

# Extragradient method with a new adaptive step size for solving non-Lipschitzian pseudo-monotone variational inequalities

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**ABSTRACT.** The purpose of this work is to develop a new version of the extragradient method for solving non-Lipschitzian and pseudo-monotone variational inequalities in real Hilbert spaces. First, we prove a sufficient condition for weak convergence of a proposed algorithm under relaxed assumptions. Next, under strong pseudomonotonicity and Lipschitz continuity assumptions, we obtain also a Q-linear convergence rate of this algorithm. Our results improve some recent contributions in the literature on the extragradient method.

## 1. INTRODUCTION

We are focus on the following classical variational inequality (VI) ([10, 11]) which consists in finding a point  $x^* \in C$  such that

$$(1.1) \quad \langle Fx^*, x - x^* \rangle \geq 0 \quad \forall x \in C,$$

where  $C$  is a nonempty closed convex subset in a real Hilbert space  $H$ , and  $F : C \rightarrow H$  is a single-valued mapping. As commonly done, we denote by  $Sol(C, F)$  the solution set of VI (1.1). Variational inequalities are fundamental in a broad range of mathematical and applied sciences; the theoretical and algorithmic foundations as well as the applications of variational inequalities have been extensively studied in the literature and continue to attract intensive research. For the current state of the art in finite dimensional setting, see for instance [9] and the extensive list of references therein.

Many authors have proposed and analyzed several iterative methods for solving the variational inequality (1.1). The simplest one is the following projection method, which can be seen an extension of the projected gradient method introduced for solving optimization problems.

$$x_{n+1} = P_C(x_n - \tau Fx_n),$$

for each  $n \geq 1$ , where  $P_C$  denotes by the metric projection from  $H$  onto  $C$ . It is shown that the convergence of this method is quite restrictive and assume that  $F$  is  $L$ -Lipschitz continuous and  $\alpha$ -strongly monotone (will be explained in the sequel), and  $\tau \in \left(0, \frac{2\alpha}{L^2}\right)$ .

To avoid the hypothesis of the strong monotonicity, Korpelevich [20] (also independently by [1]) proposed a double projection method known as the extragradient method in Euclidean space when  $F$  is monotone and  $L$ -Lipschitz continuous. The iterative step of the method is as follows.

$$(1.2) \quad x_0 \in C, \quad y_n = P_C(x_n - \tau_n Fx_n), \quad x_{n+1} = P_C(x_n - \tau_n Fy_n),$$

where  $\tau_n \in \left(0, \frac{1}{L}\right)$ .

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Received: 06.02.2021. In revised form: 05.01.2022. Accepted: 12.01.2022

2010 *Mathematics Subject Classification.* 47H09, 47J20, 47J05, 47J25.

Key words and phrases. *Extragradient method, variational inequality, pseudomonotone operator, convergence rate.*

In recent years, the extragradient method was further extended to infinite-dimensional spaces in various ways, see, e.g. [6, 3, 4, 5, 12, 15, 19, 21, 22, 25, 24, 23, 28, 29] and the references therein.

However, when the inequality variational mapping is not Lipschitz continuous or its Lipschitz constant  $L$  is difficult to compute/approximate, Korpelevich's method fails since the step-size  $\tau_n$  depends on this. Indeed, we give an academic example of a variational inequality problem in the infinite dimensional spaces which shows that the mapping is pseudo-monotone and uniformly continuous on  $C$  but it fails to be Lipschitz continuous on  $C$  with the solution set of the variational inequality mapping is nonempty. Hence, the method (1.2) cannot be applied.

**Example 1.1.** Consider the Hilbert space

$$H = l_2 := \left\{ u = (u_1, u_2, \dots, u_n, \dots) \mid \sum_{n=1}^{\infty} |u_n|^2 < +\infty \right\}$$

equipped with the inner product and induced norm on  $H$ :

$$\langle u, v \rangle = \sum_{n=1}^{\infty} u_n v_n \text{ and } \|u\| = \sqrt{\langle u, u \rangle}$$

for any  $u = (u_1, u_2, \dots, u_n, \dots), v = (v_1, v_2, \dots, v_n, \dots) \in H$ .

Consider the set and the mapping:

$$C = \{u \in H \mid \|u\| \leq \alpha\}, \quad Fu = \left( (\|u\| + \alpha) - \frac{1}{\|u\| + \alpha} \right) u$$

where  $\alpha > 1$  is a positive real number.

With this  $C$  and  $F$ , it is easy to see that  $Sol(C, F) = \{0\}$  and moreover,  $F$  is pseudo-monotone and uniformly continuous on  $C$  but  $F$  fails to be Lipschitz continuous on  $C$ .

First observe that since  $\alpha > 1$ , we get that

$$\left( (\|u\| + \alpha) - \frac{1}{\|u\| + \alpha} \right) > 0 \quad \forall u \in C.$$

Now let  $u, v \in C$  be such that  $\langle Fu, v - u \rangle \geq 0$ . This implies that  $\langle u, v - u \rangle \geq 0$ .

Consequently,

$$\begin{aligned} \langle Fv, v - u \rangle &= \left( (\|u\| + \alpha) - \frac{1}{\|u\| + \alpha} \right) \langle v, v - u \rangle \\ &\geq \left( (\|u\| + \alpha) - \frac{1}{\|u\| + \alpha} \right) (\langle v, v - u \rangle - \langle u, v - u \rangle) \\ &= \left( (\|u\| + \alpha) - \frac{1}{\|u\| + \alpha} \right) \|v - u\|^2 \geq 0 \end{aligned}$$

meaning that  $F$  is pseudo-monotone.

Now, since  $C$  is compact, the mapping  $F$  is uniformly continuous on  $C$ .

