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Dedicated to Prof. Emeritus Mihail Megan on the occasion of his 75^{th} anniversary

Feedback null controllability for a class of semilinear control systems

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ABSTRACT. For a class of semilinear control systems we get null controllability results and estimates for the minimum time function by considering appropriate feedback laws.

1. INTRODUCTION AND PRELIMINARIES

Let X and U be two Hilbert spaces and consider the nonlinear control system described by the equation

$$y'(t) = Ay(t) + f(y(t)) + Bu(t),\ y(0) = x, \tag{1.1}$$

where $A : D(A) \subseteq X \to X$ generates a C_0-semigroup $\{S(t) : X \to X;\ t \geq 0\}$, $B : U \to X$ is a linear continuous operator, $f : X \to X$ is a given function, $x \in X$ and $u(\cdot)$ is the control.

Denote by $\mathcal{U}$ the set of all admissible controls, i.e. measurable functions taking values in $B_U(0,1)$. We denoted by $B_U(0,1)$ the closed unit ball in U. For $x \in X$ and $u \in \mathcal{U}$ consider the mild solution of equation (1.1) which satisfies the initial condition $y(0) = x$, i.e. a continuous function $y : [0,\infty) \to X$ satisfying

$$y(t) = S(t)x + \int_0^t S(t-s)\left[f(y(s)) + Bu(s)\right] ds.$$

We shall denote it by $y(\cdot, x, u)$.

Denote by $\mathcal{C}(t)$ the null controllable set at time $t > 0$, i.e. the set of all points $x \in X$ for which there exists $u \in \mathcal{U}$ with $y(t,x,u) = 0$. Set $\mathcal{C} = \bigcup_{t>0} \mathcal{C}(t)$ called the null controllable set, and define the minimum time function $\mathcal{T} : X \to [0,+\infty]$ by

$$\mathcal{T}(x) = \begin{cases} \inf\{t \geq 0;\ x \in \mathcal{C}(t)\},\ \text{if } x \in \mathcal{C} \\ +\infty,\ \text{otherwise.} \end{cases}$$

Our aim here is to get a feedback law which leads to null controllability and, further, to get estimates for the minimum time function around the target.

In [4] it was studied the semilinear system (1.1) with $B = I$, in Banach spaces. The approach used there allowed to obtain an estimate of the following type: there exists a control $u(\cdot)$ such that $\|y(t,x,u)\| \leq c(t)(\|x\| - \gamma(t))$, for some positive functions $c(\cdot)$ and $\gamma(\cdot)$. Feedback controls of the form $u(t) = -sign(y(t))$, where $sign(x) = x/\|x\|$ for $x \neq 0$ and $sign(0) = B(0,1)$, were used. The above inequality provided null controllability of

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the system and estimates for the minimum time function around the target. See also [1], [2], [5], [6].

In the linear case we mention [9], [10] when B is the identity operator, [11], [12] for the wave equation.

In the present paper we will use viability results for suitable sets and evolution equations. To our knowledge, this technique of studying controllability problems by means of viability was first used in [8] (see also [7]). More precisely, viability leads to the existence of trajectories which satisfy an inequality of type $\|y(t)\| \leq \alpha(t)$, where $\alpha(t)$ will be given explicitly. This inequality leads to null controllability and, implicitly, by solving the equation $\alpha(t) = 0$, to estimates for the minimum time for which controllability holds.

In the end of this section we present some concepts and results concerning viability theory, which will be used in the next section. For more details see [7].

Let $\mathcal{X}$ be a Banach space, $\mathcal{A} : D(\mathcal{A}) \subseteq \mathcal{X} \to \mathcal{X}$ the generator of a C_0-semigroup $\{\mathcal{S}(t) : \mathcal{X} \to \mathcal{X};\ t \geq 0\}$, K a nonempty subset in $\mathcal{X}$ and $g : K \to \mathcal{X}$ a given function.

