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Some sequences of Euler type, their convergences and their stability

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ABSTRACT. The aim of this paper is to present some sequences of Euler type. We will explore the sequences $(F_n)_{n\geq 1}$, defined by $F_n(x) = \sum_{k=1}^{n} f(k) - \int_1^{n+x} f(t)\,dt$, for any $n \geq 1$ and $x \in [0,1]$, where f is a local integrable and positive function defined on $[1,\infty)$. Starting from some particular example we will find that this sequence is uniformly convergent to a constant function. Also, we present a stability result.

1. INTRODUCTION

One of the more important result from calculus says that the sequence

$$x_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} - \ln n$$

is convergent. Its limit is denoted γ and is called Euler-Mascheroni constant [15]. Today we find more papers to this topic, including [13], where it was proven the next inequalities about the asymptotic behavior of harmonic sum:

$$\frac{1}{2n+\frac{2}{5}} < 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} - \ln n - \gamma < \frac{1}{2n+\frac{1}{3}},\ (n \geq 1)$$

More authors tries to study a more general case, respectively the sequence $(y_n)_{n\geq 1}$ defined by

$$y_n = \sum_{k=1}^{n} f(k) - \int_1^n f(t)\,dt, (n \geq 1),$$

where f is a real continuous function defined on $[1,\infty)$. For example, the readers can find the papers [10], [11], [12] or [14].

Recently, a very interesting result of the same type, due to Ivan [4], was published. It is included in the following proposition.

Propostion 1.1. *Let $f : [1,\infty) \to \mathbb{R}$ a differentiable function with $\lim_{x\to\infty} f(x) = 0$. Suppose that f' is strictly monotone, does not vanish anywhere and $\lim_{n\to\infty} \frac{f'(n+1)}{f'(n)} = 1$. Then, the limit*

$$\lim_{n\to\infty} \left(\sum_{k=1}^{n} f(k) - \int_1^n f(t)\,dt \right)$$

exists and it is finite. Moreover, if we denote γ_f the value of this limit, then we have

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} f(k) - \int_1^n f(t)\,dt - \gamma_f}{f(n)} = \frac{1}{2}.$$

The readers can find the original proof in [5].

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The aim of this paper is to present our contributions to this topic. First, we present a synthesis of the most relevant results, based on the Sándor [10] or Trif [14] paper. Our contribution is represented by some improvement of the original proofs. The main results are presented on the third section where we will prove a stronger result about the convergence of this type of sequence. We will complete with a stability result about the limit of these sequences. Finally, we will obtain Ivan's result as consequences of the results from previous section.

2. Some sequences with equal limits

Throughout this paper, we consider a local integrable function $f : [1, \infty) \to (0, \infty)$. For any $x \in [0, 1]$ and any positive integer n,we define the sequence of functions $(F_n)_{n\geq 1}$ by

$$F_n(x) = \sum_{k=1}^{n} f(k) - \int_{1}^{n+x} f(t)\,dt.$$

We will obtain some important conclusion about this sequence starting from some particular cases.

First, we consider the sequences $(a_n)_{n\geq 1}$, $(b_n)_{n\geq 1}$ and $(c_n)_{n\geq 1}$ defined, for any integer $n \geq 1$, by $a_n = F_n(1)$, $b_n = F_n(0)$ and $c_n = F_n\left(\frac{1}{2}\right)$. Their properties are described in the following two theorems.

Theorem 2.1. ([10], Theorem 1) *We assume that f is a decreasing function.*
a) *For any integer $n \geq 1$, we have $a_n \leq a_{n+1} < b_{n+1} \leq b_n$;*
b) *The sequences $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ are convergences;*
c) *If $\lim_{x\to\infty} f(x) = 0$ then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.*

Proof. $a)$ First we evaluate the difference $a_{n+1} - a_n$ and we obtain

$$a_{n+1} - a_n = f(n+1) - \int_{n+1}^{n+2} f(t)\,dt.$$

Hence f is decreasing, we have

$$\int_{n+1}^{n+2} f(t)\,dt \leq \int_{n+1}^{n+2} f(n+1)\,dt = f(n+1),$$

also $a_{n+1} - a_n \geq 0$. In the same mode, we obtain $b_{n+1} - b_n = f(n+1) - \int_{n}^{n+1} f(t)\,dt \leq 0$.