Finally we show that  $F$  is not Lipschitz continuous on  $C$ . Assume to the contrary that  $F$  is Lipschitz continuous on  $C$ , i.e., there exists  $L > 0$  such that

$$\|Fu - Fv\| \leq L\|u - v\| \quad \forall u, v \in C.$$

Let  $u = (L, 0, \dots, 0, \dots)$  and  $v = (0, 0, \dots, 0, \dots)$  then

$$\|Fu - Fv\| = \|Fu\| = \left( (\|u\| + \alpha) - \frac{1}{\|u\| + \alpha} \right) \|u\| = \left( (L + \alpha) - \frac{1}{L + \alpha} \right) L.$$

Thus,  $\|Fu - Fv\| \leq L\|u - v\|$  is equivalent to

$$\left( (L + \alpha) - \frac{1}{L + \alpha} \right) L \leq L^2,$$

equivalently

$$L + \alpha \leq L + \frac{1}{L + \alpha} < L + 1,$$

which implies that  $\alpha < 1$ , and this leads to a contraction and thus  $F$  is not Lipschitz continuous on  $C$ .

Motivated by Example 1.1, it would be of interest to propose an iterative method for solving  $Sol(C, F)$  for which the underline cost function  $F$  is uniformly continuous on bounded subsets of  $C$  but not Lipschitz continuous on  $C$ . Iusem [16] proposed the following algorithm so that it can solve this problem.

**Algorithm 1.**

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**Initialization:** Given  $\gamma > 0, l \in (0, 1), \mu \in (0, 1)$ . Let  $x_0 \in C$  be arbitrary

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**Iterative Steps:** Given the current iterate  $x_n$ , calculate  $x_{n+1}$  as follows:

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**Step 1.** Compute

$$y_n = P_C(x_n - \tau_n Fx_n)$$

where  $\tau_n := \gamma l^{m_n}$  and  $m_n$  is the smallest non-negative integer  $m$  satisfying

$$\gamma l^m \|Fx_n - P_C(x_n - \gamma l^m Fx_n)\| \leq \mu \|x_n - P_C(x_n - \gamma l^m Fx_n)\|.$$

If  $x_n = y_n$  then stop and  $x_n$  is a solution of  $Sol(C, F)$ . Otherwise

**Step 2.** Compute

$$x_{n+1} = P_C(x_n - \beta_n Fy_n),$$

where

$$\beta_n := \frac{\langle Fy_n, x_n - y_n \rangle}{\|Fy_n\|^2}.$$

Set  $n := n + 1$  and go to **Step 1**.

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Recently, Cai et al. [2] constructed a modification of the subgradient extragradient method [3] which converges under a weaker condition in Hilbert spaces. To be more specific, they assumed that  $F$  is uniformly continuous pseudo-monotone operator and proved a strong convergence theorem under some appropriate conditions imposed on the parameters. In particular, they introduced the algorithm which uses a different Armijo-type line-search and then  $F$  is only assumed to be pseudo-monotone on  $C$  in the sense of Karamardian [18]. In this paper, motivated by the works of Cai et al., we propose a new extragradient method for solving variational inequality problems of pseudo-monotone and non-Lipschitz continuous operator in real Hilbert spaces. We present weak convergence and convergence rate of the proposed algorithm in real Hilbert spaces under suitable conditions. Our results improve some recent contributions in the literature.

The paper is organized as follows. We first recall some basic definitions and results in Section 2. Our algorithms are presented and analysed in Section 3.

## 2. PRELIMINARIES

Let  $H$  be a real Hilbert space and  $C$  be a nonempty, closed and convex subset of  $H$ . The weak convergence of  $\{x_n\}_{n=1}^{\infty}$  to  $x$  is denoted by  $x_n \rightharpoonup x$  as  $n \rightarrow \infty$ , while the strong convergence of  $\{x_n\}_{n=1}^{\infty}$  to  $x$  is written as  $x_n \rightarrow x$  as  $n \rightarrow \infty$ . For each  $x, y \in H$  we have

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle.$$

**Definition 2.1.** Let  $T : H \rightarrow H$  be an operator.

- (1) The operator  $T$  is called  $L$ -Lipschitz continuous with  $L > 0$  if

$$\|Tx - Ty\| \leq L\|x - y\| \quad \forall x, y \in H.$$

- (2) The operator  $T$  is called monotone if

$$\langle Tx - Ty, x - y \rangle \geq 0 \quad \forall x, y \in H.$$

- (3) The operator  $T$  is called pseudo-monotone if

$$\langle Tx, y - x \rangle \geq 0 \implies \langle Ty, y - x \rangle \geq 0 \quad \forall x, y \in H.$$

- (4) The operator  $T$  is called  $\alpha$ -strongly monotone if there exists a constant  $\alpha > 0$  such that

$$\langle Tx - Ty, x - y \rangle \geq \alpha\|x - y\|^2 \quad \forall x, y \in H.$$

- (5) The operator  $T$  is called sequentially weakly continuous if for each sequence  $\{x_n\}$  we have:  $\{x_n\}$  converges weakly to  $x$  implies  $Tx_n$  converges weakly to  $Tx$ .

It is easy to see that *general monotone operator* is pseudo-monotone but the converse is not true.

For every point  $x \in H$ , there exists a unique nearest point in  $C$ , denoted by  $P_Cx$  such that  $\|x - P_Cx\| \leq \|x - y\| \quad \forall y \in C$ .  $P_C$  is called the *metric projection* of  $H$  onto  $C$ . It is known that  $P_C$  is nonexpansive.

**Lemma 2.1.** ([14]) Let  $C$  be a nonempty closed convex subset of a real Hilbert space  $H$ . Given  $x \in H$  and  $z \in C$ . Then  $z = P_Cx \iff \langle x - z, z - y \rangle \geq 0 \quad \forall y \in C$ .

**Lemma 2.2.** ([14]) Let  $C$  be a closed and convex subset in a real Hilbert space  $H$ ,  $x \in H$ . Then

- i)  $\|P_Cx - P_Cy\|^2 \leq \langle P_Cx - P_Cy, x - y \rangle \quad \forall y \in C$ ;
- ii)  $\|P_Cx - y\|^2 \leq \|x - y\|^2 - \|x - P_Cx\|^2 \quad \forall y \in C$ ;
- iii)  $\langle (I - P_C)x - (I - P_C)y, x - y \rangle \geq \|(I - P_C)x - (I - P_C)y\|^2 \quad \forall y \in C$ .

