\dot{\phi}_{ij}(t)\dot{\phi}_{ik}(t)+A_{ijkl}e_{ij}(t)e_{kl}(t)+$$
$$+2E_{ijkl}e_{ij}(t)\varepsilon_{kl}(t)+2F_{ijklm}e_{ij}(t)\gamma_{klm}(t)+B_{ijkl}\varepsilon_{ij}(t)\varepsilon_{kl}(t)+$$
$$+2G_{ijklm}\varepsilon_{ij}(t)\gamma_{klm}(t)+C_{ijklmn}\gamma_{ijk}(t)\gamma_{lmn}(t)+$$
$$+c\kappa_{ij}\varphi_{,i}(t)\,\varphi_{,j}(t)+d\left(\vartheta(t)+\frac{h}{d}\dot{\vartheta}(t)\right)^2+h\left(a-\frac{h}{d}\right)\dot{\vartheta}^2(t)\Big]\,dV+$$
$$+\int_0^t\int_\Omega\left[\kappa_{ij}\varphi_{,i}(s)\varphi_{,j}(s)+c\left(\left(\kappa_{ij}\varphi_{,i}(s)\right)_{,j}\right)^2+(ad-h)\dot{\vartheta}^2(s)\right]dVds.$$

In the case we don't take into account the rate of conductivity temperature, that is, we consider the relation (2.6), the energy conservation law receives the form

$$W_2(t)=W_2(0),\ t\in[0,\infty), \tag{3.15}$$

where

$$W_2(t)=\frac{1}{2}\int_\Omega\Big[\rho\dot{v}_i(t)\dot{v}_i(t)+I_{jk}\dot{\phi}_{ij}(t)\dot{\phi}_{ik}(t)+A_{ijkl}e_{ij}(t)e_{kl}(t)+$$
$$+2E_{ijkl}e_{ij}(t)\varepsilon_{kl}(t)+2F_{ijklm}e_{ij}(t)\gamma_{klm}(t)+B_{ijkl}\varepsilon_{ij}(t)\varepsilon_{kl}(t)+$$
$$+2G_{ijklm}\varepsilon_{ij}(t)\gamma_{klm}(t)+C_{ijklmn}\gamma_{ijk}(t)\gamma_{lmn}(t)+$$
$$+d\left(\vartheta(t)+\frac{h}{d}\dot{\vartheta}(t)\right)^2+h\left(a-\frac{h}{d}\right)\dot{\vartheta}^2(t)\Big]dV+$$
$$+\int_0^t\int_\Omega\left[\kappa_{ij}\varphi_{,i}(s)\varphi_{,j}(s)+c\left(\left(\kappa_{ij}\varphi_{,i}(s)\right)_{,j}\right)^2+(ad-h)\dot{\vartheta}^2(s)\right]dVds.$$

Let us denote by $\mathcal{P}$ the mixed problem consists in system of equations (2.13), the boundary conditions (2.11) and the initial conditions (2.12).

Our two main results, namely, an uniqueness and an instability results for the solution of our mixed problem, will be obtained in the simpler case in which we consider equation (2.6). For uniqueness we use the usual procedure: we will show that the problem $\mathcal{P}$ admits only the null solution, if it is considered that the initial data are null.

Theorem 3.1. *The mixed initial-boundary value problem $\mathcal{P}$, in the case of null initial data, admits only the null solution.*

Proof. We denote by $(v_i,\ \phi_{ij},\ \varphi,\ \vartheta)$ the difference of two solutions of the problem $\mathcal{P}$. Our proof is based on a logarithmic convexity argument, as such we will use a function $E(t)$ with convex logarithm of the form

$$\begin{aligned} E(t) &= \frac{1}{2}\int_\Omega \left[\rho v_i(x) v_i(x) + I_{jk}\phi_{ij}(x)\phi_{ik}(x)\right] dV + \\ &+\frac{1}{2}\int_0^t\!\!\int_\Omega\Big[\kappa_{ij}\xi_{,i}(x,\tau)\xi_{,j}(x,\tau)+c\Big(\left(\kappa_{ij}\xi_{,i}(x,\tau)\right)_{,j}\Big)+(ad-h)\vartheta^2(x,\tau)\Big] dV\, d\tau, \end{aligned} \tag{3.16}$$

