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Dedicated to Prof. Emeritus Mihail Megan on the occasion of his 75^{th} anniversary

A characterization of fuzzy fractals generated by an orbital fuzzy iterated function system

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ABSTRACT. Orbital fuzzy iterated function systems are obtained as a combination of the concepts of iterated fuzzy set system and orbital iterated function system. It turns out that, for such a system, the corresponding fuzzy operator is weakly Picard, its fixed points being called fuzzy fractals. In this paper we present a structure result concerning fuzzy fractals associated to an orbital fuzzy iterated function system by proving that such an object is perfectly determined by the action of the initial term of the Picard iteration sequence on the closure of the orbits of certain elements.

1. INTRODUCTION

Fuzzy sets have their origin in Zadeh's remark that "more often than not, the classes of objects encountered in the real physical world do not have precisely defined criteria of membership" (see [16]). They have been introduced, in 1965, with the purpose of reconciling mathematical modelling and human knowledge in engineering sciences. To be precise, Zadeh was focused on their potential applications "in human thinking, particularly in the domain of pattern recognition, communication of information, and abstraction".

Theory of iterated function systems, which was initiated, in 1981, by J. Hutchinson (see [8]) and enriched by M. Barnsley and S. Demko (see [2] and [3]), has at its core the construction of deterministic fractals and measures. It has applications in image processing, stochastic growth model, random dynamical systems, bioinformatics, economics, finance, engineering sciences, human anatomy, physics etc.

The fuzzification Zadeh's idea was naturally adjusted to the Hutchinson-Barnsley theory of iterated function systems. More precisely, in 1991, C. Cabrelli, B. Forte, U. Molter and E. Vrscay (see [4] and [5]) introduced the concept of iterated fuzzy set system which consists of a finite family $(f_i)_{i\in I}$ of contractions on a compact metric space (X, d) together with a family of "grey level" maps $(\phi_i)_{i\in I}$, where $\phi_i : [0,1] \to [0,1]$. One can associate to such a system an operator on the class of normalized uppersemicontinuous fuzzy sets of X which turns out to be a contraction with respect to a metric d_∞ involving the Hausdorff-Pompeiu distances between level sets. Its unique fixed point is called the invariant fuzzy set. The relevance of this theory to image processing is mentioned in [5]. The continuity properties of the invariant fuzzy set with respect to changes in the contractions f_i and grey level maps ϕ_i are studied in [7]. Other papers dealing with iterated fuzzy set systems are [1] and [13]. Let us also mention that R. Uthayakumar and D. Easwaramoorthy (see [15]) studied the Hutchinson-Barnsley theory in the framework of the fuzzy hyperspace with respect to the Hausdorrf-Pompeiu fuzzy metric.

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Another natural generalization of Hutchinson's concept of iterated function system, termed as orbital iterated function system, was recently considered in [9], [10] and [14]. Here, the idea is to consider iterated function systems consisting of continuous functions satisfying Banach's orbital condition. The novelty of this approach is that the associated fractal operator is weakly Picard. It comes to the light that this is a genuine generalization since there exist such systems for which the fractal operator is weakly Picard, but not Picard (see Remark 4.1 from [9]). For extra properties of this kind of system see [10] and [14].

A natural continuation of research lines previously mentioned is to examine the so called orbital fuzzy iterated function systems (see [12]) which are obtained as a combination of iterated fuzzy set systems and orbital iterated function systems. It was proved (see Theorem 3.1 from [12]) that the corresponding fuzzy operator is weakly Picard. Its fixed points are called fuzzy fractals. More precisely, let us suppose that $\mathcal{S}_Z$ is such a system, where $\mathcal{S}_Z = ((X,d),(f_i)_{i\in I},(\rho_i)_{i\in I})$, $Z : \mathcal{F}^*_{\mathcal{S}} \to \mathcal{F}^*_{\mathcal{S}}$ (where $\mathcal{F}^*_{\mathcal{S}}$ is a certain class of fuzzy sets - see Section 2 for details) is the fuzzy Hutchinson-Barnsley operator associated to $\mathcal{S}_Z$ and let us arbitrarily choose an element u from $\mathcal{F}^*_{\mathcal{S}}$. Then the sequence $(Z^{[n]}(u))_{n\in\mathbb{N}}$ is convergent and its limit, denoted by $\mathbf{u}_u$, is a fuzzy fractal.

