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Dedicated to Prof. Emeritus Mihail Megan on the occasion of his 75^{th} anniversary

A Fixed Point Approach of Variational-Hemivariational Inequalities

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ABSTRACT. In this paper we provide a new approach in the study of a variational-hemivariational inequality in Hilbert space, based on the theory of maximal monotone operators and the Banach fixed point theorem. First, we introduce the inequality problem we are interested in, list the assumptions on the data and show that it is governed by a multivalued maximal monotone operator. Then, we prove that solving the variational-hemivariational inequality is equivalent to finding a fixed point for the resolvent of this operator. Based on this equivalence result, we use the Banach contraction principle to prove the unique solvability of the problem. Moreover, we construct the corresponding Picard, Krasnoselski and Mann iterations and deduce their convergence to the unique solution of the variational-hemivariational inequality.

1. INTRODUCTION

Variational-hemivariational inequalities represent a relevant class of inequalities which have both a convex and nonconvex structure. Their study is motivated by the analysis of various boundary value problems which arise in Physics, Mechanics and Engineering Sciences. It is carried out by using arguments of both convex and nonsmooth analysis, including the properties of convex subdifferential and the Clarke directional derivative for locally Lipschitz functions. Introduced in the pioneering work of Panagiotopoulos [14], the theory of variational-hemivariational inequalities grew up rapidly, as shown in [12, 13, 15] and the references therein. Recent existence, uniqueness and convergence results can be found in [6, 11, 17, 20]. The numerical analysis of variational-hemivariational inequalities was carried out by using various methods. For instance, optimization methods have been developed in [9, 10], the finite element method has been considered in [7, 8, 18] and the virtual element method has been used in [4, 16].

Motivated by the research works on the solvability of variational-hemivariational inequalities, in this paper we introduce a new approach which allows us to prove existence, uniqueness and convergence results for a class of variational-hemivariational inequalities in Hilbert spaces. An inequality of this class is denoted by *VHVI* and is formulated as follows:

VHVI : Find $u \in K$ such that

$$(Au, v-u)_H + \varphi(v) - \varphi(u) + J^0(u; v-u) \geq (f, v-u)_H \quad \forall\, v \in K. \tag{1.1}$$

Here and below H represents a real Hilbert space endowed with the inner product $(\cdot,\cdot)_H$ and the associated norm $\|\cdot\|_H$, K is a nonempty subset of H, $A : H \to H$ is a nonlinear operator, $\varphi : H \to \mathbb{R}$ and $J : H \to \mathbb{R}$ are given functions and, finally, $f \in H$.

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The function J is assumed to be locally Lipschitz and notation $J^0(u;v)$ represents the Clarke directional derivative of J at the point u, in the direction v.

Existence and uniqueness results in the study of variational-hemivariational inequality problems of the form (1.1) have been obtained in many papers, under different conditions on the data. For instance, inequality (1.1) was considered in [11] under the assumptions that A is a pseudomonotone and strongly monotone operator, φ is a convex lower semicontinuous function and the Clarke subdifferential of the function J satisfies a growth condition. The unique solvability of the problem was proved by using a surjectivity result for pseudomonotone multivalued operators. Recently, problem (1.1) was considered in [6], under the assumptions that A is strongly monotone and Lipschitz continuous and φ is a convex continuous function. The unique solvability of the problem was obtained by using a minimization principle which avoids any pseudomonotonicity argument.

Our results in this paper complete the results in [6, 11]. Indeed, here we present a new approach in the study of the variational-hemivariational inequality (1.1), based on arguments of multivalued maximal monotone operators in Hilbert spaces and fixed point. Our results are obtained under assumptions which are slightly different to those used in [6, 11]. Moreover, the method we introduce here opens the way to the approach of the solution by using the Picard, Krasnoselski and Mann iterations. Therefore, one of the novelties of our approach is that it allows us to recursively construct an approximation of the solution to *VHVI*. Extending this approach to variational-hemivariational inequalities in the framework of reflexive Banach spaces represents a challenging problem which deserves to be studied in the future.