where the ξ is an antiderivative function of the function φ, that is

$$\xi(x,s) = \int_0^s \varphi(x,\tau)d\tau.$$

Also, for (3.16) we consider the equation (2.6).
By direct calculations, we can compute the first two derivative of the function E, starting from (3.16)

$$\begin{aligned} \dot{E}(t) &= \int_\Omega \left[\rho v_i(x)\dot{v}_i(x) + I_{jk}\phi_{ij}(x)\dot{\phi}_{ik}(x)\right] dV + \\ &\quad +\frac{1}{2}\int_\Omega\Big[\kappa_{ij}\xi_{,i}(x)\xi_{,j}(x)+c\Big(\left(\kappa_{ij}\xi_{,i}(x)\right)_{,j}\Big)+(ad-h)\vartheta^2(x)\Big] dV, \\ \ddot{E}(t) &= \int_\Omega \left[\rho\left(\dot{v}_i(x)\dot{v}_i(x) + v_i(x)\ddot{v}_i(x)\right) + \right. \\ &\qquad \left. +I_{jk}\left(\dot{\phi}_{ij}(x)\dot{\phi}_{ik}(x) + \phi_{ij}(x)\ddot{\phi}_{ik}(x)\right)\right] dV + \\ &\qquad + \int_\Omega\Big[\kappa_{ij}\xi_{,i}(x)\varphi_{,j}(x)+(ad-h)\vartheta(x)\dot{\vartheta}(x)+ \\ &\qquad\quad +c\left(\left(\kappa_{ij}\xi_{,i}(x)\right)_{,j}\right)\left(\left(\kappa_{mn}\varphi_{,m}(x)\right)_{,n}\right)\Big] dV. \end{aligned} \tag{3.17}$$

Taking into account the constitutive equations (2.2), we can obtain the following identity:

$$\begin{aligned} &\int_\Omega \Big[\rho v_i(x)\ddot{v}_i(x) + I_{jk}\phi_{ij}(x)\ddot{\phi}_{ik}(x) + A_{ijkl}e_{ij}(x)e_{kl}(x) + \\ &\quad +2E_{ijkl}e_{ij}(x)\varepsilon_{kl}(x) + 2F_{ijklm}e_{ij}(x)\gamma_{klm}(x) + B_{ijkl}\varepsilon_{ij}(x)\varepsilon_{kl}(x) + \\ &\quad +2G_{ijklm}\varepsilon_{ij}(x)\gamma_{klm}(x) + C_{ijklmn}\gamma_{ijk}(x)\gamma_{lmn}(x)\Big] dV = \\ &= \int_\Omega \left(\alpha_{ij}e_{ij}(x) + \beta_{ij}\varepsilon_{ij}(x) + \delta_{ijk}\gamma_{ijk}(x)\right)\left(\vartheta(x) + a\dot{\vartheta}(x)\right) dV. \end{aligned} \tag{3.18}$$

On the other hand, starting from (2.6) we deduce:

$$\begin{aligned} &\int_\Omega \Big[\kappa_{ij}\xi_{,i}(x)\varphi_{,j}(x)+ \left(\vartheta(x) + a\dot{\vartheta}(x)\right)\left(d\vartheta(x) + h\dot{\vartheta}(x)\right) + \\ &\quad +c\left(\left(\kappa_{ij}\xi_{,i}(x)\right)_{,j}\right)\left(\left(\kappa_{mn}\varphi_{,m}(x)\right)_{,n}\right)\Big] dV = \\ &= -\int_\Omega \left(\alpha_{ij}e_{ij}(x) + \beta_{ij}\varepsilon_{ij}(x) + \delta_{ijk}\gamma_{ijk}(x)\right)\left(\vartheta(x) + a\dot{\vartheta}(x)\right) dV. \end{aligned} \tag{3.19}$$