Further,, we have

$$b_{n+1} - a_{n+1} = \int_{n+1}^{n+2} f(t)\,dt > 0,$$

hence f is non-constant and positive.

$b)$ The previous inequalities led us to the conclusion that the sequences $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ are monotone and bounded, so they are convergences.

$c)$ The equality $b_{n+1} - a_{n+1} = \int_{n+1}^{n+2} f(t)\,dt$ goes to $f(n+2) \leq b_{n+1} - a_{n+1} \leq f(n)$. The hypothesis $\lim_{x\to\infty} f(x) = 0$ give us $\lim_{n\to\infty} (b_{n+1} - a_{n+1}) = 0$ and the conclusion follows now. □

Theorem 2.2. ([14], Theorem 1) *We suppose that f is a convex and decreasing function.*
a) *The sequence $(c_n)_{n\geq 1}$ is decreasing;*
b) $a_n < c_n < b_n$;
c) *If $\lim_{x\to\infty} f(x) = 0$ then $\lim_{n\to\infty} c_n = \lim_{n\to\infty} a_n$.*

Proof. $a)$ We have

$$c_{n+1}-c_n=f\left(n+1\right)-\int_{n+1/2}^{n+3/2}f\left(t\right)dt.$$

Hence f is convex, we are using the Hermite-Hadamard inequalities, for example see [8], and we obtain

$$\int_{n+1/2}^{n+3/2}f\left(t\right)dt\geq f\left(\frac{n+\frac{1}{2}+n+\frac{3}{2}}{2}\right)=f\left(n+1\right).$$

Then $c_{n+1}-c_n\leq 0$ and $\left(c_n\right)_{n\geq 1}$ is decreasing.

$b)$ The function f is positive, so we have

$$\int_1^n f\left(t\right)dt<\int_1^{n+1/2}f\left(t\right)dt<\int_1^{n+1}f\left(t\right)dt,$$

for any positive integer n. Now the inequalities $a_n<c_n<b_n$ are clear.

$c)$ The result is consequences of the previous point and Theorem 2.1.c. □

The previous theorems shows us that, if the function f is convex and decreasing with $\lim_{x\to\infty}f\left(x\right)=0$, the sequences $\left(a_n\right)_{n\geq 1}$, $\left(b_n\right)_{n\geq 1}$ and $\left(c_n\right)_{n\geq 1}$ are convergences and have the same limit. The common limit will be denoted γ_f. It is depending exclusively by f. The next theorem present a first result involving this constant.

Theorem 2.3. ([14], relation 7) *We suppose that f is a convex and decreasing function with $\lim_{x\to\infty}f\left(x\right)=0$. For any positive integer n there exists a unique point $x_n\in\left(0,1\right)$ such that $F_n\left(x_n\right)=\gamma_f$.*

Proof. From Theorem 2.1 we find $a_n<\gamma_f<b_n$, also $F_n\left(1\right)<\gamma_f<F_n\left(0\right)$, for any positive integer n. The intermediary value theorem give us a point $x_n\in\left(0,1\right)$ such that $F_n\left(x_n\right)=\gamma_f$.

If we suppose that this point is not unique, then there exists $y_n\in\left(0,1\right)$ such that $F_n\left(y_n\right)=\gamma_f$. From $F_n\left(y_n\right)=F_n\left(x_n\right)$, we obtain $\int_1^{n+y_n}f\left(t\right)dt=\int_1^{n+x_n}f\left(t\right)dt$, also $\int_{n+x_n}^{n+y_n}f\left(t\right)dt=0$. Hence, f is positive, we find $n+y_n=n+x_n$, so $y_n=x_n$ and the proof is complete. □

Theorem 2.3 give us a sequence $\left(x_n\right)_{n\geq 1}$. The properties of this sequence are included in the following result.