The rest of the paper is organized as follows. In Section 2 we recall some basic definitions and preliminary material which will be used in the rest of the paper. In Section 3 we state and prove our main existence and uniqueness result, Theorem 3.1. Finally, in Section 4 we prove a convergence result to the solution of the *VHVI*.

2. Preliminaries

The preliminary results we present in this section can be found in many books and surveys, including [2, 3, 19]. For this reason we present them without proofs. Everywhere below we use the symbols "$\to$" and "$\rightharpoonup$" to denote the strong and the weak convergence in the space H, respectively. We use the notation H_w for the real Hilbert space H equipped with the weak topology. The limits, lower limits and upper limits are considered as $n \to \infty$, even if we do not mention it explicitly. Moreover, we use $int\, M$ for the interior of the set $M \subset H$, in the strong topology of H. Finally, we denote by I the identity map of H, by 0_H the zero element of H and by 2^H the set of parts of H. We start with the following definitions.

Definition 2.1. The operator $A : H \to H$ is said to be:

(1) demicontinuous if $u_n \to u$ in H implies $Au_n \rightharpoonup Au$ in H;

(2) strongly monotone if there exists constant $m_A > 0$ such that

$$(Au - Av, u - v)_H \geq m_A \|u - v\|_H^2 \quad \forall\, u, v \in H;$$

(3) a contraction if there exists constant $0 \leq k < 1$ such that

$$\|Au - Av\|_H \leq k\|u - v\|_H \quad \forall\, u, v \in H.$$

Definition 2.2. The Clarke directional derivative of the locally Lipschitz function $J : H \to \mathbb{R}$ at the point $u \in H$ in the direction $v \in H$ is defined by

$$J^0(u;v) = \limsup_{w \to u, \lambda \downarrow 0} \frac{J(w + \lambda v) - J(w)}{\lambda}.$$

The Clarke subdifferential of J at u, denoted by $\partial J(u)$, is the subset of the space H defined by
$$\partial J(u) = \{\, \xi \in H \mid J^0(u;v) \geq (\xi, v)_H \quad \forall\, v \in H \,\}.$$

For the Clarke subdifferential and directional derivative we recall the following properties.

Proposition 2.1. *Let $J : H \to \mathbb{R}$ be a locally Lipschitz function. Then:*

(1) *for all $u \in H$, the set $\partial J(u)$ is a nonempty convex and weakly compact subset of H;*
(2) *the graph of the Clarke subdifferential $\partial J : H \to 2^H$ is closed in the $H \times H_w$ topology;*
(3) *for all $u, v \in H$, one has*
$$J^0(u;v) = \max\{\, (\xi, v)_H \mid \xi \in \partial J(u) \,\}.$$

We now move to some basic definitions and results on convex analysis.

Definition 2.3. The subdifferential of a proper convex function $\varphi : H \to \mathbb{R} \cup \{+\infty\}$ is defined by
$$\partial^c \varphi(u) = \{\, \eta \in H \mid \varphi(v) - \varphi(u) \geq (\eta, v - u)_H \quad \forall\, v \in H \,\}.$$

Definition 2.4. Given a nonempty subset K of H, the indicator function of set K is defined by
$$I_K(u) = \begin{cases} 0 & \text{if } u \in K, \\ +\infty & \text{if } u \notin K. \end{cases}$$

It is well known that if the subset K of H is nonempty closed and convex, then the indicator function I_K is proper convex and lower semicontinuous. We also recall the following result proved in [5].