Clearly, considering (3.18) and (3.19) we deduce the equation:

$$\begin{aligned}
&\int_\Omega \Big[\rho v_i(x)\ddot{v}_i(x) + I_{jk}\phi_{ij}(x)\ddot{\phi}_{ik}(x) + A_{ijkl}e_{ij}(x)e_{kl}(x) + \\
&\quad +2E_{ijkl}e_{ij}(x)\varepsilon_{kl}(x) + 2F_{ijklm}e_{ij}(x)\gamma_{klm}(x) + B_{ijkl}\varepsilon_{ij}(x)\varepsilon_{kl}(x) + \\
&\quad +2G_{ijklm}\varepsilon_{ij}(x)\gamma_{klm}(x) + C_{ijklmn}\gamma_{ijk}(x)\gamma_{lmn}(x)]\, dV + \\
&\quad +\int_\Omega \Big[\kappa_{ij}\xi_{,i}(x)\varphi_{,j}(x) + \Big(\vartheta(x) + a\dot{\vartheta}(x)\Big)\Big(d\vartheta(x) + h\dot{\vartheta}(x)\Big) + \\
&\quad +c\left(\left(\kappa_{ij}\xi_{,i}(x)\right)_{,j}\right)\left(\left(\kappa_{mn}\varphi_{,m}(x)\right)_{,n}\right)\Big]dV = 0.
\end{aligned} \tag{3.20}$$

It is easy to observe that:

$$\begin{aligned}
\Big(\vartheta(x) + a\dot{\vartheta}(x)\Big)\Big(d\vartheta(x) + h\dot{\vartheta}(x)\Big) &= (ad - h)\vartheta(x)\dot{\vartheta}(x) + \\
&+ \frac{1}{d}\Big(d\vartheta(x) + h\dot{\vartheta}(x)\Big)^2 + \frac{h}{d}(ad - h)\Big(\dot{\vartheta}(x)\Big)^2,
\end{aligned}$$

so that (3.20) receives the following form:

$$\begin{aligned}
&\int_\Omega \Big[\rho v_i(x)\ddot{v}_i(x) + I_{jk}\phi_{ij}(x)\ddot{\phi}_{ik}(x) + A_{ijkl}e_{ij}(x)e_{kl}(x) + \\
&\quad +2E_{ijkl}e_{ij}(x)\varepsilon_{kl}(x) + 2F_{ijklm}e_{ij}(x)\gamma_{klm}(x) + B_{ijkl}\varepsilon_{ij}(x)\varepsilon_{kl}(x) + \\
&\quad +2G_{ijklm}\varepsilon_{ij}(x)\gamma_{klm}(x) + C_{ijklmn}\gamma_{ijk}(x)\gamma_{lmn}(x)]\, dV + \\
&\quad +\int_\Omega \Big[\kappa_{ij}\xi_{,i}(x)\varphi_{,j}(x) + c\left(\left(\kappa_{ij}\xi_{,i}(x)\right)_{,j}\right)\left(\left(\kappa_{mn}\varphi_{,m}(x)\right)_{,n}\right)\Big]dV + \\
&\quad +\int_\Omega \left[\frac{h}{d}(ad - h)\Big(\dot{\vartheta}(x)\Big)^2 + (ad - h)\vartheta(x)\dot{\vartheta}(x)\right] dV + \\
&\quad +\int_\Omega \left[\frac{1}{d}\Big(d\vartheta(x) + h\dot{\vartheta}(x)\Big)^2\right] dV = 0,
\end{aligned}$$

and this equation can be restated in the form that follows:

$$\begin{aligned}
&\int_\Omega \Big[\rho v_i(x)\ddot{v}_i(x) + I_{jk}\phi_{ij}(x)\ddot{\phi}_{ik}(x) + (ad - h)\vartheta(x)\dot{\vartheta}(x)\Big]\, dV + \\
&+\int_\Omega \Big[\kappa_{ij}\xi_{,i}(x)\varphi_{,j}(x) + c\left(\left(\kappa_{ij}\xi_{,i}(x)\right)_{,j}\right)\left(\left(\kappa_{mn}\varphi_{,m}(x)\right)_{,n}\right)\Big]dV = \\
&= -\int_\Omega [A_{ijkl}e_{ij}(x)e_{kl}(x) + 2E_{ijkl}e_{ij}(x)\varepsilon_{kl}(x) + \\
&\quad +2F_{ijklm}e_{ij}(x)\gamma_{klm}(x) + B_{ijkl}\varepsilon_{ij}(x)\varepsilon_{kl}(x) + \\
&+2G_{ijklm}\varepsilon_{ij}(x)\gamma_{klm}(x) + C_{ijklmn}\gamma_{ijk}(x)\gamma_{lmn}(x)]\, dV - \\
&-\int_\Omega \left[\frac{h}{d}(ad - h)\Big(\dot{\vartheta}(x)\Big)^2 + \frac{1}{d}\Big(d\vartheta(x) + h\dot{\vartheta}(x)\Big)^2\right] dV.
\end{aligned} \tag{3.21}$$

With the help of (3.21) the second derivative of the function $E(t)$, from $(3.17)_2$, receives the form:

$$\begin{aligned}\ddot{E}(t) = &\int_\Omega \left[\rho v_i(x)\ddot{v}_i(x) + I_{jk}\phi_{ij}(x)\ddot{\phi}_{ik}(x)\right] dV - \\ &- \int_\Omega [A_{ijkl}e_{ij}(x)e_{kl}(x) + 2E_{ijkl}e_{ij}(x)\varepsilon_{kl}(x) + \\ &+2F_{ijklm}e_{ij}(x)\gamma_{klm}(x) + B_{ijkl}\varepsilon_{ij}(x)\varepsilon_{kl}(x) + \\ &+2G_{ijklm}\varepsilon_{ij}(x)\gamma_{klm}(x) + C_{ijklmn}\gamma_{ijk}(x)\gamma_{lmn}(x)]\, dV - \\ &- \int_\Omega \left[\frac{h}{d}(ad-h)\left(\dot{\vartheta}(x)\right)^2 + \frac{1}{d}\left(d\vartheta(x) + h\dot{\vartheta}(x)\right)^2\right] dV.\end{aligned} \tag{3.22}$$

If we take into account the conservation law (3.15), from (3.21) we obtain:

$$\ddot{E}(t) = 2\int_\Omega \left[\rho v_i(x)\ddot{v}_i(x) + I_{jk}\phi_{ij}(x)\ddot{\phi}_{ik}(x)\right] dV +$$
$$2\int_0^t \int_\Omega \left[(ad-h)\left(\dot{\vartheta}(x)\right)^2 + \kappa_{ij}\varphi_{,i}(x)\varphi_{,j}(x) + c\left(\left(\kappa_{ij}\varphi_{,i}(x)\right)_{,j}\right)^2\right] dV ds,$$

so that we can deduce that:

$$E(t)\ddot{E}(t) - \left(\dot{E}(t)\right)^2 \geq 0,\ \forall t \geq 0,$$

and, as a consequence, we have the following inequality

$$\frac{d^2}{dt^2}(\ln E(t)) \geq 0,$$

from where we deduce that the function $\ln E(t)$ is convex, regarding the time variable t. If we consider that the maximum domain of definition of the solution is $[0, t_0]$, we integrate the above inequality to obtain:

$$E(t)\left(\frac{E(0)}{E(t_0)}\right)^{t/t_0} \leq E(0). \tag{3.23}$$

But we considered the initial data in their homogeneous form so that from (3.16) we obtain $E(0) = 0$ and from (3.23) we deduce

$$E(t) = 0,\ \forall t \in [0, t_0] \Rightarrow v_i(t) = \phi_{ij}(t) = \varphi(t) = \vartheta(t) = 0,\ \forall t \in [0, t_0],$$