Definition 1.1. The set K is viable with respect to $\mathcal{A} + f$ if for each $\xi \in K$ there exists $T > 0$ such that the Cauchy problem (1.2) has at least one solution $y : [0,T] \to K$.

By a solution of the Cauchy problem

$$\begin{cases} y'(t) = \mathcal{A}y(t) + g(y(t)) \\ y(0) = \xi \end{cases} \tag{1.2}$$

on $[0,T]$ we mean a continuous function $y : [0,T] \to K$ such that the mapping $s \mapsto g(y(s))$ is Bochner integrable on $[0,T]$ and

$$y(t) = \mathcal{S}(t)\xi + \int_0^t \mathcal{S}(t-s)\, g(y(s))\, ds,$$

for any $t \in [0,T]$.

Definition 1.2. We say that $\eta \in \mathcal{X}$ is $\mathcal{A}$-tangent to K at $\xi \in K$ if

$$\liminf_{h \downarrow 0} \frac{1}{h} dist\left(\mathcal{S}(h)\xi + h\eta; K\right) = 0.$$

We denote by $\mathcal{T}_K^{\mathcal{A}}(\xi)$ the set of all $\mathcal{A}$-tangent elements to K at $\xi \in K$.

The following characterization by sequences will prove to be useful later.

Remark 1.1. $\eta \in \mathcal{X}$ is $\mathcal{A}$-tangent to K at $\xi \in K$ if and only if there exist two sequences (h_n) in $\mathbb{R}_+$ with $h_n \downarrow 0$ and (p_n) in $\mathcal{X}$ with $\lim_{n\to\infty} p_n = 0$ such that

$$\mathcal{S}(h_n)\xi + h_n(\eta + p_n) \in K$$

for each $n \in \mathbb{N}$.

We present now two viability results from [7] that will be used in the next section. In fact, in [7] there are given several viability results based on this tangency concept, under different assumptions on $\mathcal{A}$, g and K.

Theorem 1.1. [7, Theorem 8.2.3] *Let $\mathcal{X}$ be a Banach space, $\mathcal{A} : D(\mathcal{A}) \subseteq \mathcal{X} \to \mathcal{X}$ the generator of a compact C_0-semigroup $\{\mathcal{S}(t) : \mathcal{X} \to \mathcal{X};\ t \geq 0\}$, K a nonempty and locally closed subset in $\mathcal{X}$ and $g : K \to \mathcal{X}$ a continuous function. Then K is viable with respect to $\mathcal{A} + g$ if and only if, for each $\xi \in K$, the tangency condition*

$$g(\xi) \in \mathcal{T}_K^{\mathcal{A}}(\xi) \tag{1.3}$$

is satisfied.

Theorem 1.2. [7, Theorem 8.2.6] *Let $\mathcal{X}$ be a Banach space, $\mathcal{A} : D(\mathcal{A}) \subseteq \mathcal{X} \to \mathcal{X}$ the generator of a C_0-semigroup $\{\mathcal{S}(t) : \mathcal{X} \to \mathcal{X};\ t \geq 0\}$, K a nonempty and locally closed subset in $\mathcal{X}$ and $g : K \to \mathcal{X}$ a locally Lipschitz function. Then K is viable with respect to $\mathcal{A} + g$ if and only if, for each $\xi \in K$, the tangency condition (1.3) is satisfied.*

2. Main results

In this section we provide an admissible control $u(\cdot)$ in order to reach the origin starting from the initial point x in some time T, by mild solutions of equation (1.1). More precisely, we use a feedback control of the type $-B^* y(t) / \|B^* y(t)\|$.

We assume that B is surjective. This implies that there exists $\gamma > 0$ such that

$$\|B^* x\| \geq \gamma \|x\| \tag{2.4}$$

for any $x \in X$.

First, we prove the following result.