Proposition 2.2. *Let C be a nonempty closed convex subset of H, $C^* \subset H$ a nonempty closed convex bounded subset of H, $\varphi : H \to \mathbb{R} \cup \{+\infty\}$ a proper convex lower semicontinuous function and $u \in C$. Assume that for each $v \in C$ there exists $u^*(v) \in C^*$ such that*
$$(u^*(v), v - u)_H \geq \varphi(u) - \varphi(v).$$
Then, there exists $u^* \in C^*$ such that
$$(u^*, v - u)_H \geq \varphi(u) - \varphi(v) \quad \forall\, v \in C.$$

We now proceed with some results concerning multivalued operators defined on the space H. To this end we recall that, given a multivalued operator $T : H \to 2^H$, its domain $D(T)$, range $R(T)$ and graph $Gr(T)$ are the sets defined by
$$D(T) = \{\, v \in H \,|\, Tv \neq \emptyset \,\},$$
$$R(T) = \{\, f \in H \,|\, \exists\, v \in D(T) \text{ s.t. } f \in Tv \,\},$$
$$Gr(T) = \{\, (v, v^*) \in D(T) \times H \,|\, v^* \in Tv \,\}.$$

Definition 2.5. The operator $T : H \to 2^H$ is said to be:

(1) relaxed monotone if there exists constant $\alpha_T > 0$ such that
$$(u_1^* - u_2^*, u_1 - u_2)_H \geq -\alpha_T \|u_1 - u_2\|_H^2 \quad \forall\, (u_1, u_1^*),\ (u_2, u_2^*) \in Gr(T);$$
(2) monotone if
$$(u_1^* - u_2^*, u_1 - u_2)_H \geq 0 \quad \forall\, (u_1, u_1^*),\ (u_2, u_2^*) \in Gr(T);$$
(3) maximal monotone if it is monotone and the following implication holds:
$$(u^* - v^*, u - v)_H \geq 0 \ \ \forall\, u \in D(T),\ u^* \in Tu \implies v \in D(T) \text{ and } v^* \in Tv.$$

Proposition 2.3. *Let $T : D(T) \to 2^H$ be a maximal monotone operator and let $\lambda > 0$. Then $R(I + \lambda T) = H$. Moreover, for any $f \in H$ there exists a unique element $u \in D(T)$ such that $u + \lambda Tu \ni f$.*

Proposition 2.3 allows us to consider the resolvent operator $T_\lambda : H \to D(T)$ defined by

$$T_\lambda f = u \iff u \in D(T) \text{ and } f \in u + \lambda Tu \tag{2.2}$$

for all $f \in H$. Note that the resolvent operator exists for each $\lambda > 0$ and is a single valued operator. A proof can be found in [3], together with the following results which represent sufficient conditions for an operator to be maximal monotone.

Proposition 2.4. *Assume that $\varphi : H \to \mathbb{R} \cup \{+\infty\}$ is a proper convex lower semicontinuous function. Then the subdifferential operator $\partial^c\varphi : H \to 2^H$ is maximal monotone.*

Proposition 2.5. *Assume that $T : H \to 2^H$ is a monotone operator such that for every $v \in H$, Tv is nonempty convex and weakly closed set. Moreover, assume that for all $u, v \in H$, the mapping $\lambda \mapsto T(\lambda u + (1 - \lambda)v)$ has a graph which is closed in $[0, 1] \times H_w$. Then the operator T is maximal monotone.*

Proposition 2.6. *Let T_1, $T_2 : H \to 2^H$ be two maximal monotone operators such that $int\, D(T_1) \cap D(T_2) \neq \emptyset$. Then the sum $T_1 + T_2 : H \to 2^H$ is a maximal monotone operator.*

3. Main result

In this section we state and prove our main result in the study of the variational-hemivariational inequality (1.1). To this end we consider the following assumptions on the data.