Theorem 2.4. ([14], Theorem 2) *We suppose that f is a convex and decreasing function with $\lim_{x\to\infty}f\left(x\right)=0$.*

a) *For any $n\geq 1$, we have* $\frac{1}{2}\leq x_n\leq\frac{1}{4}\left(1+\frac{f(n)}{f(n+1)}\right)$;

b) *If* $\lim_{n\to\infty}\frac{f(n+1)}{f(n)}=1$ *then* $\lim_{n\to\infty}x_n=\frac{1}{2}$.

Proof. $a)$ From Theorem 2.2 we obtain $c_n\geq\gamma_f$, also $F_n\left(\frac{1}{2}\right)\geq F_n\left(x_n\right)$. The definition of F_n goes to $\int_1^{n+1/2}f\left(t\right)dt\leq\int_1^{n+x_n}f\left(t\right)dt$, also $\int_{n+1/2}^{n+x_n}f\left(t\right)dt\geq 0$. Hence f is positive, we obtain $x_n\geq\frac{1}{2}$.

Hence f is a convex function then the Hermite-Hadamard inequalities goes to

$$\int_{n+1/2}^{n+x_n}f\left(t\right)dt\geq f\left(\frac{n+1/2+n+x_n}{2}\right)\left(x_n-\frac{1}{2}\right).$$

The monotonicity of f lead us to

$$f\left(\frac{n+1/2+n+x_n}{2}\right)\geq f\left(\frac{n+1/2+n+1}{2}\right)=f\left(n+\frac{3}{4}\right)\geq f\left(n+1\right).$$

Now, we consider the sequence $(y_n)_{n\geq 1}$ defined by $y_n = c_n + \frac{f(n+1)-f(n)}{4}$, $n \geq 1$. We have

$$\begin{aligned} y_{n+1} - y_n &= (c_{n+1} - c_n) + \left(\frac{f(n+2) - f(n+1)}{4} - \frac{f(n+1) - f(n)}{4} \right) \\ &= f(n+1) - \int_{n+1/2}^{n+3/2} f(t)\,dt + \frac{f(n+2) - 2f(n+1) + f(n)}{4} \\ &= \frac{1}{2} \cdot \frac{f(n+2) + f(n+1)}{2} + \frac{1}{2} \cdot \frac{f(n+1) + f(n)}{4} - \int_{n+1/2}^{n+3/2} f(t)\,dt \\ &\geq \frac{1}{2} \left(f\left(n + \frac{3}{2}\right) + f\left(n + \frac{1}{2}\right) \right) - \int_{n+1/2}^{n+3/2} f(t)\,dt \\ &\geq 0. \end{aligned}$$

The last inequality is true due to the Hermite-Hadamard inequalities. We obtain that the sequence $(y_n)_{n\geq 1}$ is increasing. Hence $\lim_{n\to\infty} y_n = \lim_{n\to\infty} c_n = \gamma_f$, we obtain that $y_n \leq \gamma_f$, also $c_n + \frac{f(n+1)-f(n)}{4} \leq \gamma_f$. This is equivalent with

$$\int_{n+1/2}^{n+x_n} f(t)\,dt \leq \frac{f(n) - f(n+1)}{4}.$$

With the previous relations, we obtain

$$f(n+1)\left(x_n - \frac{1}{2}\right) \leq \frac{f(n) - f(n+1)}{4}$$

and the conclusion follow now.

$b)$ It is clear from the previous point. □

We conclude this section with a theorem that present the convergence order of the sequences $(a_n)_{n\geq 1}$, $(b_n)_{n\geq 1}$ and $(c_n)_{n\geq 1}$.

Theorem 2.5. *We assume that the function f is decreasing, convex and $\lim_{x\to\infty} f(x) = 0$. If $\lim_{n\to\infty} \frac{f(n+1)}{f(n)} = 1$ then*

a) $\lim_{n\to\infty} \frac{a_n - \gamma_f}{f(n)} = -\frac{1}{2}$;

b) $\lim_{n\to\infty} \frac{b_n - \gamma_f}{f(n)} = \frac{1}{2}$;

c) $\lim_{n\to\infty} \frac{c_n - \gamma_f}{f(n)} = 0$.