Theorem 2.3. *Assume that the semigroup $\{S(t) : X \to X;\ t \geq 0\}$ is compact and satisfies the condition $\|S(t)\| \leq e^{\omega t}$, $\omega \in \mathbb{R}$, B satisfies (2.4) and $f : X \to X$ is a continuous function satisfying*

$$\langle y, f(y) \rangle \leq L \|y\|^2 \tag{2.5}$$

for any $y \in X$. Then, for any $x \in X$ with $\|x\| > 0$ there exist $\sigma > 0$ and $y : [0, \sigma) \to X$ solution of

$$\begin{cases} y' = Ay + f(y) - B\dfrac{B^* y}{\|B^* y\|} \\ y(0) = x \end{cases} . \tag{2.6}$$

which satisfies

$$\|y(t)\| \leq \|x\| - \gamma t + (L + \omega) \int_0^t \|y(s)\|\, ds \tag{2.7}$$

and $y(t) \neq 0$ for every $t \in [0, \sigma)$.

Proof. Consider the space $\mathcal{X} = \mathbb{R} \times X$, the operator $\mathcal{A} = (0, A)$ which generates the compact C_0-semigroup $(1, S(t))$ on $\mathbb{R} \times X$, the set

$$K = \{(\lambda, x) \in \mathbb{R}_+ \times X \setminus \{0\};\ \|x\| \leq \lambda\},$$

which is locally closed, and the function $F : \mathbb{R}_+ \times X \setminus \{0\} \to \mathbb{R} \times X$ given by

$$F(z, x) = \left((L + \omega) \|x\| - \gamma, f(x) - B\frac{B^* x}{\|B^* x\|} \right).$$

We show that K is viable with respect to

$$\begin{cases} z'(t) = (L + \omega) \|y(t)\| - \gamma \\ y'(t) = Ay(t) + f(y(t)) - B\dfrac{B^* y(t)}{\|B^* y(t)\|} \end{cases} . \tag{2.8}$$

To this end we will apply Theorem 1.1 and we have to verify that the tangency condition

$$F(z, x) \in \mathcal{T}_K^{\mathcal{A}}(z, x)$$

holds for any $(z, x) \in K$. Therefore, let $(\lambda, x) \in K$. By Remark (1.1) we have to show that there exist $(h_n), (\theta_n)$ in $\mathbb{R}$, $h_n \downarrow 0$, $\theta_n \to 0$ such that

$$\left\| S(h_n) x + h_n \left(f(x) - B\frac{B^* x}{\|B^* x\|} \right) \right\| \leq \lambda + h_n ((L + \omega) \|x\| - \gamma) + h_n \theta_n \tag{2.9}$$

for any $n \in \mathbb{N}$. First, for $h > 0$ we have that

$$\left\| S(h)x + h\left(f(x) - B\frac{B^*x}{\|B^*x\|}\right)\right\| =$$

$$\left\| S(h)x + hS(h)\left(f(x) - B\frac{B^*x}{\|B^*x\|}\right) + \right.$$

$$\left. + h\left[f(x) - B\frac{B^*x}{\|B^*x\|} - S(h)\left(f(x) - B\frac{B^*x}{\|B^*x\|}\right)\right]\right\|$$

$$\leq \left\| S(h)x + hS(h)\left(f(x) - B\frac{B^*x}{\|B^*x\|}\right)\right\| +$$

$$+h\left\| f(x) - B\frac{B^*x}{\|B^*x\|} - S(h)\left(f(x) - B\frac{B^*x}{\|B^*x\|}\right)\right\|$$

$$\leq e^{\omega h}\left\| x + h\left(f(x) - B\frac{B^*x}{\|B^*x\|}\right)\right\| +$$

$$+h\left\| f(x) - B\frac{B^*x}{\|B^*x\|} - S(h)\left(f(x) - B\frac{B^*x}{\|B^*x\|}\right)\right\|.$$