$$K \text{ is a nonempty closed convex subset of } H. \tag{3.3}$$

$$A\colon H \to H \text{ is demicontinuous and strongly monotone with } m_A > 0. \tag{3.4}$$

$$\varphi : H \to \mathbb{R} \text{ is convex lower semicontinuous.} \tag{3.5}$$

$$\left\{\begin{array}{l} J\colon H \to \mathbb{R} \text{ is such that} \\ \quad \text{(a) } J \text{ is locally Lipschitz continuous;} \\ \quad \text{(b) } J^0(u_1; u_2 - u_1) + J^0(u_2; u_1 - u_2) \le \alpha_J \, \|u_1 - u_2\|_H^2 \\ \qquad \forall\, u_1, u_2 \in H \quad \text{with} \quad \alpha_J > 0. \end{array}\right. \tag{3.6}$$

$$f \in H. \tag{3.7}$$

$$m_A > \alpha_J. \tag{3.8}$$

Remark 3.1. We note that a convex function $\varphi : H \to \mathbb{R}$ satisfies condition (3.5) if and only if it is continuous. Moreover, $(3.6)(b)$ is equivalent to the relaxed monotonicity condition of ∂J with constant α_J. A proof of this equivalence can be found in [15, p.124].

Under these assumptions we have the following existence and uniqueness result.

Theorem 3.1. *Assume* (3.3)–(3.8). *Then the variational-hemivariational inequality VHVI has a unique solution on K.*

The proof of Theorem 3.1 requires some preliminary results that we present in what follows.

Lemma 3.1. *Assume* (3.3)–(3.8). *Then the operator $S : H \to 2^H$ defined by*

$$Su = Au + \partial J(u) + \partial^c(\varphi + I_K)(u) - f \quad \forall\, u \in H \tag{3.9}$$

is maximal monotone.

Proof. The proof of this lemma is carried out in several steps.

i) First, we claim that operator $A + \partial J : H \to 2^H$ is monotone. The proof of this claim follows directly from the strong monotonicity of operator A with constant m_A, the relaxed monotonicity with α_J of the operator ∂J, guaranteed by Remark 3.1 and the smallness condition $m_A > \alpha_J$.

ii) Next, we claim that for all $v \in H$ the set $Av + \partial J(v)$ is a nonempty convex and weakly closed subset in H. Indeed, this statement follows from Proposition 2.1 (1).

iii) We now prove that the mapping $\lambda \mapsto (A + \partial J)(\lambda u + (1-\lambda)v)$ has a closed graph in $[0,1] \times H_w$. To this end let $u, v \in H$ and assume that $\lambda_n \to \lambda$ in $[0,1]$, $x_n \rightharpoonup x$ in H as $n \to \infty$ and

$$x_n \in (A + \partial J)(\lambda_n u + (1-\lambda_n)v),$$

for each $n \in \mathbb{N}$. Then, it follows that

$$x_n - A(\lambda_n u + (1-\lambda_n)v) \in \partial J(\lambda_n u + (1-\lambda_n)v).$$

Since $\lambda_n \to \lambda$, $x_n \rightharpoonup x$ and the operator A is demicontinuous, we deduce that

$$x_n - A(\lambda_n u + (1-\lambda_n)v) \rightharpoonup x - A(\lambda u + (1-\lambda)v).$$

Moreover, the closedness of the graph of $\partial J(\cdot)$ in $H \times H_w$ implies that

$$x - A(\lambda u + (1-\lambda)v) \in \partial J(\lambda u + (1-\lambda)v),$$

i.e., $x \in (A + \partial J)(\lambda u + (1-\lambda)v)$. We conclude from here that the mapping $\lambda \mapsto (A + \partial J)(\lambda u + (1-\lambda)v)$ has a closed graph in $[0,1] \times H_w$.

iv) We now use the steps i)-iii) above and Proposition 2.5 to obtain that the operator $A + \partial J : H \to 2^H$ is maximal monotone.

v) To proceed, we claim that operator $A + \partial J + \partial^c(\varphi + I_K) : H \to 2^H$ is maximal monotone. Indeed, using (3.3), (3.5) and Proposition 2.4 we deduce that the operator $\partial^c(\varphi + I_K)$ is maximal monotone. Moreover, Proposition 2.1(1) shows that $D(A + \partial J) = H$ and, on the other hand, it is obvious to see that $D(\partial^c(\varphi + I_K)) = K$. This implies that $intD(A + \partial J) \cap D(\partial^c(\varphi + I_K)) = K \neq \emptyset$. We are now in a position to use Proposition 2.6 in order to deduce that the operator $A + \partial J + \partial^c(\varphi + I_K) : H \to 2^H$ is maximal monotone.

