CARPATHIAN J. MATH.
Volume **38** (2022), No. 2,
Pages 517 - 522

Online version at https://semnul.com/carpathian/
Print Edition: ISSN 1584 - 2851; Online Edition: ISSN 1843 - 4401
DOI: https://doi.org/10.37193/CJM.2022.02.20

On the maximum modulus principle and the identity theorem in arbitrary dimension

VLAD TIMOFTE

ABSTRACT. We prove an identity theorem for Gâteaux holomorphic functions on polygonally connected 2-open sets, which yields a very general maximum norm principle and a sublinear "max-min" principle. All results apply in particular to vector-valued functions which are holomorphic (in any sense that implies Gâteaux holomorphy) on domains in Hausdorff locally convex spaces.

1. INTRODUCTION

For any domain (that is, a connected open set) $\Omega \subset \mathbb{C}$ and holomorphic function $f \in \mathcal{H}(\Omega)$, the classical maximum modulus principle (MMP) states that *if* $|f(\cdot)|$ *has a maximum on* Ω*, then* f *is constant*. The conclusion still holds if we only require $|f(\cdot)|$ to have a local maximum; in this case f is constant on a neighborhood of some point, and hence on Ω, by the classical identity theorem. For a complex Banach (or just normed) space Y and $f \in \mathcal{H}(\Omega, Y)$ such that $\|f(\cdot)\|$ has a maximum, it is well-known ([5, p.230]) that $\|f(\cdot)\|$ is constant, while f may be nonconstant. In [12, Th. 3.1] Thorp and Whitley have shown that MMP holds for holomorphic functions $f \in \mathcal{H}(\Omega, Y)$, if and only if Y is a strictly c-convex Banach space (also see [3, Th. 4.4, p.164]).

Let us recall that a point e in a convex subset K of a complex Banach space Y is said to be *complex extreme for* K, if and only if $e + B_{\mathbb{C}}(0,1) \cdot u \subset K$ (with $u \in Y$) implies $u = 0$. The space Y is called *strictly c-convex*, if and only if every point of its unit sphere is complex extremal for the closed unit ball. Since these notions were introduced by [12], various Banach spaces have been investigated for characterizing the complex extreme points of the closed unit ball or for strict c-convexity (weaker than the usual strict convexity) or uniform c-convexity. For domain $\Omega \subset \mathbb{C}$ and arbitrary complex Banach space Y, the MMP from [12, Th. 3.1] was generalized by Globevnik [6] through a complete characterization of the functions $f \in \mathcal{H}(\Omega, Y)$ with constant norm $\|f(\cdot)\|$. Several other results on operator-valued holomorphic functions with constant norm can be found in Globevnik and Vidav [7], Daniluk [2], and Rovnyak [10]. For matrix-valued holomorphic functions including a spectral version of MMP we refer the reader to Condori [1]. All results from the cited papers hold for holomorphic functions on a domain $\Omega \subset \mathbb{C}$.

The following straightforward version of MMP for a domain $\Omega \subset \mathbb{C}^n$ and a strictly c-convex Banach space Y seems to be new (we found no reference for such a result; since it is obvious, we assume it to be "folklore"):

Theorem. *If* $f \in \mathcal{H}(\Omega, Y)$ *and* $\|f(\cdot)\|$ *has a local maximum, then* f *is constant*.

Indeed, if the local maximum is attained at $a \in \Omega$, then for a sufficiently small open ball $B \subset \Omega$ centered at a the restriction $\|f(\cdot)|_B\|$ has a maximum, and so, according to the cited

Received: 16.03.2021. In revised form: 21.07.2021. Accepted: 28.07.2021
2010 *Mathematics Subject Classification.* 32A10, 32K12, 46G20.
Key words and phrases. *maximum modulus, identity theorem, strictly c-convex.*
Corresponding author: Vlad Timofte; vlad.timofte@imar.ro

result from [12], $f|_{B\cap L} \equiv f(a)$ for every complex one-dimensional linear variety $L \subset \mathbb{C}^n$ containing a. Hence $f|_B \equiv f(a)$ and the conclusion follows by the identity theorem.