and the proof of uniqueness is finished. □

In our second main result we wish to prove that the solution of the mixed problem $\mathcal{P}$ is exponentially unstable, if some specific conditions are satisfied.
Specifically, we will assume that the initial energy of the system is not strictly positive.
First, we present a helpful auxiliary result.
Let us note with $\nu(x)$ the function that satisfies the following boundary value problem:

$$\begin{aligned}&\left(\kappa_{ij}\nu_{,i}(x)\right)_{,j} = d\vartheta^1 + h\vartheta^0 - \left(\alpha_{ij}e^0_{ij} + \beta_{ij}\epsilon^0_{ij} + \delta_{ijk}\gamma^0_{ijk}\right),\ x \in \Omega,\\ &\nu(x) = 0,\ x \in \partial\Omega.\end{aligned} \tag{3.24}$$

The fact that the boundary value problem (3.24) has a solution can be deduced from the usual properties of the boundary value problems attached to elliptic equations.
From (3.24) we deduce that the function ν satisfies the equation:

$$\begin{aligned}&d\vartheta(x) + h\dot{\vartheta}(x) - \left[\kappa_{ij}\left(\nu_{,i}(x) + \xi_{,i}(x)\right)\right]_{,j} = \\ &\quad = \alpha_{ij}e_{ij}(x) + \beta_{ij}\epsilon_{ij}(x) + \delta_{ijk}\gamma_{ijk}(x),\end{aligned}$$

where the function $\xi(x)$ is defined after (3.16).
Now, we consider the above problem $\mathcal{P}$ in the situation of the nonhomogeneous initial data (2.12) and homogeneous boundary data (2.11).

Theorem 3.2.. *Suppose that the conditions* (2.9) *and* (2.10) *hold.*
If the mixed initial-boundary value problem $\mathcal{P}$, in the case of null boundary data, admits a solution for which $W_2(0) \leq 0$, then this solution is exponentially unstable.

Proof. For the demonstration we will use a function, defined in a manner similar to that defined in [15] and [16], from the study of which we will obtain the exponential growth of the solutions of the problem $\mathcal{P}$.
Let us define the function $\Gamma(t)$ by:

$$\Gamma(t) = w\,(t+t_0)^2 + \frac{1}{2}\int_\Omega (\rho v_i v_i + I_{jk}\phi_{ij}\phi_{ik})\, dV +$$

$$+\frac{1}{2}\int_0^t\int_\Omega\left[(ad-h)\vartheta^2 + \kappa_{ij}\,(\nu_{,i}+\xi_{,i})\,\xi_{,j} + c\left(\kappa_{ij}\,(\nu_{,i}+\xi_{,i})_{,j}\right)^2\right] dV ds, \tag{3.25}$$

in which the choice of the positive constants w and t_0 is at our disposal.
By direct calculation we obtain the first two derivatives of the function $\Gamma(t)$:

$$\dot{\Gamma}(t) = 2w\,(t+t_0) + \int_\Omega \left(\rho v_i \dot{v}_i + I_{jk}\phi_{ij}\dot{\phi}_{ik}\right) dV +$$

$$+\int_0^t\int_\Omega\Big[(ad-h)\vartheta\dot{\vartheta} + \kappa_{ij}\,(\nu_{,i}+\xi_{,\,i})\,\varphi_{,\,j} + c\Big((\kappa_{ij}\varphi_{,\,i})_{,\,j}\Big)\Big((\kappa_{mn}\,(\nu_{,m}+\xi_{,m}))_{,n}\Big)\Big] dV ds$$

$$+\frac{1}{2}\int_\Omega\left[(ad-h)\,(\vartheta^0)^2 + \kappa_{ij}\varphi^0_{,i}\varphi^0_{,j} + c\left(\left(\kappa_{ij}\varphi^0_{,i}\right)_{,j}\right)^2\right] dV, \tag{3.26}$$