vi) As a consequence of step v) we deduce that the operator S defined by (3.9) is also maximal monotone, which concludes the proof. □

Lemma 3.1 and Proposition 2.3 allow us to consider resolvent operator $S_\lambda : H \to K$ defined by

$$S_\lambda u = (I + \lambda S)^{-1} u \quad \forall\, u \in H, \tag{3.10}$$

for each $\lambda > 0$. In the next step we state the equivalence between inequality *VHVI* and the problem of finding a fixed point of the operator $S_\lambda : H \to K$ defined by (3.10).

Lemma 3.2. *Assume* (3.3)–(3.8) *and let* $u \in H$, $\lambda > 0$. *Then the following statements are equivalent:*

(1) *u is a solution of inequality VHVI;*
(2) *u is a fixed point of the resolvent operator S_λ, i.e., $u = S_\lambda u$.*

Proof. Assume that u is a solution to *VHVI*. Then $u \in K$ and we deduce from Proposition 2.1 (3) that for each $v \in K$ there exists $\xi(v) \in \partial J(u)$ such that

$$(Au + \xi(v) - f, v - u)_H + \varphi(v) - \varphi(u) \geq 0 \quad \forall\, v \in K.$$

Moreover, Proposition 2.1 (1), shows that the set $C^* = \{ Au + \xi - f \ : \ \xi \in \partial J(u) \}$ is a nonempty closed convex bounded subset of H. Therefore, using Proposition 2.2 with $C = K$ we find that there exists $\xi^* \in \partial J(u)$, which does not depend on v, such that

$$(Au + \xi^* - f, v - u)_H + \varphi(v) - \varphi(u) \geq 0 \quad \forall\, v \in K.$$

So,

$$\varphi(v) - \varphi(u) + I_K(v) - I_K(u) \geq (-Au - \xi^* + f, v - u)_H \quad \forall\, v \in H.$$

Therefore, by the definition of subdifferential of convex function and inclusion $\xi^* \in \partial J(u)$ we have

$$f \in Au + \partial J(u) + \partial^c(\varphi + I_K)(u). \tag{3.11}$$

This implies that $0_H \in \lambda S u$ or, equivalently, $u \in u + \lambda S u$. We now use the equivalence (2.2) to see that $S_\lambda u = u$ which shows that u is a fixed point of operator S_λ.

Conversely, let u be a fixed point of operator S_λ, i.e., $u = S_\lambda u$. It follows from the definitions (3.9) and (3.10) that (3.11) holds. Then, there exist $\eta \in \partial^c(\varphi + I_K)(u)$ and $\xi \in \partial J(u)$ such that $f = Au + \xi + \eta$. Therefore, we deduce from the definitions of the Clarke directional derivative and the subdifferential of convex function that u is a solution to inequality *VHVI*, which concludes the proof. □

We now proceed with the following result.

Lemma 3.3. *Assume* (3.3)–(3.8). *Then, for each* $\lambda > 0$ *the operator* $S_\lambda : H \to K$ *is a contraction on* H.