In this paper we prove a maximum norm principle for vector-valued Gâteaux holomorphic functions on polygonally connected 2-open sets in arbitrary complex vector spaces; this setting includes all versions of holomorphy in infinite dimension. In order to accomplish this, we first prove a needed identity theorem which is of independent interest, as it will be a key ingredient in the proof of the Hartogs extension theorem for Gâteaux holomorphic functions on finitely open sets in arbitrary dimension (forthcoming paper).

2. Linear cuts and the identity theorem

Setting. From now on we consider a complex vector space X and a complex Hausdorff locally convex space $Y \neq \{0\}$.

Let us first define or recall some needed notions; these derive from the vector space structure of X and will allow us to state our results in full generality.

On complex linear varieties of finite dimension we always consider the (unique) Euclidean topology.

Definition 2.1. *A set $A \subset X$ is called*

- **(a):** d-open *(with $d \in \{1, 2\}$), if and only if for every linear variety $L \subset X$ of complex dimension at most d, the set $A \cap L$ is open in L.*
- **(b):** sufficiently bounded, *if and only if for every $a \in \Omega$, there exists a complex one-dimensional linear variety $L_a \subset X$, such that $a \in L_a$ and $\Omega \cap L_a$ is bounded in L_a.*
- **(c):** polygonally connected, *if and only if any to points from A can be joined by a polygonal chain $\Lambda \subset A$.*
- **(d):** real-absorbing, *if and only if for every $x \in X$, there exists $\varepsilon > 0$, such that $[0, \varepsilon]\cdot x \subset A$ (this condition holds in particular for 1-open sets containing the origin of X).*

The above notions (among which the first two are new) do not require any topological structure on X. Nonetheless, let us note that a subset $A \subset X$ is d-open, if and only if A is open in the translation invariant topology $\tau_{(d)}$ defined by all d-open subsets of X.

Remark 2.1. *Both topologies $\tau_{(1)} \geq \tau_{(2)}$ are stronger than any linear topology on X. For open sets in topological vector spaces, connectedness is equivalent to polygonal connectedness.*

Even in normed spaces of finite dimension, d-open sets may not be open:

Example 2.1 (d-open sets which are not open)**.**

- **(a):** *For arbitrary set $A \subset \mathbb{C}$, let*
$$K_A = \{(z, z^2, z^3) \in \mathbb{C}^3 \mid z \in A\} \subset \mathbb{C}^3.$$
Then $\mathbb{C}^3 \setminus K_A$ is 2-open. For $A = \mathbb{Q}$, the set $\mathbb{C}^3 \setminus K_{\mathbb{Q}}$ is not open.
- **(b):** *For arbitrary infinite set T, let us consider the direct sum vector space*
$$\mathbb{C}^{(T)} = \{u : T \to \mathbb{C} \mid u^{-1}(\mathbb{C} \setminus \{0\}) \text{ is finite}\}$$
equipped with the supremum norm, a function $\rho : T \to]0, \infty[$, and the set
$$\Omega_\rho = \{u \in \mathbb{C}^{(T)} \mid |u| < \rho \text{ pointwise}\} \subset \mathbb{C}^{(T)}.$$
Then Ω_ρ is 2-open. If $\inf \rho(T) = 0$, then $\mathring{\Omega}_\rho = \emptyset$, and so Ω_ρ is not open.

The usual definition of Gâteaux holomorphy requires the domain of the function to be a finitely open set (that is, every intersection with a linear variety $L \subset X$ of finite dimension is open in L; see [3, Def. 2.2, p.54], [4, Def. 3.1, p.144], [8, Def. 2.3.1, p.35]). Since our results hold in a more general setting, we next slightly modify this definition as follows:

Definition 2.2 (Gâteaux holomorphy)**.**

(a): *A function $f:\Omega\to Y$ defined on a* 1*-open set $\Omega\subset X$ is called* Gâteaux holomorphic, *if and only if for all $a\in\Omega$, $v\in X$, and $\varphi\in Y^*$ (the continuous dual of Y), there exists $r>0$, such that the function*

$$B_{\mathbb{C}}(0,r)\ni\lambda\mapsto(\varphi\circ f)(a+\lambda v)\in\mathbb{C}$$

is holomorphic. Let $\mathcal{H}_{\mathrm{G}}(\Omega,Y)$ denote the complex vector space consisting of all such Y-valued functions on Ω. Set $\mathcal{H}_{\mathrm{G}}(\Omega):=\mathcal{H}_{\mathrm{G}}(\Omega,\mathbb{C})$. Obviously,

$$f\in\mathcal{H}_{\mathrm{G}}(\Omega,Y)\iff\varphi\circ f\in\mathcal{H}_{\mathrm{G}}(\Omega)\ \textit{for every}\ \varphi\in Y^*.$$