For properties of the metric projection, the interested reader could be referred to [14, Section 3].

The following Lemmas are useful for the convergence of our proposed methods.

**Lemma 2.3.** ([8]) For  $x \in H$  and  $\alpha \geq \beta > 0$  the following inequalities hold.

$$\frac{\|x - P_C(x - \alpha Ax)\|}{\alpha} \leq \frac{\|x - P_C(x - \beta Ax)\|}{\beta},$$

$$\|x - P_C(x - \beta Ax)\| \leq \|x - P_C(x - \alpha Ax)\|.$$

**Lemma 2.4.** ([17]) Let  $H_1$  and  $H_2$  be two real Hilbert spaces. Suppose  $A : H_1 \rightarrow H_2$  is uniformly continuous on bounded subsets of  $H_1$  and  $M$  is a bounded subset of  $H_1$ . Then  $A(M)$  (the image of  $M$  under  $A$ ) is bounded.

**Lemma 2.5.** [7, Lemma 2.1] Consider the problem  $VI(C, A)$  with  $C$  being a nonempty, closed, convex subset of a real Hilbert space  $H$  and  $A : C \rightarrow H$  being pseudo-monotone and continuous. Then,  $x^*$  is a solution of  $VI(C, A)$  if and only if

$$\langle Ax, x - x^* \rangle \geq 0 \quad \forall x \in C.$$

**Lemma 2.6.** ([26]) Let  $C$  be a nonempty set of  $H$  and  $\{x_n\}$  be a sequence in  $H$  such that the following two conditions hold:

- i) for every  $x \in C$ ,  $\lim_{n \rightarrow \infty} \|x_n - x\|$  exists;
  - ii) every sequential weak cluster point of  $\{x_n\}$  is in  $C$ .
- Then  $\{x_n\}$  converges weakly to a point in  $C$ .

**Definition 2.2.** [27] Let  $\{x_n\}$  be a sequence in  $H$ .

- i)  $\{x_n\}$  is said to converge  $R$ -linearly to  $x^*$  with rate  $\rho \in [0, 1)$  if there is a constant  $c > 0$  such that

$$\|x_n - x^*\| \leq c\rho^n \quad \forall n \in \mathbb{N}.$$

- ii)  $\{x_n\}$  is said to converge  $Q$ -linearly to  $x^*$  with rate  $\rho \in [0, 1)$  if

$$\|x_{n+1} - x^*\| \leq \rho \|x_n - x^*\| \quad \forall n \in \mathbb{N}.$$

### 3. MAIN RESULTS

In this section, we introduce the new modification of extragradient method solving the VI (1.1). We present the weak convergence and convergence rate of the sequence generated by the proposed method under the suitable assumptions. The proposed algorithm is of the form:

**Algorithm 2.** Given  $\gamma > 0, l, \mu \in (0, 1)$ . Let  $v_0 \in C$  be arbitrary. Compute

$$\begin{aligned} y_n &= P_C(v_n - \tau_n Fv_n), \\ v_{n+1} &= P_C(v_n - \tau_n Fy_n), \end{aligned}$$

with  $\tau_n := \gamma l^{m_n}$  where  $m_n$  is the smallest nonnegative integer  $m$  satisfying

$$\begin{aligned} &\gamma l^m \langle FP_C(v_n - \gamma l^m Fv_n) - Fv_n, P_C(v_n - \gamma l^m Fv_n) - P_C(v_n - \gamma l^m FP_C(v_n - \gamma l^m Fv_n)) \rangle \\ (3.3) \quad &\leq \frac{\mu}{2} (\|v_n - P_C(v_n - \gamma l^m Fv_n)\|^2 + \|P_C(v_n - \gamma l^m Fv_n) - P_C(v_n - \gamma l^m FP_C(v_n - \gamma l^m Fv_n))\|^2). \end{aligned}$$

**Remark 3.1.** (3.3) can be rewritten as follows

$$\gamma l^m \langle Fy_n - Fv_n, y_n - v_{n+1} \rangle \leq \frac{\mu}{2} (\|v_n - y_n\|^2 + \|y_n - v_{n+1}\|^2).$$

**3.1. Weak convergence.** To prove the weak convergence we need the following conditions:

*Condition 1.* The feasible set  $C$  of the VI (1.1) is a nonempty, closed, and convex subset of the real Hilbert space  $H$ .

*Condition 2.* The operator  $F : C \rightarrow H$  is a pseudo-monotone, uniformly continuous on bounded subsets of  $C$ . Moreover, the mapping  $F : C \rightarrow H$  satisfies the following condition

$$(3.4) \quad \text{whenever } \{x_n\} \subset C, x_n \rightharpoonup z, \text{ one has } \|Fz\| \leq \liminf_{n \rightarrow \infty} \|Fx_n\|.$$

**Condition 3.** The solution set of the VI (1.1) is nonempty, that is  $Sol(C, F) \neq \emptyset$ .

We start the algorithm's convergence analysis by proving that (3.3) terminates after finite steps.

**Lemma 3.7.** Assume that the mapping  $F : C \rightarrow H$  is uniformly continuous on bounded subsets of  $C$ . The Armijo-line search rule (3.3) is well defined. In addition, we have  $\tau_n \leq \gamma$ .