Proof. Let $\lambda > 0$ and let $\sigma,\ u \in H$. We claim that $u = S_\lambda \sigma$ if and only if $u \in K$ and

$$(Au, v - u)_H + \varphi(v) - \varphi(u) + J^0(u; v - u) \geq (f + \frac{\sigma - u}{\lambda}, v - u)_H \quad \forall\, v \in K. \tag{3.12}$$

Indeed, we use (2.2) and (3.9) to see that the following equivalences hold:

$$\begin{aligned} u = S_\lambda \sigma \quad &\Longleftrightarrow \quad u \in K \text{ and } \sigma \in (I + \lambda S)u \\ &\Longleftrightarrow \quad u \in K \text{ and } f + \frac{\sigma - u}{\lambda} \in Au + \partial J(u) + \partial^c(\varphi + I_K)(u) \\ &\Longleftrightarrow \quad u \in K \text{ and} \\ &\qquad\quad (Au, v - u)_H + \varphi(v) - \varphi(u) + J^0(u; v - u) \geq (f + \frac{\sigma - u}{\lambda}, v - u)_H, \end{aligned}$$

for all $v \in K$. Note that the last equivalence above follows from arguments similar to those used in Lemma 3.2, replacing f with $f + \frac{\sigma - u}{\lambda}$.

Assume now that $\sigma_1, \sigma_2 \in H$ and denote $u_1 = S_\lambda \sigma_1$ and $u_2 = S_\lambda \sigma_2$. Then, using the equivalence (3.12) we have

$$u_1 \in K,\ (Au_1, v - u_1)_H + \varphi(v) - \varphi(u_1) + J^0(u_1; v - u_1) \geq (f + \frac{\sigma_1 - u_1}{\lambda}, v - u_1)_H, \tag{3.13}$$

$$u_2 \in K,\ (Au_2, v - u_2)_H + \varphi(v) - \varphi(u_2) + J^0(u_2; v - u_2) \geq (f + \frac{\sigma_2 - u_2}{\lambda}, v - u_2)_H, \tag{3.14}$$

for each $v \in K$. We now take $v = u_2$ in (3.13), $v = u_1$ in (3.14) and add the resulting inequalities to find that

$$\begin{aligned} &\frac{1}{\lambda}\|u_1 - u_2\|_H^2 + (Au_1 - Au_2, u_1 - u_2)_H \\ &\qquad \leq J^0(u_1; u_2 - u_1) + J^0(u_2; u_1 - u_2) + \frac{1}{\lambda}(\sigma_1 - \sigma_2, u_1 - u_2)_H. \end{aligned}$$

We now use the strong monotonicity of operator A, assumption (3.6)(b) and Cauchy-Schwarz's inequality to deduce that

$$(\frac{1}{\lambda} + m_A - \alpha_J)\|u_1 - u_2\|_H^2 \leq \frac{1}{\lambda}\|\sigma_1 - \sigma_2\|_H \|u_1 - u_2\|_H.$$

Therefore, the smallness assumption (3.8) yields

$$\|u_1 - u_2\|_H \leq k\|\sigma_1 - \sigma_2\|_H$$

with $k = \frac{1}{1+\lambda(m_A - \alpha_J)} \in (0,1)$, which concludes the proof. □

We now are in a position to provide the proof of our main existence and uniqueness result, Theorem 3.1.

Proof. Let $\lambda > 0$. We use Lemma 3.3 and the Banach fixed point theorem to see that the operator S_λ has a unique fixed point $u \in K$. Hence, Lemma 3.2 guarantees that u is the unique solution of the variational-hemivariational inequality $VHVI$, which concludes the proof. □

4. CONVERGENCE RESULTS

The results in the previous section show that Problem *VHVI* has a fixed point structure since its unique solvability is based on Lemma 3.3 and the Banach fixed point theorem. In this section we exploit this structure in order to provide the following convergence result.

Theorem 4.2. *Assume* (3.3)–(3.8), $\lambda > 0$, $\alpha_0 \in (0,1]$, $\{a_n\} \subset [\alpha_0, 1]$, $u_0 \in K$ *and consider the sequence* $\{u_n\}$ *defined by*

(4.15) $$u_{n+1} = (1 - a_n)u_n + a_n S_\lambda u_n \qquad \forall\, n \in \mathbb{N}.$$

Then

(4.16) $$u_n \to u \quad \text{in} \quad H \quad \text{as} \quad n \to \infty,$$

where u *is the unique solution of Problem VHVI obtained in Theorem 3.1.*

Proof. Following [1, p.16], (4.15) represent the Mann iterations associated to the operator S_λ. Therefore, since Lemma 3.3 guarantees that S_λ is a contraction, the convergence result (4.16) follows from standard arguments. Nevertheless, for the convenience of the reader, we present below its proof.