(b): *If Ω is an open subset of a Hausdorff complex locally convex space, we may consider the vector spaces of all* holomorphic functions *(*Fréchet*)*

$$\mathcal{H}(\Omega,Y):=\{f\in\mathcal{H}_{\mathrm{G}}(\Omega,Y)\,|\,f\ \textit{is continuous}\},\quad\mathcal{H}(\Omega):=\mathcal{H}(\Omega,\mathbb{C}).$$

(c): *For linear variety of finite dimension $L\subset X$ (which is a translated vector subspace) and* 1*-open subset $D\subset L$, we define in the natural way the vector spaces $\mathcal{H}_{\mathrm{G}}(D,Y)$ and $\mathcal{H}_{\mathrm{G}}(D)$. For open subset $D\subset L$, we may write these spaces as $\mathcal{H}(D,Y)$ and $\mathcal{H}(D)$.*

Let us note that every function $f\in\mathcal{H}_{\mathrm{G}}(\Omega,Y)$ is Gâteaux differentiable, that is, the limit $\lim_{\mathbb{C}\ni\lambda\to0}\frac{f(a+\lambda v)-f(a)}{\lambda}$ exists in the completion of Y, for all $a\in\Omega$ and $v\in X$ (this follows at once from Dineen [4, Lemma 3.3, p.149]).

A widely-known identity theorem (see for instance Scheidemann [11, p.10]) states that if $f\in\mathcal{H}(\Omega)$ vanishes on an open subset of the domain $\Omega\subset\mathbb{C}^n$, then $f\equiv0$. Our maximum norm principle (Theorem 3.2 below) requires a much more general identity theorem. Since every topology $\tau_{(d)}$ on X contains at least all open sets from all possible linear topologies on X, the requirement on a set $\Omega\subset X$ to be 2-open is very convenient.

Theorem 2.1 (identity)**.** *Let a polygonally connected* 2*-open set $\Omega\subset X$ and a subset $C\subset\Omega$, such that $C-c$ is real-absorbing for some $c\in C$. Then*

$$f(\Omega)-f(c)\subset\overline{\mathrm{Sp}}(f(C)-f(c)),\quad\textit{for every}\ f\in\mathcal{H}_{\mathrm{G}}(\Omega,Y)$$

(in particular, $f\equiv0$ if and only if $f|_C\equiv0$).

Proof. Let us first prove the last equivalence. Assume $f|_C\equiv0$. For $c\in C$ as in the theorem, let us consider a linear segment $[c,a]\subset\Omega$. We claim that $f|_A\equiv0$ for some 1-open subset $A\subset\Omega$, such that $a\in A$ (then $A-a$ is real-absorbing). Set

$$A:=\{x\in X\,|\,[c,x]\subset\Omega\}.$$

Hence $a\in A\subset\Omega$. Suppose there exists a complex one-dimensional linear variety $L\subset X$, such that $A\cap L$ is not open in L. Consequently, there exists a sequence $(x_n)_{n\in\mathbb{N}}\subset L\setminus A$, which converges in L to some $a_0\in A\cap L$. Hence $[c,a_0]\subset\Omega$. For every $n\in\mathbb{N}$ we have $x_n\notin A$, that is, $\xi_n:=(1-t_n)c+t_nx_n\notin\Omega$ for some $t_n\in[0,1]$. By taking a convergent subsequence, we may assume that $\lim_{n\to\infty}t_n=s\in[0,1]$. Clearly, $L\cup\{c\}\subset L'$ for some complex two-dimensional linear variety $L'\subset X$. As $\Omega\cap L'$ is open in L' and $(\xi_n)_{n\in\mathbb{N}}\subset L'\setminus\Omega$, a passage to the limit in L' yields $(1-s)c+sa_0\in L'\setminus\Omega$, which contradicts $[c,a_0]\subset\Omega$. We conclude that A is 1-open, and hence that $A-a$ is a real-absorbing set. In order to show that $f|_A\equiv0$, let us fix $x\in A$ and $\theta\in Y^*_{\mathbb{R}}$ (the continuous dual of Y considered as a real normed space). Thus $[c,x]\subset\Omega$ and the function