*Proof.* If  $v_n \in Sol(C, F)$  then  $v_n = P_C(v_n - \gamma Fv_n)$  and  $m_n = 0$ . We consider the situation  $v_n \notin Sol(C, F)$  and assume the contrary that for all  $m$  we have

$$\begin{aligned} & \gamma^{l^m} \langle FP_C(v_n - \gamma^{l^m} Fv_n) - Fv_n, P_C(v_n - \gamma^{l^m} Fv_n) - P_C(v_n - \gamma^{l^m} FP_C(v_n - \gamma^{l^m} Fv_n)) \rangle \\ (3.5) \quad & > \frac{\mu}{2} (\|v_n - P_C(v_n - \gamma^{l^m} Fv_n)\|^2 + \|P_C(v_n - \gamma^{l^m} Fv_n) - P_C(v_n - \gamma^{l^m} FP_C(v_n - \gamma^{l^m} Fv_n))\|^2). \end{aligned}$$

By Cauchy-Schwartz inequality, we have

$$\begin{aligned} & \gamma^{l^m} \langle FP_C(v_n - \gamma^{l^m} Fv_n) - Fv_n, P_C(v_n - \gamma^{l^m} Fv_n) - P_C(v_n - \gamma^{l^m} FP_C(v_n - \gamma^{l^m} Fv_n)) \rangle \\ (3.6) \quad & \leq \gamma^{l^m} \|FP_C(v_n - \gamma^{l^m} Fv_n) - Fv_n\| \|P_C(v_n - \gamma^{l^m} Fv_n) - P_C(v_n - \gamma^{l^m} FP_C(v_n - \gamma^{l^m} Fv_n))\|, \end{aligned}$$

and

$$\begin{aligned} & (\|v_n - P_C(v_n - \gamma^{l^m} Fv_n)\|^2 + \|P_C(v_n - \gamma^{l^m} Fv_n) - P_C(v_n - \gamma^{l^m} FP_C(v_n - \gamma^{l^m} Fv_n))\|^2) \\ (3.7) \quad & \geq 2\|v_n - P_C(v_n - \gamma^{l^m} Fv_n)\| \|P_C(v_n - \gamma^{l^m} Fv_n) - P_C(v_n - \gamma^{l^m} FP_C(v_n - \gamma^{l^m} Fv_n))\|. \end{aligned}$$

Combining (3.5) and (3.6) and (3.7) we find

$$(3.8) \quad \gamma^{l^m} \|FP_C(v_n - \gamma^{l^m} Fv_n) - Fv_n\| > \mu \|P_C(v_n - \gamma^{l^m} Fv_n) - v_n\|.$$

This implies that

$$(3.9) \quad \|Fv_n - FP_C(v_n - \gamma^{l^m} Fv_n)\| > \mu \frac{\|v_n - P_C(v_n - \gamma^{l^m} Fv_n)\|}{\gamma^{l^m}}.$$

Since  $v_n \in C$  for all  $n$  and  $P_C$  is continuous, we have  $\lim_{m \rightarrow \infty} \|v_n - P_C(v_n - \gamma^{l^m} Fv_n)\| = 0$ . From the uniform continuity of the operator  $F$  on bounded subsets of  $C$  it implies that

$$(3.10) \quad \lim_{m \rightarrow \infty} \|Fv_n - FP_C(v_n - \gamma^{l^m} Fv_n)\| = 0.$$

Combining (3.9) and (3.10) we get

$$(3.11) \quad \lim_{m \rightarrow \infty} \frac{\|v_n - P_C(v_n - \gamma^{l^m} Fv_n)\|}{\gamma^{l^m}} = 0.$$

Assume that  $z_m = P_C(x_n - \gamma^{l^m} Fv_n)$  we have

$$\langle z_m - v_n + \gamma^{l^m} Fv_n, x - z_m \rangle \geq 0 \quad \forall x \in C.$$

This implies that

$$(3.12) \quad \langle \frac{z_m - v_n}{\gamma^{l^m}}, x - z_m \rangle + \langle Fv_n, x - z_m \rangle \geq 0 \quad \forall x \in C.$$

Taking the limit  $m \rightarrow \infty$  in (3.12) and using (3.11) we obtain

$$\langle Fv_n, x - v_n \rangle \geq 0 \quad \forall x \in C,$$

which implies that  $v_n \in Sol(C, F)$  this is a contraction.  $\square$

**Lemma 3.8.** *Assume that Conditions 1–3 hold and let  $\{v_n\}$  be any sequence generated by Algorithm 2. If there exists a subsequence  $\{v_{n_k}\}$  of  $\{v_n\}$  such that  $\{v_{n_k}\}$  converges weakly to  $z \in C$  and  $\lim_{k \rightarrow \infty} \|v_{n_k} - y_{n_k}\| = 0$  then  $z \in \text{Sol}(C, F)$ .*

*Proof.* We have  $y_{n_k} = P_C(v_{n_k} - \tau_{n_k} Fv_{n_k})$  thus,

$$\langle v_{n_k} - \tau_{n_k} Fv_{n_k} - y_{n_k}, x - y_{n_k} \rangle \leq 0 \quad \forall x \in C.$$

or equivalently

$$\frac{1}{\tau_{n_k}} \langle v_{n_k} - y_{n_k}, x - y_{n_k} \rangle \leq \langle Fv_{n_k}, x - y_{n_k} \rangle \quad \forall x \in C.$$

This implies that

$$(3.13) \quad \frac{1}{\tau_{n_k}} \langle v_{n_k} - y_{n_k}, x - y_{n_k} \rangle + \langle Fv_{n_k}, y_{n_k} - v_{n_k} \rangle \leq \langle Fv_{n_k}, x - v_{n_k} \rangle \quad \forall x \in C.$$

Now, we show that

$$(3.14) \quad \liminf_{k \rightarrow \infty} \langle Fv_{n_k}, x - v_{n_k} \rangle \geq 0.$$

For showing this, we consider two possible cases. Suppose first that  $\liminf_{k \rightarrow \infty} \tau_{n_k} > 0$ . We have  $\{v_{n_k}\}$  is a bounded sequence,  $F$  is uniformly continuous on bounded subsets of  $C$ . By Lemma 2.5, we get that  $\{Fv_{n_k}\}$  is bounded. Taking  $k \rightarrow \infty$  in (3.13) since  $\|v_{n_k} - y_{n_k}\| \rightarrow 0$ , we get

$$\liminf_{k \rightarrow \infty} \langle Fv_{n_k}, x - v_{n_k} \rangle \geq 0.$$

Now, we assume that  $\liminf_{k \rightarrow \infty} \tau_{n_k} = 0$ . Assume  $z_{n_k} = P_C(v_{n_k} - \tau_{n_k} l^{-1} Fv_{n_k})$ , we have  $\tau_{n_k} l^{-1} > \tau_{n_k}$ . Applying Lemma 2.3, we obtain

$$\|v_{n_k} - z_{n_k}\| \leq \frac{1}{l} \|v_{n_k} - y_{n_k}\| \rightarrow 0 \text{ as } k \rightarrow \infty.$$