$$\ddot{\Gamma}(t) = 2w + \int_\Omega \left[\rho\,(v_i\ddot{v}_i + \dot{v}_i\dot{v}_i) + I_{jk}\left(\phi_{ij}\ddot{\phi}_{ik} + \dot{\phi}_{ij}\dot{\phi}_{ik}\right)\right] dV +$$

$$+\int_\Omega\Big[(ad-h)\vartheta\dot{\vartheta} + \kappa_{ij}\,(\nu_{,i}+\xi_{,\,i})\,\varphi_{,\,j} + c\Big((\kappa_{ij}\varphi_{,\,i})_{,\,j}\Big)\Big((\kappa_{mn}\,(\nu_{,\,m}+\xi_{,\,m}))_{,\,n}\Big)\Big] dV.$$

Taking into account the expression of the initial energy of the system, W_2, the second derivative of the function Γ receives the following form:

$$\ddot{\Gamma}(t) = 2\,(w - W_2(0)) + 2\int_\Omega \left(\rho\dot{v}_i\dot{v}_i + I_{jk}\dot{\phi}_{ij}\dot{\phi}_{ik}\right) dV +$$

$$+2\int_0^t\int_\Omega\left[(ad-h)\left(\dot{\vartheta}\right)^2 + \kappa_{ij}\varphi_{,i}\varphi_{,j} + c\Big((\kappa_{ij}\varphi_{,i})_{,j}\Big)^2\right] dV ds. \tag{3.27}$$

From (3.26) and (3.27) we can obtain the following inequality:

$$\Gamma(t)\ddot{\Gamma}(t) - \left(\dot{\Gamma}(t) - \frac{1}{2}I\right)^2 \geq 2\,(W_2(0)+w)\,\Gamma(t), \tag{3.28}$$

where, to simplify the writing, we have the note with I the following integral:

$$I = \int_\Omega\left[(ad-h)\,(\vartheta^0)^2 + \kappa_{ij}\varphi^0_{,i}\varphi^0_{,j} + c\Big(\left(\kappa_{ij}\varphi^0_{,i}\right)_{,j}\Big)^2\right] dV.$$

According to the theorem hypothesis, we assumed that $W_2(0) \leq 0$, so we can take $w = -W_2(0)$ and we choose t_0 large enough to be sure that $\dot{\Gamma}(0) > I$. As such, from (3.28) we deduce:

$$\Gamma(t)\ddot{\Gamma}(t) - \dot{\Gamma}(t)\left(\dot{\Gamma}(t) - I\right) \geq 0.$$

From this inequality we can deduce that the function

$$\frac{\dot{\Gamma}(t) - I}{\Gamma(t)}$$

is an increasing application in relation to the time variable t. As such, we deduce that:

$$\frac{\dot{\Gamma}(t) - I}{\Gamma(t)} \geq \frac{\dot{\Gamma}(0) - I}{\Gamma(0)}, \quad \forall t \geq 0,$$

and this inequality can be restated in the form:

$$\dot{\Gamma}(t) \geq \frac{\dot{\Gamma}(t) - I}{\Gamma(0)} \Gamma(t) + I.$$

Finally, we integrate the last inequality on the interval and obtain:

$$\Gamma(t) \geq \frac{\Gamma(0)\dot{\Gamma}(0)}{\dot{\Gamma}(0) - I}\, e^{\frac{\dot{\Gamma}(0)-I}{\Gamma(0)}t} - \frac{\Gamma(0)}{\dot{\Gamma}(0) - I}.$$

From this inequality we are led to the conclusion that the solutions exponential growth, that is, the solutions of the problem $\mathcal{P}$ are exponentially unstable. With this the proof of Theorem 3.2 ends. □

Remark 3.1. It should be noted that the two results, both uniqueness and instability, can be obtained using a very similar procedure in the case we consider the rate of the conductive temperature, that is, we consider the two-temperatures relation (2.7). In this situation, the initial energy $W_2(t)$ is replaced by $W_1(t)$.

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