This implies that

$$\frac{1}{h}\left(\left\| S(h)x + h\left(f(x) - B\frac{B^*x}{\|B^*x\|}\right)\right\| - \|x\|\right)$$

$$\leq e^{\omega h}\cdot\frac{\left\| x + h\left(f(x) - B\frac{B^*x}{\|B^*x\|}\right)\right\| - \|x\|}{h} + \frac{e^{\omega h}-1}{h}\|x\|$$

$$+\left\| f(x) - B\frac{B^*x}{\|B^*x\|} - S(h)\left(f(x) - B\frac{B^*x}{\|B^*x\|}\right)\right\|.$$

Denoting the right hand side by Π_h, letting $h \downarrow 0$ and using (2.5) and (2.4) we get

$$\lim_{h\downarrow 0}\Pi_h \leq \left[x, f(x) - B\frac{B^*x}{\|B^*x\|}\right]_+ + \omega\|x\| = \frac{1}{\|x\|}\left\langle x, f(x) - B\frac{B^*x}{\|B^*x\|}\right\rangle + \omega\|x\|$$

$$= \frac{1}{\|x\|}\langle x, f(x)\rangle - \frac{1}{\|x\|}\left\langle x, B\frac{B^*x}{\|B^*x\|}\right\rangle + \omega\|x\|$$

$$\frac{1}{\|x\|}\langle x, f(x)\rangle - \frac{1}{\|x\|\,\|B^*x\|}\langle B^*x, B^*x\rangle + \omega\|x\|.$$

$$\leq L\|x\| - \frac{\|B^*x\|}{\|x\|} + \omega\|x\| \leq L\|x\| - \gamma + \omega\|x\|.$$

This clearly implies the existence of (h_n) and (θ_n) which satisfy (2.9). Hence, for each $x \in X$, $x \neq 0$, there exist $\sigma > 0$ and a noncontinuable solution on $[0,\sigma)$ of the Cauchy problem (2.8) with $z(0) = \|x\|$, $y(0) = x$, which satisfies $(z(t), y(t)) \in K$ for any $t \in [0,\sigma)$. Clearly, $y(\sigma)$ exists and equals 0. Hence,

$$\|y(t)\| \leq \|x\| + (L+\omega)\int_0^t \|y(s)\|\,ds - \gamma t$$

for any $t \in [0,\sigma)$. □

Theorem 2.4. *Assume that the semigroup $\{S(t) : X \to X;\ t \geq 0\}$ satisfies the condition $\|S(t)\| \leq e^{\omega t}$, $\omega \in \mathbb{R}$, B satisfies (2.4) and $f : X \to X$ is a locally Lipschitz function satisfying (2.5) for any $y \in X$. Then, for any $x \in X$ with $\|x\| > 0$ there exist $\sigma > 0$ and a unique solution $y : [0,\sigma) \to X$ of (2.6) which satisfies (2.7) and $y(t) \neq 0$ for every $t \in [0,\sigma)$.*

Proof. First, observe that the function $x \mapsto B\dfrac{B^*x}{\|B^*x\|}$ is Lipschitz continuous on $X \backslash B(0,R)$, for each $R > 0$. It is used that B and B^* are linear and bounded operators.

Then, we apply Theorem 1.2 to show that the unique solution of (2.6) reaches $B(0,R)$ in time $\widetilde{t}$ and satisfies (2.7) for $t \in [0,\widetilde{t})$. If $R_1 < R_2$ then the corresponding times $\widetilde{t}_1$ and $\widetilde{t}_2$ satisfy $\widetilde{t}_2 < \widetilde{t}_1$ and the corresponding solutions $y_1(\cdot)$ and $y_2(\cdot)$ satisfy $y_1(t) = y_2(t)$ on $\left[0,\widetilde{t}_2\right]$. Hence, the solution of (2.6) does not depend on $R > 0$ untill it reaches $B(0,R)$. Taking a sequence $R_n \downarrow 0$ we get a sequence of times $\widetilde{t}_n$ inscreasing to some σ and the inequality (2.7) is satisfied on $[0,\sigma)$. □

From Theorems 2.3 and 2.4 we get estimates for the minimum time function.