Consequently,  $z_{n_k} \rightharpoonup z \in C$ , this implies that  $\{z_{n_k}\}$  is bounded, which the uniform continuity of the operator  $A$  on bounded subsets of  $C$  it follows that

$$(3.15) \quad \|Fv_{n_k} - Fz_{n_k}\| \rightarrow 0 \text{ as } k \rightarrow \infty.$$

By the Armijo line-search rule (3.3) and the proof is similar to the inequality (3.8) we have

$$\tau_{n_k} l^{-1} \|Fv_{n_k} - FP_C(v_{n_k} - \tau_{n_k} l^{-1} Fv_{n_k})\| > \mu \|v_{n_k} - P_C(v_{n_k} - \tau_{n_k} l^{-1} Fv_{n_k})\|.$$

That is,

$$(3.16) \quad \frac{1}{\mu} \|Fv_{n_k} - Fz_{n_k}\| > \frac{\|v_{n_k} - z_{n_k}\|}{\tau_{n_k} l^{-1}}.$$

Combining (3.15) and (3.16) we obtain

$$\lim_{k \rightarrow \infty} \frac{\|v_{n_k} - z_{n_k}\|}{\tau_{n_k} l^{-1}} = 0.$$

Furthermore, we have

$$\langle v_{n_k} - \tau_{n_k} l^{-1} Fv_{n_k} - z_{n_k}, x - z_{n_k} \rangle \leq 0 \quad \forall x \in C.$$

This implies that

$$(3.17) \quad \frac{1}{\tau_{n_k} l^{-1}} \langle v_{n_k} - z_{n_k}, x - z_{n_k} \rangle + \langle Fv_{n_k}, z_{n_k} - v_{n_k} \rangle \leq \langle Fv_{n_k}, x - v_{n_k} \rangle \quad \forall x \in C.$$

Taking the limit  $k \rightarrow \infty$  in (3.17) we get

$$\liminf_{k \rightarrow \infty} \langle Fv_{n_k}, x - v_{n_k} \rangle \geq 0.$$

Therefore, the inequality (3.14) is proved. Next, we show that  $z \in \text{Sol}(C, F)$ .

Now we choose a sequence  $\{\epsilon_k\}$  of positive numbers decreasing and tending to 0. For each  $k$ , we denote by  $N_k$  the smallest positive integer such that

$$(3.18) \quad \langle Fv_{n_j}, x - v_{n_j} \rangle + \epsilon_k \geq 0 \quad \forall j \geq N_k,$$

where the existence of  $N_k$  follows from (3.14). Since  $\{\epsilon_k\}$  is decreasing, it is easy to see that the sequence  $\{N_k\}$  is increasing. Furthermore, for each  $k$ , since  $\{v_{N_k}\} \subset C$  we have  $Fv_{N_k} \neq 0$  and, setting

$$u_{N_k} = \frac{Fv_{N_k}}{\|Fv_{N_k}\|^2},$$

we have  $\langle Fv_{N_k}, v_{N_k} \rangle = 1$  for each  $k$ . Now, we can deduce from (3.18) that for each  $k$

$$\langle Fv_{N_k}, x + \epsilon_k u_{N_k} - v_{N_k} \rangle \geq 0.$$

Since the fact that  $F$  is pseudo-monotone, we get

$$\langle F(x + \epsilon_k u_{N_k}), x + \epsilon_k u_{N_k} - v_{N_k} \rangle \geq 0.$$

This implies that

$$(3.19) \quad \langle Fx, x - v_{N_k} \rangle \geq \langle Fx - F(x + \epsilon_k u_{N_k}), x + \epsilon_k u_{N_k} - v_{N_k} \rangle - \epsilon_k \langle Fx, u_{N_k} \rangle.$$

Now, we show that  $\lim_{k \rightarrow \infty} \epsilon_k u_{N_k} = 0$ . Indeed, we have  $v_{n_k} \rightarrow z$  as  $k \rightarrow \infty$ . Since  $F$  satisfies the condition (3.4). We have

$$0 < \|Fz\| \leq \liminf_{k \rightarrow \infty} \|F_{n_k}\| \quad (\text{note that } Fz \neq 0 \text{ otherwise, } z \text{ is a solution}).$$

Since  $\{v_{N_k}\} \subset \{v_{n_k}\}$  and  $\epsilon_k \rightarrow 0$  as  $k \rightarrow \infty$ , we obtain

$$0 \leq \limsup_{k \rightarrow \infty} \|\epsilon_k u_{N_k}\| = \limsup_{k \rightarrow \infty} \left( \frac{\epsilon_k}{\|Fv_{N_k}\|} \right) \leq \frac{\limsup_{k \rightarrow \infty} \epsilon_k}{\liminf_{k \rightarrow \infty} \|Fv_{n_k}\|} = 0,$$

which implies that  $\lim_{k \rightarrow \infty} \epsilon_k u_{N_k} = 0$ .

Now, letting  $k \rightarrow \infty$ , then the right hand side of (3.19) tends to zero by  $F$  is uniformly continuous,  $\{v_{N_k}\}, \{u_{N_k}\}$  are bounded and  $\lim_{k \rightarrow \infty} \epsilon_k u_{N_k} = 0$ . Thus, we get

$$\liminf_{k \rightarrow \infty} \langle Fx, x - v_{N_k} \rangle \geq 0.$$

Hence, for all  $x \in C$  we have

$$\langle Fx, x - z \rangle = \lim_{k \rightarrow \infty} \langle Fx, x - v_{N_k} \rangle = \liminf_{k \rightarrow \infty} \langle Fx, x - v_{N_k} \rangle \geq 0.$$

By Lemma 2.5 we obtain  $z \in \text{Sol}(C, F)$  and the proof is complete.  $\square$

**Remark 3.2.** When the function  $F$  is monotone, it is not necessary to impose the sequential weak continuity on  $F$ .