Let $n \in \mathbb{N}$ and recall that there exists $k \in (0,1)$ which does not depend on n such that

(4.17) $$\|S_\lambda v - S_\lambda w\|_H \leq k\|v - w\|_H \quad \forall\, v, w \in H.$$

We use (4.17) and (4.15) to see that

$$\begin{aligned}\|S_\lambda u_n - u_n\|_H &\leq \|S_\lambda u_n - S_\lambda u_{n-1}\|_H + \|S_\lambda u_{n-1} - u_n\|_H \\ &\leq k\|u_{n-1} - u_n\|_H + (1 - a_{n-1})\|S_\lambda u_{n-1} - u_{n-1}\|_H \\ &\leq \big(1 + (k-1)a_{n-1}\big)\|S_\lambda u_{n-1} - u_{n-1}\|_H,\end{aligned}$$

which implies that

$$\|S_\lambda u_n - u_n\|_H \leq \big(1 + (k-1)a_{n-1}\big)\cdots\big(1 + (k-1)a_0\big)\|S_\lambda u_0 - u_0\|_H.$$

Using now the inequalities $0 < 1 + (k-1)a_m \leq 1 + (k-1)\alpha_0$ with $m = 0, 1, \ldots, n-1$ we find that

$$\|S_\lambda u_n - u_n\|_H \leq \big(1 + (k-1)\alpha_0\big)^n \|S_\lambda u_0 - u_0\|_H.$$

Note that $1 + (k-1)\alpha_0 < 1$ and, therefore, this inequality shows that

(4.18) $$\|S_\lambda u_n - u_n\|_H \to 0 \qquad \text{as} \quad n \to \infty.$$

Next, using equality $u = S_\lambda u$ and (4.17), again, we find that

$$\|u_n - u\|_H \le \|u_n - S_\lambda u_n\|_H + \|S_\lambda u_n - u\|_H \le \|u_n - S_\lambda u_n\|_H + k\|u_n - u\|_H,$$

which implies that

$$\|u_n - u\|_H \le \frac{1}{1-k}\|u_n - S_\lambda u_n\|_H. \tag{4.19}$$

We now combine (4.18) and (4.19) to deduce the convergence (4.16). □

Note that for $a_n = a \in (0,1]$ the Mann iterations (4.15) reduce to Krasnoselski iterations

$$u_{n+1} = (1-a)u_n + aS_\lambda u_n \qquad \forall\, n \in \mathbb{N} \tag{4.20}$$

and for $a = 1$ the Krasnoselski iterations (4.20) reduce to Picard iterations

$$u_{n+1} = S_\lambda u_n \qquad \forall\, n \in \mathbb{N}. \tag{4.21}$$

Theorem 4.2 shows that the sequences defined by (4.20) and (4.21) converge to the solution of the variational-hemivariational inequality *VHVI*. Finally, from the equivalence (3.12) it follows that the Picard iteration can be defined, recursively, as follows: $u_0 \in K$ and, for all $n \in \mathbb{N}$, u_{n+1} is the solution of the variational-hemivariational inequality

$$\begin{aligned} u_{n+1} \in K, \quad (Au_{n+1}, v - u_{n+1})_H + \frac{1}{\lambda}(u_{n+1}, v - u_{n+1})_H + \varphi(v) - \varphi(u_{n+1}) \\ + J^0(u_{n+1}; v - u_{n+1}) \ge (f + \frac{u_n}{\lambda}, v - u_{n+1})_H \qquad \forall\, v \in K. \end{aligned}$$

This iterative scheme could be used in the numerical approximation of the inequality problem *VHVI*.

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