The goal of the present paper is to provide, for each $u \in \mathcal{F}^*_{\mathcal{S}}$, a description of the fuzzy fractal $\mathbf{u}_u$ in terms of certain fuzzy fractals $\mathbf{u}_x$ obtained as the limit of the Picard iteration sequence which starts with a fuzzy set u^x associated to u and $x \in X$ such that $u(x) > 0$. More precisely, Theorem 2.1, which is our main result, states that $\mathbf{u}_u = \max\limits_{x \text{ such that } u(x)>0} \mathbf{u}_x = \max\limits_{x \text{ such that } u(x)=1} \mathbf{u}_x$.

A. Basic notations and terminology

By $\mathbb{N}$ we mean the set $\{1,2,...\}$.

For a family of functions $(f_i)_{i\in I}$, where $f_i : X \to \mathbb{R}$, we shall use the following notation:

$$\sup_{i\in I} f_i \overset{\text{not}}{=} \bigvee_{i\in I} f_i.$$

For a function $f : X \to X$ and $n \in \mathbb{N}$, the composition of f by itself n times is denoted by $f^{[n]}$.

A function $f : X \to X$, where (X,d) is a metric space, is called weakly Picard operator if the sequence $(f^{[n]}(x))_{n\in\mathbb{N}}$ is convergent for every $x \in X$ and the limit (which may depend on x) is a fixed point of f. A weakly Picard operator having a unique fixed point is called Picard operator.

For a subset A of a metric space (X,d), by $diam(A)$ we mean the diameter of A i.e. $\sup\limits_{x,y\in A} d(x,y)$.

For a metric space (X,d), we shall use the following notations:

$$\{A \subseteq X \mid A \neq \emptyset \text{ and } A \text{ is bounded}\} \overset{\text{not}}{=} P_b(X),$$

$$\{A \subseteq X \mid A \neq \emptyset \text{ and } A \text{ is closed}\} \overset{\text{not}}{=} P_{cl}(X),$$

$$P_b(X) \cap P_{cl}(X) \overset{\text{not}}{=} P_{b,cl}(X),$$

$$\{A \subseteq X \mid A \neq \emptyset \text{ and } A \text{ is compact}\} \overset{\text{not}}{=} P_{cp}(X).$$

For a metric space (X,d), by h we designate the Hausdorff-Pompeiu metric on X, i.e. the function $h : P_{b,cl}(X) \times P_{b,cl}(X) \to [0,\infty)$, described by

$$h(K_1,K_2) = \max\left\{\sup_{x\in K_1} d(x,K_2), \sup_{x\in K_2} d(x,K_1)\right\},$$

for every $K_1, K_2 \in P_{b,cl}(X)$.

Remark 1.1. $(P_{cp}(X), h)$ is a complete metric space, provided that (X, d) is complete. If $(A_n)_{n\in\mathbb{N}} \subseteq P_{cp}(X)$ is Cauchy, then $\lim\limits_{n\to\infty} A_n = \{x \in X \mid$ there exists a strictly increasing sequence $(n_k)_{k\in\mathbb{N}} \subseteq \mathbb{N}$ and $x_{n_k} \in A_{n_k}$ for every $k \in \mathbb{N}$ such that $\lim\limits_{k\to\infty} x_{n_k} = x\}$.

B. Fuzzy sets

For a set X, we shall use the following notation:

$$\{u : X \to [0,1]\} \overset{\text{not}}{=} \mathcal{F}_X.$$

The elements of $\mathcal{F}_X$ are called fuzzy subsets of X.

A non-zero function $\rho : [0,1] \to [0,1]$ is called a grey level map.

To every grey level map ρ and $u \in \mathcal{F}_X$ one could associate the element of $\mathcal{F}_X$, denoted by $\rho(u)$, given by $\rho \circ u$.

$u \in \mathcal{F}_X$ is called normal if there exists $x \in X$ such that $u(x) = 1$.

For $u \in \mathcal{F}_X$ and $\alpha \in (0,1]$, we shall use the following notations:

$$\{x \in X \mid u(x) \geq \alpha\} \overset{\text{not}}{=} [u]^\alpha,$$

$$\{x \in X \mid u(x) > 0\} \overset{\text{not}}{=} [u]^*.$$

Given a metric space (X, d), $u \in \mathcal{F}_X$ is called compactly supported if supp$u := \overline{[u]^*} \overset{\text{not}}{=} [u]^0 \in P_{cp}(X)$.

For a metric space (X, d), we shall use the following notations:

$$\{u \in \mathcal{F}_X \mid u \text{ is normal and compactly supported}\} \overset{\text{not}}{=} \mathcal{F}_X^{**},$$

$$\{u \in \mathcal{F}_X^{**} \mid u \text{ is upper semicontinuous}\} \overset{\text{not}}{=} \mathcal{F}_X^{*}.$$

For a metric space (X, d) and $x \in X$, we consider $\delta_x \in \mathcal{F}_X^*$ given by

$$\delta_x(t) = \begin{cases} 1, \text{ if } t = x \\ 0, \text{ if } t \neq x \end{cases},$$

for every $t \in X$.