**Theorem 3.1.** Assume that Conditions 1–3 hold. Then any sequence  $\{v_n\}$  generated by Algorithm 2 converges weakly to an element of  $\text{Sol}(C, F)$ .

*Proof.* Let  $u^*$  be a certain solution of (VIP). We divide the proof into two claims.

**Claim 1.** There exists  $N_1 \in \mathbb{N}$  such that

$$\|v_{n+1} - u^*\|^2 \leq \|v_n - u^*\|^2 - (1 - \mu)\|y_n - v_n\|^2 - (1 - \mu)\|v_{n+1} - y_n\|^2.$$



Indeed, from  $u^* \in \text{Sol}(C, F) \subset C$ , we have

$$\begin{aligned}
 \|v_{n+1} - u^*\|^2 &= \|P_{T_n}(v_n - \tau_n F y_n) - P_{T_n} u^*\|^2 \leq \langle v_{n+1} - u^*, v_n - \tau_n F y_n - u^* \rangle \\
 &= \frac{1}{2} \|v_{n+1} - u^*\|^2 + \frac{1}{2} \|v_n - \tau_n F y_n - u^*\|^2 - \frac{1}{2} \|v_{n+1} - v_n + \tau_n F y_n\|^2 \\
 &= \frac{1}{2} \|v_{n+1} - u^*\|^2 + \frac{1}{2} \|v_n - u^*\|^2 + \frac{1}{2} \tau_n^2 \|F y_n\|^2 - \langle v_n - u^*, \tau_n F y_n \rangle \\
 &\quad - \frac{1}{2} \|v_{n+1} - v_n\|^2 - \frac{1}{2} \tau_n^2 \|F y_n\|^2 - \langle v_{n+1} - v_n, \tau_n F y_n \rangle \\
 &= \frac{1}{2} \|v_{n+1} - u^*\|^2 + \frac{1}{2} \|v_n - u^*\|^2 - \frac{1}{2} \|v_{n+1} - v_n\|^2 - \langle v_{n+1} - u^*, \tau_n F y_n \rangle.
 \end{aligned}$$

This implies that

$$(3.20) \quad \|v_{n+1} - u^*\|^2 \leq \|v_n - u^*\|^2 - \|v_{n+1} - v_n\|^2 - 2\langle v_{n+1} - u^*, \tau_n F y_n \rangle.$$

Since  $u^*$  is a solution of the problem (VI), we have  $\langle F u^*, x - u^* \rangle \geq 0$  for all  $x \in C$ . By the pseudomonotonicity of  $F$  on  $C$ , we have  $\langle F x, x - u^* \rangle \geq 0$  for all  $x \in C$ . Taking  $x := y_n \in C$ , we get

$$\langle F y_n, u^* - y_n \rangle \leq 0.$$

Thus we have

$$(3.21) \quad \langle F y_n, u^* - v_{n+1} \rangle = \langle F y_n, u^* - y_n \rangle + \langle F y_n, y_n - v_{n+1} \rangle \leq \langle F y_n, y_n - v_{n+1} \rangle.$$

From (3.20) and (3.21), we obtain

$$\begin{aligned}
 \|v_{n+1} - u^*\|^2 &\leq \|v_n - u^*\|^2 - \|v_{n+1} - v_n\|^2 + 2\tau_n \langle F y_n, y_n - v_{n+1} \rangle \\
 &= \|v_n - u^*\|^2 - \|v_{n+1} - y_n\|^2 - \|y_n - v_n\|^2 - 2\langle v_{n+1} - y_n, y_n - v_n \rangle \\
 &\quad + 2\tau_n \langle F y_n, y_n - v_{n+1} \rangle \\
 (3.22) \quad &= \|v_n - u^*\|^2 - \|v_{n+1} - y_n\|^2 - \|y_n - v_n\|^2 + 2\langle v_n - \tau_n F y_n - y_n, v_{n+1} - y_n \rangle.
 \end{aligned}$$

Since  $y_n = P_C(v_n - \tau_n F v_n)$  and  $v_{n+1} \in C$ , we have

$$\begin{aligned}
 2\langle v_n - \tau_n F y_n - y_n, v_{n+1} - y_n \rangle &= 2\langle v_n - \tau_n F v_n - y_n, v_{n+1} - y_n \rangle + 2\tau_n \langle F v_n - F y_n, v_{n+1} - y_n \rangle \\
 (3.23) \quad &\leq 2\tau_n \langle F v_n - F y_n, v_{n+1} - y_n \rangle.
 \end{aligned}$$

Combining (3.3) and (3.23), we obtain

$$(3.24) \quad 2\langle v_n - \tau_n F y_n - y_n, v_{n+1} - y_n \rangle \leq \mu \|v_n - y_n\|^2 + \mu \|y_n - v_{n+1}\|^2.$$

Substituting (3.24) into (3.22), we obtain

$$\|v_{n+1} - u^*\|^2 \leq \|v_n - u^*\|^2 - (1 - \mu) \|y_n - v_n\|^2 - (1 - \mu) \|v_{n+1} - y_n\|^2.$$

**Claim 2.** Now, we show that  $\{v_n\}$  converges weakly to an element of  $\text{Sol}(C, F)$ . Thanks to **Claim 1** we have

$$\|v_{n+1} - u^*\| \leq \|v_n - u^*\|.$$

This implies that for all  $u^* \in \text{Sol}(C, F)$  then  $\lim_{n \rightarrow \infty} \|v_n - u^*\|$  exists, thus the sequence  $\{v_n\}$  is bounded. Consequently,  $\{v_n\}$  is bounded.