$$g:[0,1]\to\mathbb{R},\qquad g(t)=(\theta\circ f)((1-t)c+tx),$$

is real-analytic. As $C-c$ is a real-absorbing set, we have $[0,\varepsilon]\cdot(x-c)\subset C-c$ for some $\varepsilon\in]0,1]$, and so $g|_{[0,\varepsilon]}\equiv0$. By the identity theorem for real-analytic functions we get

$g|_{[0,1]} \equiv 0$. It follows that $\theta(f(x)) = 0$ for every $\theta \in Y^*_{\mathbb{R}}$, which yields $f(x) = 0$. Our claim is proved. Since Ω is polygonally connected, an easy induction (on the number of linear segments from a polygonal chain in Ω joining c to other points $x \in \Omega$) based on the above claim shows that $f \equiv 0$. We thus have proved the equivalence. In order to show the inclusion for $f(\Omega)$, let the function $g := f - f(c) \in \mathcal{H}_{\mathrm{G}}(\Omega, Y)$, the closed vector subspace $Y_0 := \overline{\mathrm{Sp}}(g(C)) \subset Y$, the quotient Hausdorff locally convex space $Y/Y_0 = \{\hat{y} \,|\, y \in Y\}$, and the standard continuous linear surjection $s : Y \to Y/Y_0$. Since $s \circ g \in \mathcal{H}_{\mathrm{G}}(\Omega, Y/Y_0)$ and $(s \circ g)|_C \equiv \hat{0}$, by the already proved equivalence we get $s \circ g \equiv \hat{0}$, that is, $g(\Omega) \subset Y_0$. This yields $f(\Omega) - f(c) \subset \overline{\mathrm{Sp}}(f(C) - f(c))$. □

3. THE MAXIMUM NORM AND MAX-MIN SEMINORM PRINCIPLES.

The identity theorem yields a maximum norm principle for Gâteaux holomorphic functions on polygonally connected 2-open sets.

Theorem 3.2 (maximum norm principle). *Assume $(Y, \|\;\|)$ is a strictly c-convex Banach space. Let a polygonally connected 2-open set $\Omega \subset X$ and $f \in \mathcal{H}_{\mathrm{G}}(\Omega, Y)$. If $\|f(\cdot)\|$ has a $\tau_{(1)}$-local maximum, then f is constant.*

Proof. Assume $\|f(\cdot)\|$ has a $\tau_{(1)}$-local maximum at $c \in \Omega$. Let Γ_c denote the set of all complex one-dimensional linear varieties $L \subset X$, such that $c \in L$. For every $L \in \Gamma_c$, by the $\tau_{(1)}$-local maximum hypothesis there exists a connected open neighborhood $C_L \subset \Omega \cap L$ of c in L, such that $\|f(c)\| = \max_{x \in C_L} \|f(x)\|$. As $f|_{C_L} \in \mathcal{H}(C_L, Y)$, by [12, Th. 3.1] we see that $f|_{C_L} \equiv f(c)$. Set $C := \bigcup_{L \in \Gamma_c} C_L \subset \Omega$. Since $C - c$ is an absorbing set (hence also real-absorbing) and $f|_C \equiv f(c)$, by Theorem 2.1 we conclude that $f \equiv f(c)$. □

For $\Omega \subset \mathbb{C}$ and strictly convex Banach space Y, a direct proof (not using the theorem from [12]) of the maximum norm principle can be found in [9, Th 1].