Proof. First, we observe that $\frac{F_n(x)-\gamma_f}{f(n)} = \frac{1}{f(n)} \int_{n+x}^{n+x_n} f(t)\,dt$, for any $x \in [0,1]$ and $n \geq 1$.

$a)$ Hence $a_n = F_n(1)$ we obtain $\frac{a_n-\gamma_f}{f(n)} = -\frac{1}{f(n)} \int_{n+x_n}^{n+1} f(t)\,dt$. The function f is decresing so

$$(n+1)(1 - x_n) \leq \int_{n+x_n}^{n+1} f(t)\,dt \leq f(n + x_n)(1 - x_n) \leq f(n)(1 - x_n).$$

We obtain

$$\frac{f(n+1)}{f(n)}(1 - x_n) \leq \frac{1}{f(n)} \int_{n+x_n}^{n+1} f(t)\,dt \leq 1 - x_n$$

and the conclusion follows due to hypothesis and Theorem 2.4.b.

$b)$ Since $b_n = F_n(0)$ we obtain $\frac{b_n - \gamma_f}{f(n)} = \frac{1}{f(n)} \int_n^{n+x_n} f(t)\,dt$. The function f is decresing so

$$f(n + x_n)\,x_n \leq \int_n^{n+x_n} f(t)\,dt \leq f(n)\,x_n.$$

Since $f(n + x_n) \geq f(n+1)$ we obtain

$$\frac{f(n+1)}{f(n)} x_n \leq \frac{1}{f(n)} \int_n^{n+x_n} f(t)\,dt \leq x_n.$$

Now the conclusion follows due to hypothesis and Theorem 2.4.b.

$c)$ In the same mode, we obtain $\frac{c_n - \gamma_f}{f(n)} = -\frac{1}{f(n)} \int_{n+1/2}^{n+x_n} f(t)\,dt$. The function f is decreasing so

$$f\left(n + \frac{1}{2}\right)\left(x_n - \frac{1}{2}\right) \leq \int_{n+1/2}^{n+x_n} f(t)\,dt \leq f(n + x_n)\left(x_n - \frac{1}{2}\right).$$

Moreover, we obtain

$$f(n+1)\left(x_n - \frac{1}{2}\right) \leq \int_{n+1/2}^{n+x_n} f(t)\,dt \leq f(n)\left(x_n - \frac{1}{2}\right).$$

Then

$$\frac{f(n+1)}{f(n)}\left(x_n - \frac{1}{2}\right) \leq \frac{1}{f(n)} \int_{n+1/2}^{n+x_n} f(t)\,dt \leq \left(x_n - \frac{1}{2}\right).$$

Now the conclusion follows due to the hypothesis and Theorem 2.4.b. □

3. The main results

This section is reserved to the main results of this paper. First, we will prove that the sequence of functions $(F_n)_{n\geq 1}$, defined on the second section, is uniformly convergent and we include the results from Theorem 2.2 and 2.5 in a more general case. Recall that the common limit of the sequences $(F_n(0))_{n\geq 1}$, $\left(F_n\left(\frac{1}{2}\right)\right)_{n\geq 1}$ and $(F_n(1))_{n\geq 1}$ is denoted γ_f.

Theorem 3.6. *We assume that f is a decreasing and convex function with* $\lim_{x\to\infty} f(x) = 0$.

a) *The sequence of functions $(F_n)_{n\geq 1}$ is uniformly convergent to γ_f;*

b) *If* $\lim_{n\to\infty} \frac{f(n+1)}{f(n)} = 1$, *then, for any $x \in [0,1]$, we have* $\lim_{n\to\infty} \frac{F_n(x) - \gamma_f}{f(n)} = \frac{1}{2} - x$.