Corollary 2.1. *Under the hypotheses of Theorem* 2.3 *or Theorem* 2.4, *the following properties hold.*
(i) *In case* $L + \omega \leq 0$, *for any* $x \in X$, $x \neq 0$, *there exist an admissible control* $u(\cdot)$ *and a mild solution* $y(\cdot)$ *of* (1.1) *that reaches the origin of* X *in some time* $T \leq \|x\|/\gamma$ *and satisfies*

$$\|y(t)\| \leq \|x\| - \gamma t$$

for any $t \in [0,T]$. *Therefore,*

$$\mathcal{T}(x) \leq \|x\|/\gamma$$

for any $x \in X$.
(ii) *In case* $L+\omega > 0$, *for every* $x \in X$ *satisfying* $0 < \|x\| < \gamma/(L+\omega)$, *there exist an admissible control* $u(\cdot)$ *and a mild solution* $y(\cdot)$ *of* (1.1) *that reaches the origin of* X *in some time*

$$T \leq \frac{1}{L+\omega}\log\frac{\gamma}{\gamma-(L+\omega)\|x\|}$$

and satisfies

$$\|y(t)\| \leq e^{(L+\omega)t}\left(\|x\| - \frac{\gamma}{L+\omega}\right) + \frac{\gamma}{L+\omega}$$

for any $t \in [0,T]$. *Considering* $0 < \rho < \dfrac{\gamma}{L+\omega}$ *and defining* $r = \dfrac{1}{\gamma-(L+\omega)\rho}$ *we have*

$$\mathcal{T}(x) \leq r\|x\|$$

for any $x \in X$ *with* $0 < \|x\| \leq \rho$.

Proof. Let $y : [0,\sigma) \to X$ be the solution of (2.6) given by Theorem 2.3, which satisfies (2.7) and $y(t) \neq 0$ for any $t \in [0,\sigma)$ and $y(\sigma) = 0$. Then $y(\cdot)$ is a solution of (1.1) with $u(t) = -B^*y(t)/\|B^*y(t)\|$ for $t \in [0,\sigma)$.

(i) In the case when $L+\omega \leq 0$, by (2.7) we get that $\|y(t)\| \leq \|x\| - \gamma t$ for any $t \in [0,\sigma]$. Hence $\mathcal{T}(x) \leq \|x\|/\gamma$.

(ii) If $L+\omega > 0$, from (2.7), using Gronwall inequality, we get

$$\|y(t)\| \leq e^{(L+\omega)t}\left(\|x\| - \frac{\gamma}{L+\omega}\right) + \frac{\gamma}{L+\omega}$$

for any $t \in [0,\sigma]$. Further, we get that $\mathcal{T}(x) \leq \dfrac{1}{L+\omega}\log\dfrac{\gamma}{\gamma-(L+\omega)\|x\|}$ for any $x \in X$ with $0 < \|x\| < \gamma/(L+\omega)$. Using now the elementary inequality $\log(1+y) < y$, for every $y > 0$, we get

$$\mathcal{T}(x) \leq \frac{\|x\|}{\gamma-(L+\omega)\|x\|} \leq r\|x\|$$

for any $x \in X$ with $0 < \|x\| \leq \rho$. □

Proposition 2.1. *Assume that the semigroup $\{S(t) : X \to X;\ t \geq 0\}$ satisfies the condition $\|S(t)\| \leq e^{\omega t}$, $\omega \in \mathbb{R}$, B satisfies (2.4) and $f : X \to X$ is λ-Lipschitz and satisfies (2.5) for any $y \in X$.*

(i) Suppose $L + \omega > 0$, let $0 < \rho < \dfrac{\gamma}{L+\omega}$ and define $r = \dfrac{1}{\gamma - (L+\omega)\rho}$. If $x \in \mathcal{C}$ and $z \in X$ is such that

$$\|z - x\| \leq \rho e^{-(\lambda+\omega)T(x)}, \tag{2.10}$$

then $z \in \mathcal{C}$ and

$$\mathcal{T}(z) \leq \mathcal{T}(x) + re^{(\lambda+\omega)\mathcal{T}(x)} \|x - z\|.$$

Moreover, $\mathcal{C}$ is open, $\mathcal{T}(\cdot)$ is locally Lipschitz continuous on $\mathcal{C}$ if $\lambda + \omega > 0$ and globally Lipschitz continuous on $\mathcal{C}$ if $\lambda + \omega \leq 0$.