Corollary 3.1 (minimum modulus principle). *Let a polygonally connected 2-open set $\Omega \subset X$ and $f \in \mathcal{H}_{\mathrm{G}}(\Omega)$. If $|f|$ has a $\tau_{(1)}$-local minimum at $c \in \Omega$ and $f(c) \neq 0$, then f is constant.*

Proof. Let us consider Γ_c as in the proof of Theorem 3.2. For every $L \in \Gamma_c$, by the $\tau_{(1)}$-local minimum hypothesis there is a connected open neighborhood $C_L \subset \Omega \cap L$ of c in L, such that $\min_{x \in C_L} |f(x)| = |f(c)| > 0$. As $g := \frac{1}{f|_{C_L}} \in \mathcal{H}(C_L)$ and $|g|$ has a maximum at c, the classical maximum modulus principle yields $g \equiv g(c)$, that is, $f|_{C_L} \equiv f(c)$. As in the proof of Theorem 3.2 we conclude that $f \equiv f(c)$. □

Without the strict c-convexity of Y, we still can show that if $\|f(\cdot)\|$ has a $\tau_{(1)}$-local maximum at $c \in \Omega$, then c is also a global minimum point and $f(\Omega)$ has empty interior (and therefore is not a domain), and if $f(c) \neq 0$, then f vanishes nowhere on Ω. A similar result holds with the norm replaced by a continuous sublinear functional $p : Y \to \mathbb{R}$ (subadditive and positively homogeneous), and in particular by a continuous seminorm on Y, which is assumed to be a locally convex space.

Theorem 3.3 (sublinear max-min principle). *Let a polygonally connected 2-open set $\Omega \subset X$, a function $f \in \mathcal{H}_{\mathrm{G}}(\Omega, Y)$, and a continuous sublinear functional $p \not\equiv 0$ on Y. If $p \circ f$ has a $\tau_{(1)}$-local maximum at $c \in \Omega$, then $p \circ f$ also has a global minimum at c and $\overset{\circ}{\overline{f(\Omega)}} = \emptyset$. In particular, if $p(f(c)) > 0$, then $0 \notin f(\Omega)$.*

Proof. As $p \not\equiv 0$, according to the Hahn-Banach theorem, there exists $\varphi \in Y^*_{\mathbb{R}} \setminus \{0\}$ ($Y^*_{\mathbb{R}}$ denotes the continuous dual of the real locally convex space $Y_{\mathbb{R}}$), such that $\varphi(f(c)) = p(f(c))$ and $\varphi \leq p$ on Y. For the associated $\mathbb{C}$-linear functional $\tilde{\varphi} \in Y^*$ defined by $\tilde{\varphi}(y) = \varphi(y) - \mathrm{i}\varphi(\mathrm{i}y)$, we claim that $g := \tilde{\varphi} \circ f \in \mathcal{H}_{\mathrm{G}}(\Omega)$ is constant. For Γ_c as in the proof of Theorem 3.2, let us fix $L \in \Gamma_c$. By the $\tau_{(1)}$-local maximum hypothesis there exists a connected open neighborhood $C_L \subset \Omega \cap L$ of c in L, such that $p(f(c)) = \max_{x \in C_L} p(f(x))$. For every $x \in C_L$, we have

$$\varphi(f(x)) \leq p(f(x)) \leq p(f(c)) = \varphi(f(c)).$$

By the open mapping theorem it follows that $g|_{C_L} \in \mathcal{H}(C_L)$ is constant (its real part $\varphi \circ f|_{C_L}$ has a maximum at c, and so $g(C_L)$ is not a neighborhood of $g(c)$). Hence $g|_{C_L} \equiv g(c)$. Set $C := \bigcup_{L \in \Gamma_c} C_L \subset \Omega$. Since $C - c$ is an absorbing set and $g|_C \equiv g(c)$, by Theorem 3.2 we deduce that $g \equiv g(c)$. Our claim is proved. It follows that $f(\Omega) \subset f(c) + \ker \tilde{\varphi}$, which yields $\overset{\circ}{\overline{f(\Omega)}} = \emptyset$. For every $x \in \Omega$ we have

$$p(f(x)) \geq \varphi(f(x)) = \varphi(f(c)) = p(f(c)).$$

Hence $p \circ f$ has a global minimum at c. If $p(f(c)) > 0$, then $0 \notin p(f(\Omega))$, and so $0 \notin f(\Omega)$. $\square$

There is a bijective correspondence between the set of all nonnegative continuous sublinear functionals on Y and the set of all convex open neighborhoods of the origin in Y. This leeds us to the following interpretation of Theorem 3.3.