On the other hand, according to **Claim 1**, we get

$$(1 - \mu) \|v_n - y_n\|^2 \leq \|v_n - u^*\|^2 - \|v_{n+1} - u^*\|^2$$

which implies that

$$(3.25) \quad \lim_{n \rightarrow \infty} \|v_n - y_n\| = 0.$$

Since  $\{v_n\}$  is a bounded sequence, there exists the subsequence  $\{v_{n_k}\}$  of  $\{v_n\}$  such that  $\{v_{n_k}\}$  converges weakly to  $z \in C$ . It implies from Lemma 3.8 and (3.25) that  $z \in \text{Sol}(C, F)$ .

Therefore, we showed that:

- i) For every  $u^* \in \text{Sol}(C, A)$ , then  $\lim_{n \rightarrow \infty} \|v_n - u^*\|$  exists;
- ii) Every sequential weak cluster point of the sequence  $\{v_n\}$  is in  $\text{Sol}(C, F)$ .

By Lemma 2.6 the sequence  $\{v_n\}$  converges weakly to an element of  $\text{Sol}(C, F)$ .  $\square$

**Remark 3.3.** Our results are more general some related results in the literature and hence might be applied for a wider class of mappings. For example, we next present the advantage of our method compared with the recent result [31, Theorem 3.1].

1. In Theorem 3.1,  $F : C \rightarrow H$  is assumed to be uniformly continuous on bounded subsets instead of Lipschitz continuous in [31].

2. Note that in our work we use condition (3.4), which is strictly weaker than the sequential weak continuity of the operator  $F$ , an assumption which has frequently been used in recent articles [2, 31] on pseudomonotone variational inequality problems.

**3.2. Convergence rate.** In this section, we provide a result on the convergence rate of the iterative sequence generated by Algorithm 2.

**Theorem 3.2.** Assume that  $F : C \rightarrow H$  is  $L$ -Lipschitz continuous and  $\delta$ -strongly pseudomonotone on  $C$ . Then the sequence  $\{x_n\}$  generated by Algorithm 2 converges strongly to  $u^*$  with a  $Q$ -linear rate, where  $u^*$  is a unique solution of VI.

*Proof.* We have  $\langle Fu^*, y_n - u^* \rangle \geq 0$ , by the strong pseudomonotonicity of  $F$  on  $C$  we get

$$(3.26) \quad \langle Fy_n, y_n - u^* \rangle \geq \delta \|y_n - u^*\|^2.$$

Using (3.26), we have

$$(3.27) \quad \langle Fy_n, u^* - v_{n+1} \rangle = \langle Fy_n, u^* - y_n \rangle + \langle Fy_n, y_n - v_{n+1} \rangle \leq -\delta \|y_n - u^*\|^2 + \langle Fy_n, y_n - v_{n+1} \rangle.$$

Substituting (3.27) into (3.20), we obtain

$$\|v_{n+1} - u^*\|^2 \leq \|v_n - u^*\|^2 - \|v_{n+1} - v_n\|^2 - 2\tau_n \delta \|y_n - u^*\|^2 + 2\tau_n \langle Fy_n, y_n - v_{n+1} \rangle.$$

Thus

$$(3.28) \quad \begin{aligned} \|v_{n+1} - u^*\|^2 &\leq \|v_n - u^*\|^2 - \|v_{n+1} - v_n\|^2 - 2\tau_n \delta \|y_n - u^*\|^2 + 2\tau_n \langle Fy_n, y_n - v_{n+1} \rangle \\ &= \|v_n - u^*\|^2 - \|v_{n+1} - y_n\|^2 - \|y_n - v_n\|^2 - 2\langle v_{n+1} - y_n, y_n - v_n \rangle \\ &\quad - 2\tau_n \delta \|y_n - u^*\|^2 + 2\tau_n \langle Fy_n, y_n - v_{n+1} \rangle \\ &= \|v_n - u^*\|^2 - \|v_{n+1} - y_n\|^2 - \|y_n - v_n\|^2 - 2\tau_n \delta \|y_n - u^*\|^2 \\ &\quad + 2\langle v_n - \tau_n Fy_n - y_n, v_{n+1} - y_n \rangle. \end{aligned}$$

Substituting (3.24) into (3.28), we obtain

$$(3.29) \quad \|v_{n+1} - u^*\|^2 \leq \|v_n - u^*\|^2 - (1 - \mu) \|y_n - v_n\|^2 - (1 - \mu) \|v_{n+1} - y_n\|^2 - 2\tau_n \delta \|y_n - u^*\|^2.$$

Now we show that  $\tau_n > \frac{\mu l}{L}$  for all  $n$ . Indeed, by the search rule (3.3), we know that  $\frac{\tau_n}{l}$  must violate inequality (3.3), i.e.,

$$\langle Fy_n - Fv_n, y_n - v_{n+1} \rangle > \frac{\mu}{2\tau_n} (\|v_n - y_n\|^2 + \|y_n - v_{n+1}\|^2).$$

This follows that

$$L\|y_n - v_n\|\|y_n - v_{n+1}\| > \frac{\mu}{2\tau_n}(\|v_n - y_n\|^2 + \|y_n - v_{n+1}\|^2).$$

Thus

$$\frac{L}{2}(\|y_n - v_n\|^2 + \|y_n - v_{n+1}\|^2) > \frac{\mu}{2\tau_n}(\|v_n - y_n\|^2 + \|y_n - v_{n+1}\|^2).$$

This implies that

$$(3.30) \quad \tau_n > \frac{\mu l}{L} \quad \forall n.$$

Combining (3.29) and (3.30) we get

$$(3.31) \quad \begin{aligned} \|v_{n+1} - u^*\|^2 &\leq \|v_n - u^*\|^2 - (1 - \mu)\|y_n - v_n\|^2 - (1 - \mu)\|v_{n+1} - y_n\|^2 - 2\frac{\mu l}{L}\delta\|y_n - u^*\|^2 \\ &\leq \|v_n - u^*\|^2 - (1 - \mu)\|y_n - v_n\|^2 - 2\frac{\mu l}{L}\delta\|y_n - u^*\|^2. \end{aligned}$$