Proof. $a)$ We have

$$\begin{aligned} |F_n(x) - \gamma_f| &= |F_n(x) - F_n(0) + F_n(0) - \gamma_f| \\ &\leq |F_n(x) - F_n(0)| + |F_n(0) - \gamma_f| \\ &= \int_n^{n+x} f(t)\,dt + |F_n(0) - \gamma_f| \\ &\leq f(n)\,x + |F_n(0) - \gamma_f| \\ &\leq f(n) + |F_n(0) - \gamma_f|. \end{aligned}$$

Since $\lim_{x\to\infty} f(x) = 0$ and $\lim_{n\to\infty} F_n(0) = \gamma_f$, we obtain the conclusion.

$b)$ We have

$$\begin{aligned}\lim_{n\to\infty}\frac{F_n(x)-\gamma_f}{f(n)} &= \lim_{n\to\infty}\frac{F_n(x)-F_n(0)+F_n(0)-\gamma_f}{f(n)}\\ &= \lim_{n\to\infty}\frac{F_n(0)-\gamma_f}{f(n)}+\lim_{n\to\infty}\frac{F_n(x)-F_n(0)}{f(n)}\\ &= \lim_{n\to\infty}\frac{b_n-\gamma_f}{f(n)}-\lim_{n\to\infty}\frac{\int_n^{n+x}f(t)\,dt}{f(n)}\\ &= \frac{1}{2}-\lim_{n\to\infty}\frac{\int_n^{n+x}f(t)\,dt}{f(n)}.\end{aligned}$$

The function f is decreasing, so $f(n+x)\cdot x\le\int_n^{n+x}f(t)\,dt\le f(n)\cdot x$. Hence $f(n+1)\le f(n+x)$, we obtain $\frac{f(n+1)}{f(n)}\cdot x\le\frac{1}{f(n)}\int_n^{n+x}f(t)\,dt\le x$. As consequences we obtain $\lim_{n\to\infty}\frac{\int_n^{n+x}f(t)dt}{f(n)}=x$ and $\lim_{n\to\infty}\frac{F_n(x)-\gamma_f}{f(n)}=\frac{1}{2}-x$. □

Further, we present a stability result involving the constant γ_f. The parents of the stability concept is considered Ulam and Hyers (see [2] and [3]). In fact, it was proven that, for any $\varepsilon>0$ and for any function $f:U\to V$ between two Banach spaces and satisfying

$$\|f(x+y)-f(x)-f(y)\|<\varepsilon,$$

for all $x,y\in U$, there exists $\delta>0$ and a unique additive function $A:U\to V$ such that

$$\|A(x)-f(x)\|<\delta,$$

for all $x\in U$.

This result opened a new research direction and a new mathematical concept was born. Now it is said that the Cauchy additive functional equation, $f(x+y)=f(x)+f(y)$, satisfies the Hyers–Ulam stability. This concept has influenced more mathematicians studying the stability problems of functional equations (a large collection of results can be found in [6]). Today, the terminologies Hyers-Ulam stability is also applicable to the case of other mathematical objects as linear recurrences [9] or the intermediary point arising from the mean value theorems [7].

Now, we will prove that the γ_f are stable relating to function f. Denote A the set of all convex and decreasing functions $h:[1,\infty)\to(0,\infty)$ with the property $\lim_{x\to\infty}h(x)=0$. Also, let γ_h the constant associated to h by Theorem 3.6.a.

Theorem 3.7. *For any function $f\in A$ and any $\varepsilon>0$ there exists $\delta>0$ such that for any function $g\in A$, satisfying the condition $|g(t)-f(t)|<\delta$, for all $t\in[1,\infty)$, we have*

$$|\gamma_g-\gamma_f|<\varepsilon.$$

Proof. First, let $f\in A$. Then, for any positive integer n, there exists $x_n\in(0,1)$ such that

$$\gamma_f=\sum_{k=1}^{n}f(k)-\int_1^{n+x_n}f(t)\,dt.$$

Further, let $g\in A$ and $\delta>0$ such that $|g(t)-f(t)|<\delta$, for all $t\in[1,\infty)$. For any positive integer n, there exists $y_n\in(0,1)$ such that