Corollary 3.2 (convex max-min principle). *Let a polygonally connected* 2*-open set* $\Omega \subset X$*, a function* $f \in \mathcal{H}_{\mathrm{G}}(\Omega, Y)$*, and a convex open set* $V \subset Y$*.*

- **(a):** *Let a* $\tau_{(1)}$*-open set* $C \subset \Omega$*, such that* $f(C) \subset \overline{V}$ *and* $f(C) \not\subset V$ *(both conditions on* $f(C)$ *hold in particular if* $C \neq \emptyset$ *and* $f(C) \subset \partial V$*). Then* $f(\Omega) \cap V = \emptyset$*.*
- **(b):** *We have the equivalence*

$$f(\Omega) \subset \overline{V},\ f(\Omega) \cap \partial V \neq \emptyset \iff f(\Omega) \subset \partial V.$$

Proof. (a). Since $f(C) \subset \overline{V}$ and $f(C) \not\subset V$ yield $\emptyset \neq V \neq Y$, we may assume $0 \in V$. The gauge $p_V \not\equiv 0$ of V is sublinear and continuous, and

$$V = p_V^{-1}([0,1[), \qquad \overline{V} = p_V^{-1}([0,1]), \qquad \partial V = p_V^{-1}(\{1\}).$$

There exists $c \in C$, such that $f(c) \in \overline{V} \setminus V = \partial V$. Hence $p_V \circ f|_C \leq 1 = p_V(f(c))$. By Theorem 3.3 we deduce that $p_V \circ f \geq 1$, and the conclusion follows.
(b). Clearly, we only need to prove the implication "$\Rightarrow$". Assume $f(\Omega) \subset \overline{V}$ and $f(\Omega) \cap \partial V \neq \emptyset$. Applying the already proved part (a) for $C = \Omega$ yields $f(\Omega) \cap V = \emptyset$. Hence $f(\Omega) \subset \overline{V} \setminus V = \partial V$. $\square$

A well-known version of the maximum modulus principle states for a bounded domain $\Omega \subset \mathbb{C}$ and for $f \in \mathcal{H}(\Omega) \cap \mathcal{C}(\overline{\Omega})$, that $\max_{x \in \Omega} |f(x)| = \max_{x \in \partial\Omega} |f(x)|$. For unbounded domains this is no longer true (e.g. for $\Omega = \mathbb{C}$, since $\partial\Omega = \emptyset$). We next show that a similar version of the principle still holds for vector-valued functions on sufficiently bounded domains in topological vector spaces.

Theorem 3.4 (boundary sublinear maximum principle). *Assume* X *is a Hausdorff topological vector space. Let a sufficiently bounded* 1*-open set* $\Omega \subset X$*, a function* $f \in \mathcal{H}_{\mathrm{G}}(\Omega, Y) \cap \mathcal{C}(\overline{\Omega}, Y)$*, and a continuous sublinear* $p : Y \to \mathbb{R}$*. Then*

$$\sup_{x \in \Omega} p(f(x)) = \sup_{x \in \partial\Omega} p(f(x)).$$

Proof. As $p \circ f$ is continuous, we have $\sup p(f(\Omega)) = \sup p(f(\overline{\Omega})) \geq \sup p(f(\partial\Omega))$. In order to prove the converse inequality, let us fix $a \in \Omega$. Since Ω is sufficiently bounded, there exists a complex one-dimensional linear variety $L \subset X$, such that $a \in L$ and $D := \Omega \cap L$ is bounded in L. Hence in L the set D is open, $\overline{D}$ is compact, $\partial D \subset \partial\Omega \cap L$, and $f|_{\overline{D}} \in$

$\mathcal{H}(D,Y) \cap \mathcal{C}(\overline{D},Y)$. Thus $p(f(a)) \leq \sup p(f(\overline{D})) = p(f(c))$ for some $c \in \overline{D} = D \cup \partial D$. We next analyze two cases.

Case 1. If $c \in \partial D \subset \partial\Omega$, then $p(f(a)) \leq \sup p(f(\partial\Omega))$.