(ii) If $L + \omega \leq 0$, then, for any $x, z \in X$ we have

$$\mathcal{T}(z) \leq \mathcal{T}(x) + \frac{1}{\gamma} e^{(\lambda+\omega)\mathcal{T}(x)} \|x - z\|.$$

Moreover, $\mathcal{T}(\cdot)$ is locally Lipschitz continuous on $\mathcal{C}$ if $\lambda+\omega > 0$ and globally Lipschitz continuous on $\mathcal{C}$ if $\lambda + \omega \leq 0$.

Proof. (i) The proof follows the same ideea as the one of [3, Proposition 3.3]. First, let us remark that, for any $x, z \in X$ and any control u, we have

$$\|y(t, x, u) - y(t, z, u)\| \leq e^{(\omega+\lambda)t} \|x - z\|, \tag{2.11}$$

for any $t \geq 0$. To simplify the exposition we assume the existence of an optimal control u for x, i.e. $y(\mathcal{T}(x), x, u) = 0$. Using (2.11) and (2.10), we get

$$\|y(\mathcal{T}(x), z, u)\| \leq e^{(\omega+\lambda)\mathcal{T}(x)} \|x - z\| \leq \rho.$$

By Corollary 2.1 we get that $y(\mathcal{T}(x), z, u) \in \mathcal{C}$ and

$$\mathcal{T}(y(\mathcal{T}(x), z, u)) \leq \frac{1}{L+\omega} \log \frac{\gamma}{\gamma - (L+\omega)\|y(\mathcal{T}(x), z, u)\|}.$$

Further, using the inequality $\log(1+y) < y$ for every $y > 0$, we get

$$\mathcal{T}(y(\mathcal{T}(x), z, u)) \leq \frac{\|y(\mathcal{T}(x), z, u)\|}{\gamma - (L+\omega)\|y(\mathcal{T}(x), z, u)\|} \leq r\|y(\mathcal{T}(x), z, u)\|$$

$$\leq re^{(\omega+\lambda)\mathcal{T}(x)} \|x - z\|.$$

Applying now the dynamic programming principle we obtain that

$$\mathcal{T}(z) \leq \mathcal{T}(x) + \mathcal{T}(y(\mathcal{T}(x), z, u)) \leq \mathcal{T}(x) + re^{(\omega+\lambda)\mathcal{T}(x)} \|x - z\|.$$

Suppose that $\lambda + \omega > 0$. Let $x_0 \in \mathcal{C}$ and take $\delta = \dfrac{\rho}{2} e^{-(\omega+\lambda)(\mathcal{T}(x_0)+r\rho)}$. Then, from the first part of the proof, we get

$$\mathcal{T}(x_1) - \mathcal{T}(x_2) \leq re^{(\omega+\lambda)(\mathcal{T}(x_0)+r\rho)} \|x_1 - x_2\|$$

for any $x_1, x_2 \in B(x_0, \delta)$. If $\lambda + \omega \leq 0$, the conclusion follows immediately.

The proof of (ii) is similar. □

3. AN EXTENSION

In this section we assume that there exists $\gamma > 0$ such that

$$\|B^* x\| \geq \frac{\gamma \|x\|}{\varphi'(\|x\|)} \tag{3.12}$$

for any $x \in X$ with $0 < \|x\| \leq \eta$, for some $\eta > 0$, where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is derivable, strictly increasing and $\varphi(0) = 0$. For example, $\varphi(t) = t^\beta$ with $0 < \beta < 1$.