Let  $\beta := \min\left\{\frac{1 - \mu}{2}, \frac{\mu l}{L}\delta\right\}$ . From (3.31), we get

$$\begin{aligned} \|v_{n+1} - u^*\|^2 &\leq \|v_n - u^*\|^2 - 2\beta\|y_n - v_n\|^2 - 2\beta\|y_n - u^*\|^2 \\ &\leq \|v_n - u^*\|^2 - \beta\|v_n - u^*\|^2 \\ &= (1 - \beta)\|v_n - u^*\|^2. \end{aligned}$$

This implies that

$$\|v_{n+1} - u^*\| \leq \sqrt{(1 - \beta)}\|v_n - u^*\|.$$

Since  $\sqrt{(1 - \beta)} \in (0, 1)$ , it implies that the sequence  $\{v_n\}$  converges strongly to  $u^*$  with a Q-linear rate.  $\square$

#### 4. NUMERICAL ILLUSTRATIONS

In this section, we perform numerical experiments to show the behaviors of Algorithm 2 and compare it with other algorithms. All the numerical experiments are performed on an HP laptop with Intel(R) Core(TM)i5-6200U CPU 2.3GHz with 4 GB RAM. All the programs are written in Matlab2015a. The followings are the experiment in detail. **Example 4.1** Let  $F(x) := Mx + q$  where

$$M = BB^T + C + D,$$

and  $B$  is an  $m \times m$  matrix,  $C$  is an  $m \times m$  skew-symmetric matrix,  $D$  is an  $m \times m$  diagonal matrix, whose diagonal entries are nonnegative (so  $M$  is positive semidefinite),  $q$  is a vector in  $\mathbb{R}^m$ . The feasible set  $C \subset \mathbb{R}^m$  is a box constraints in  $\mathbb{R}^m$  defined by

$$C := \{x \in \mathbb{R}^m : 0 \leq x \leq 1\}.$$

It is clear that  $F$  is monotone and Lipschitz-continuous with constant  $L = \|M\|$ . Let  $q = 0$ . Then, we obtain the solution set  $\Gamma = \{0\}$ . The parameters are chosen as follows:

Algorithm 2:  $\gamma = 0.01$ ,  $\mu = 0.5$ ,  $l = 0.5$ ,  $\beta_n = \frac{1}{n+2}$

Algorithm 3.1 in [2]:  $f(x) = 0.5x$ ,  $\gamma = 0.01$ ,  $\mu = 0.5$ ,  $l = 0.5$ ,  $\beta_n = \frac{1}{n+2}$ .

Algorithm 3.2 in [13]:  $f(x) = 0.5x$ ,  $\lambda = 0.01$ ,  $\gamma = 1$ ,  $\mu = 0.5$ ,  $l = 0.5$ ,  $\beta_n = \frac{1}{n+2}$ .

Algorithm 3.1 in [30]:  $\tau_0 = 0.001$ ,  $\mu = 0.5$ ,  $\alpha_n = \frac{1}{n+2}$ ,  $\beta_n = 0.99(1 - \alpha_n)$ .

For experiment, all entries of  $B$ ,  $C$  and  $D$  are generated randomly from a normal distribution with mean zero and unit variance. We use stopping rule  $\|x_n\| < 10^{-7}$  or

$Iter \geq 1000$  for all algorithms. The numerical results are described in Table 1 and Fig. 1, Fig. 2.

TABLE 1. Numerical results obtained by other algorithms

Methods	m=100			m=200		
	Sec.	Iter.	Error.	Sec.	Iter.	Error.
Algorithm 2	0.04	82	8.6284e-08	0.0980	75	9.3728e-08
Algorithm 3.1 in [2]	1.016	1000	0.0014	3.8010	1000	0.0023
Algorithm 3.2 in [13]	0.2410	777	9.9722e-08	1.5350	1000	2.2974e-07
Algorithm 3.1 in [30]	0.1470	1000	4.8767e-04	0.4960	1000	5.6404e-04

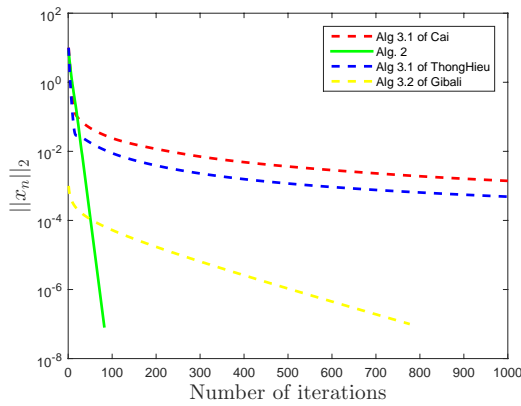


FIGURE 1. Comparison of all algorithms with  $m = 100$

Fig. 1-2 and Table 1 demonstrate that Algorithm 2 performs better than Algorithm 3.1 in [2], Algorithm 3.2 in [13] and Algorithm 3.1 in [30].

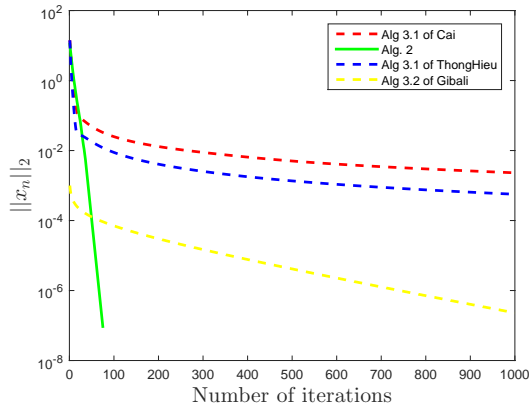


FIGURE 2. Comparison of all algorithms with  $m = 200$

## 5. CONCLUSIONS

In this paper we proposed a modification of extragradient method for solving non-Lipschitzian pseudo-monotone variational inequalities in real Hilbert spaces. Under suitable conditions we establish weak convergence and convergence rate of the proposed scheme. Our work extend and generalize some existing results in the literature.

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