Case 2. If $c \in D$, let C denote the connected component of c in D. Since $p \circ f|_C$ has a global maximum at c, by Theorem 3.3 $p \circ f|_C$ also has a global minimum at c, and so $p \circ f|_{\overline{C}} \equiv p(f(c))$. Hence $p(f(a)) \leq p(f(c)) = \sup p(f(\partial C)) \leq \sup p(f(\partial\Omega))$.

As in both cases we obtained $p(f(a)) \leq \sup p(f(\partial\Omega))$ and a was arbitrary, we conclude that $\sup p(f(\Omega)) \leq \sup p(f(\partial\Omega))$. □

Example 3.2. *Let $n \geq 2$ and a nonzero linear functional $\varphi : \mathbb{C}^n \to \mathbb{C}$. Then the unbounded open set $\Omega := \{x \in \mathbb{C}^n \mid \|x\| < |\varphi(x)|^2\}$ is sufficiently bounded, and so*

$$\sup_{x\in\Omega} |f(x)| = \sup_{x\in\partial\Omega} |f(x)| \quad \textit{for every } f \in \mathcal{H}(\Omega) \cap \mathcal{C}(\overline{\Omega}).$$

Indeed, for every $a \in \Omega$ we have $\Omega \cap (a + \ker\varphi) \subset B_{\mathbb{C}^n}(0, |\varphi(a)|^2)$. Since for a complex one-dimensional linear variety $L_a \subset \mathbb{C}^n$ such that $a \in L_a \subset a + \ker\varphi$, the set $\Omega \cap L_a = B_{\mathbb{C}^n}(0, |\varphi(a)|^2) \cap L_a$ is bounded, Ω is sufficiently bounded.

Let us finally note that all results of this paper apply in particular to vector-valued functions which are holomorphic in any sense (for instance Mackey/Silva, hypoanalytic, Fréchet, or locally bounded) that implies Gâteaux holomorphy.

Acknowledgments. The author is grateful to the referees whose valuable suggestions helped to improve the exposition of the paper.

References

[1] Condori, A. A. Maximum principles for matrix-valued analytic functions. *Amer. Math. Monthly* **127** (2020), no. 4, 331–343.

[2] Daniluk, A. The maximum principle for holomorphic operator functions. *Integral Equations Operator Theory* **69** (2011), no. 3, 365–372.

[3] Dineen, S., Complex analysis in locally convex spaces. *North-Holland Mathematics Studies*, 57, North-Holland Publishing Co., Amsterdam-New York, 1981.

[4] Dineen, S. Complex analysis on infinite dimensional spaces. *Monographs in Mathematics*, Springer-Verlag, 1999.

[5] Dunford, N.; Schwartz, J. T. *Linear Operators. Vol. I.* Interscience Publishers Inc, New-York, 1958.

[6] Globevnik, J. On vector-valued analytic functions with constant norm. *Studia Math.* **53** (1975), no. 1, 29–37

[7] Globevnik, J.; Vidav, I. On operator-valued analytic functions with constant norm. *J. Functional Analysis* **15** (1974), 394–403.

[8] Hervé, M., Analyticity in infinite-dimensional spaces. *De Gruyter Studies in Mathematics*, 10, Walter de Gruyter Co., Berlin, 1989.

[9] Moore, R. L.; Trent, T. T. A strict maximum modulus theorem for certain Banach spaces. *Monatsh. Math.* **92** (1981), no. 3, 197–201.

[10] Rovnyak, J. Operator-valued analytic functions of constant norm. *Czechoslovak Math. J.* **39(114)** (1989), no. 1, 165–168.

[11] Scheidemann, V., Introduction to complex analysis in several variables. *Birkh́auser Verlag*, Basel, 2005.

[12] Thorp, E.; Robert W. *The strong maximum modulus theorem for analytic functions into a Banach space.* Proceedings of the American Mathematical Society **18** (1967), no. 4, 640–646.

INSTITUTE OF MATHEMATICS "SIMION STOILOW" OF THE ROMANIAN ACADEMY
21 CALEA GRIVIŢEI, 010702 BUCHAREST, ROMANIA
Email address: vlad.timofte@imar.ro