Theorem 3.5. *Assume that the semigroup $\{S(t) : X \to X;\ t \geq 0\}$ is compact and satisfies the condition $\|S(t)\| \leq e^{\omega t}$, $\omega \in \mathbb{R}$, B satisfies (3.12) and $f : X \to X$ is a continuous function satisfying (2.5) for any $y \in X$. Then, for any $x \in X$ with $0 < \|x\| < \eta$ there exist $\sigma > 0$ and $y : [0, \sigma) \to X$ solution of (2.6) which satisfies*

$$\varphi(\|y(t)\|) \leq \varphi(\|x\|) + \int_0^t (L + \omega)\varphi'(\|y(s)\|)\|y(s)\|\, ds - \gamma t \tag{3.13}$$

and $y(t) \neq 0$ for every $t \in [0, \sigma)$.

Proof. We take

$$K = \{(\lambda, x) \in \mathbb{R}_+ \times X \setminus \{0\};\ \|x\| < \eta \text{ and } \varphi(\|x\|) \leq \lambda\},$$

and show that it is viable with respect to

$$\begin{cases} z'(t) = (L + \omega)\varphi'(\|y(t)\|)\|y(t)\| - \gamma \\ y'(t) = Ay(t) + f(y(t)) - B\dfrac{B^* y(t)}{\|B^* y(t)\|} \end{cases}. \tag{3.14}$$

To this end, let $(\lambda, x) \in K$, hence $0 < \|x\| < \eta$ and $\varphi(\|x\|) \leq \lambda$. We have to show that there exist $(h_n), (\theta_n)$ in $\mathbb{R}$, $h_n \downarrow 0$, $\theta_n \longrightarrow 0$ such that

$$(\lambda, S(h_n)x) + h_n\left((L + \omega)\varphi'(\|x\|)\|x\| - \gamma + \theta_n, f(x) - B\frac{B^* x}{\|B^* x\|}\right) \in K$$

for each $n \in \mathbb{N}$, that is

$$\left\|S(h_n)x + h_n\left(f(x) - B\frac{B^* x}{\|B^* x\|}\right)\right\| < \eta \tag{3.15}$$

and

$$\varphi\left(\left\|S(h_n)x + h_n\left(f(x) - B\frac{B^* x}{\|B^* x\|}\right)\right\|\right) \leq \lambda + (L + \omega)\varphi'(\|x\|)\|x\| h_n - \gamma h_n + h_n \theta_n \tag{3.16}$$

for each $n \in \mathbb{N}$. First, observe that, since φ is derivable,

$$\liminf_{h \downarrow 0} \frac{\varphi\left(\left\|S(h)x + h\left(f(x) - B\frac{B^* x}{\|B^* x\|}\right)\right\|\right) - \varphi(\|x\|)}{h} =$$

$$= \varphi'(\|x\|) \liminf_{h \downarrow 0} \frac{\left\|S(h)x + h\left(f(x) - B\frac{B^* x}{\|B^* x\|}\right)\right\| - \|x\|}{h}.$$

Further, as in the proof of Theorem 2.3, using (3.12), we get that

$$\liminf_{h \downarrow 0} \frac{\varphi\left(\left\|S(h)x + h\left(f(x) - B\frac{B^* x}{\|B^* x\|}\right)\right\|\right) - \varphi(\|x\|)}{h} \leq$$

$$\leq \varphi'(\|x\|)\left(L\|x\| - \frac{\|B^*x\|}{\|x\|} + \omega\|x\|\right) \leq -\gamma + (L+\omega)\varphi'(\|x\|)\|x\|.$$

This implies the existence of (h_n) and (θ_n) which satisfy (3.16) . Clearly, (h_n) can be chosen such that (3.15) holds. Hence, for each $x \in X$, with $0 < \|x\| < \eta$, there exist $\sigma > 0$ and a noncontinuable solution on $[0,\sigma)$ of the Cauchy problem (3.14) with $z(0) = \varphi(\|x\|)$, $y(0) = x$ which satisfies $(z(t), y(t)) \in K$ for any $t \in [0,\sigma)$. Clearly, $y(\sigma)$ exists and equals 0. Hence,

$$\varphi(\|y(t)\|) \leq \varphi(\|x\|) + \int_0^t (L+\omega)\varphi'(\|y(s)\|)\|y(s)\|\,ds - \gamma t$$

for any $t \in [0,\sigma)$. □

From Theorem 3.5, using a similar argument as in the proof of Corollary 2.1, we get the following result.