$$\gamma_g=\sum_{k=1}^{n}g(k)-\int_1^{n+y_n}g(t)\,dt.$$

Then, we have

$$\begin{aligned}
|\gamma_g - \gamma_f| &= \left|\sum_{k=1}^{n} g(k) - \int_1^{n+y_n} g(t) - \sum_{k=1}^{n} f(k) + \int_1^{n+x_n} f(t)\,dt\right| \\
&= \left|\sum_{k=1}^{n} (g(k) - f(k)) - \int_1^{n+y_n} (g(t) - f(t))\,dt + \int_{n+y_n}^{n+x_n} f(t)\,dt\right| \\
&\leq \sum_{k=1}^{n} |g(k) - f(k)| + \int_1^{n+y_n} |g(t) - f(t)|\,dt + \left|\int_{n+y_n}^{n+x_n} f(t)\,dt\right| \\
&\leq n\delta + \int_1^{n+1} |g(t) - f(t)|\,dt + \int_n^{n+1} f(t)\,dt,
\end{aligned}$$

that led us to

$$|\gamma_g - \gamma_f| < 2n\delta + f(n),$$

for any positive integer n.

Now, let $\varepsilon > 0$. Hence $\lim_{n\to\infty} f(n) = 0$, there exists a positive integer M such that $f(n) < \frac{\varepsilon}{2}$, for any positive integer $n \geq M$. We consider $\delta = \frac{\varepsilon}{4M}$. From previous inequality, we obtain

$$\begin{aligned}
|\gamma_g - \gamma_f| &< 2M\delta + f(M) \\
&< 2M \cdot \frac{\varepsilon}{4M} + \frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}$$

and the proof is complete. □

We can remark that the result from previous theorem is not only a stability result. If we consider the functional $T : A \to \mathbb{R}$, defined by $T(f) = \gamma_f$ and a sequence of function $(f_n)_{n\geq 1}$ from A, uniformly convegent to f, then $\lim_{n\to\infty} T(f_n) = T(f)$.

Now, we conclude our paper by presenting our proof to Proposition 1.1. from the first section.

Proof of Proposition 1.1. Hence $f'(x) \neq 0$, for any $x \in [1, \infty)$, then we can assume that f' is negative. If $\lim_{x\to\infty} f(x) = 0$, we obtain f is positive. If we assume that f' is decreasing, we obtain that f is concave which contradicts with f positive and $\lim_{x\to\infty} f(x) = 0$. It remains that f' is increasing and f is convex.

Further, we will proof that $\lim_{n\to\infty} \frac{f(n+1)}{f(n)} = 1$. First, we evaluate $\lim_{n\to\infty} \frac{f(n+2)-f(n+1)}{f(n+1)-f(n)}$. From mean value theorem we find $u_n \in (n, n+1)$ and $v_n \in (n+1, n+2)$ such that

$$\frac{f(n+2) - f(n+1)}{f(n+1) - f(n)} = \frac{f'(v_n)}{f(u_n)}.$$

Hence $u_n < v_n$ and f' is increasing, we obtain $\frac{f'(v_n)}{f(u_n)} > 1$. By the other side, we have

$$\frac{f'(v_n)}{f(u_n)} < \frac{f'(n+2)}{f'(n)} = \frac{f'(n+2)}{f'(n+1)} \cdot \frac{f'(n+1)}{f'(n)}.$$

Hence $\lim_{n\to\infty} \frac{f'(n+1)}{f'(n)} = 1$, we obtain $\lim_{n\to\infty} \frac{f'(v_n)}{f(u_n)} = 1$. As conclusion, we find

$$\lim_{n\to\infty} \frac{f(n+2) - f(n+1)}{f(n+1) - f(n)} = 1.$$

The Cesaro-Stolz lemma on the case $\frac{0}{0}$ (see [1]) concludes that $\lim_{n\to\infty} \frac{f(n+1)}{f(n)} = 1$.

Now the sequence $\sum_{k=1}^{n} f(k) - \int_1^n f(t)\,dt$ has finite limit as consequence of Theorem 2.1.

If f' is positive, we use similar reasoning for $-f$.

The second result is consequences of Theorem 2.1.b and Theorem 2.5.b. $\square$

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