Corollary 3.2. *Assume that $L+\omega \leq 0$. Under the hypotheses of Theorem 3.5, for any $x \in X$, with $0 < \|x\| < \eta$, there exist an admissible control $u(\cdot)$ and a mild solution $y(\cdot)$ of (1.1) that reaches the origin of X in some time $T \leq \varphi(\|x\|)/\gamma$ and satisfies*

$$\varphi(\|y(t)\|) \leq \varphi(\|x\|) - \gamma t$$

for any $t \in [0,T]$. Further,

$$\mathcal{T}(x) \leq \varphi(\|x\|)/\gamma$$

for any $x \in X$, with $0 < \|x\| < \eta$.

Corollary 3.3. *Assume that $L+\omega > 0$ and $\varphi(t) = t^\beta$, with $0 < \beta < 1$. Under the hypotheses of Theorem 3.5, for any $x \in X$ satisfying $0 < \|x\| < \eta$ and $\varphi(\|x\|) < \frac{\gamma}{\beta(L+\omega)}$, there exist an admissible control $u(\cdot)$ and a mild solution $y(\cdot)$ of (1.1) that reaches the origin of X in some time $T \leq \frac{1}{\beta(L+\omega)}\log\frac{\gamma}{\gamma-\beta(L+\omega)\varphi(\|x\|)}$ and satisfies*

$$\varphi(\|y(t)\|) \leq e^{\beta(L+\omega)t}\left(\varphi(\|x\|) - \frac{\gamma}{\beta(L+\omega)}\right) + \frac{\gamma}{\beta(L+\omega)}$$

for any $t \in [0,T]$. Considering $0 < \delta < \frac{\gamma}{\beta(L+\omega)}$ and defining $\kappa = \frac{1}{\gamma-\beta(L+\omega)\delta}$ we have

$$\mathcal{T}(x) \leq \kappa\varphi(\|x\|)$$

for any $x \in X$ with $0 < \|x\| < \eta$ and $\varphi(\|x\|) \leq \delta$.

Proof. Let $y : [0,\sigma) \to X$ be the solution of (2.6) given by Theorem 3.5, which satisfies (3.13) and $y(t) \neq 0$ for any $t \in [0,\sigma)$ and $y(\sigma) = 0$. Then $y(\cdot)$ is a solution of (1.1) with $u(t) = -B^*y(t)/\|B^*y(t)\|$ for $t \in [0,\sigma)$.

Clearly, $\varphi'(t)t = \beta\varphi(t)$ for any $t > 0$. Hence, (3.13) becomes

$$\varphi(\|y(t)\|) \leq \varphi(\|x\|) + \int_0^t (L+\omega)\beta\varphi(\|y(s)\|)\,ds - \gamma t$$

and, using Gronwall inequality, we get

$$\varphi(\|y(t)\|) \leq e^{\beta(L+\omega)t}\left(\varphi(\|x\|) - \frac{\gamma}{\beta(L+\omega)}\right) + \frac{\gamma}{\beta(L+\omega)}$$

for any $t \in [0,\sigma]$. Further, we get that $\mathcal{T}(x) \leq \frac{1}{\beta(L+\omega)}\log\frac{\gamma}{\gamma-\beta(L+\omega)\varphi(\|x\|)}$. Finally, we use the inequality $\log(1+y) < y$, for every $y > 0$, to complete the proof, as in Corollary 2.1. □

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