Remark 1.2.

$$\text{supp}\delta_x = \{x\},$$

for every $x \in X$.

To every $f : X \to Y$ and $u \in \mathcal{F}_X$ one could associate an element of $\mathcal{F}_Y$, denoted by $f(u)$, which is described in the following way:

$$f(u)(y) = \begin{cases} \sup\limits_{x\in f^{-1}(\{y\})} u(x), \text{ if } f^{-1}(\{y\}) \neq \emptyset \\ 0, \text{ if } f^{-1}(\{y\}) = \emptyset \end{cases},$$

for every $y \in Y$.

For a metric space (X, d), the function $d_\infty : \mathcal{F}_X^{**} \times \mathcal{F}_X^{**} \to [0, \infty]$, given by

$$d_\infty(u, v) \overset{\text{def}}{=} \sup_{\alpha\in[0,1]} h([u]^\alpha, [v]^\alpha) \overset{\text{Lemma 2.5 from [12]}}{=} \sup_{\alpha\in(0,1]} h([u]^\alpha, [v]^\alpha),$$

for every $u, v \in \mathcal{F}_X^{**}$, is semidistance on $\mathcal{F}_X^{**}$. Its restriction to $\mathcal{F}_X^* \times \mathcal{F}_X^*$ is a metric on $\mathcal{F}_X^*$ (see [6]), which, for the sake of simplicity, will be also denoted by d_∞. Moreover $(\mathcal{F}_X^*, d_\infty)$ is a complete metric space provided that the metric space (X, d) is complete.

C. Iterated function systems

An iterated function system (IFS for short) consists of:

i) a complete metric space (X, d);

ii) a finite family of contractions $f_i : X \to X$, with $i \in I$.

We denote by $\mathcal{S} = ((X, d), (f_i)_{i\in I})$ such an IFS.

One can associate to such a system $\mathcal{S} = ((X, d), (f_i)_{i\in I})$ the function $F_{\mathcal{S}} : P_{cp}(X) \to P_{cp}(X)$, given by

$$F_{\mathcal{S}}(K) = \underset{i\in I}{\cup} f_i(K),$$

for all $K \in P_{cp}(X)$, which is called the fractal operator associated to $\mathcal{S}$.

It turns out (see [8]) that $F_{\mathcal{S}}$ is a Banach contraction on the complete metric space $(P_{cp}(X), h)$, so it is a Picard operator with respect to h and its fixed point (which is denoted by $A_{\mathcal{S}}$) is called the attractor of $\mathcal{S}$.

D. Iterated fuzzy function systems

An iterated fuzzy function system consists of:

i) an iterated function system $\mathcal{S} = ((X, d), (f_i)_{i\in I})$;

ii) an admissible system of grey level maps $(\rho_i)_{i\in I}$ i.e. $\rho_i(0) = 0$, ρ_i is nondecreasing and right continuous for every $i \in I$ and there exists $j \in I$ such that $\rho_j(1) = 1$.

We denote by $\mathcal{S}_Z = ((X, d), (f_i)_{i\in I}, (\rho_i)_{i\in I})$ such a system.

One can associate to such a system $\mathcal{S}_Z = ((X, d), (f_i)_{i\in I}, (\rho_i)_{i\in I})$ the function $Z : \mathcal{F}_X^* \to \mathcal{F}_X^*$, given by

$$Z(u) = \underset{i\in I}{\vee} \rho_i(f_i(u)),$$

for all $u \in \mathcal{F}_X^*$, which is called the fuzzy Hutchinson-Barnsley operator associated to $\mathcal{S}_Z$. Note that Z is well defined (see Proposition 2.12 from [13]).

It turns out (see Theorem 2.14 from [13]) that Z is a Banach contraction on the complete metric space $(\mathcal{F}_X^*, d_\infty)$ (so it is a Picard operator) whose unique fixed point is called the fuzzy fractal generated by $\mathcal{S}_Z$ (note that its support is a subset of $A_{\mathcal{S}}$ - see Theorem 2.4.2 from [4] or Theorem 2.21 from [13]).

E. Orbital iterated function systems

An orbital iterated function system consists of:

i) a complete metric space (X, d);

ii) a finite family of continuous functions $f_i : X \to X$, $i \in I$, having the property that there exists $C \in [0, 1)$ such that $d(f_i(y), f_i(z)) \le Cd(y, z)$ for every $i \in I$, $x \in X$ and $y, z \in \mathcal{O}(x)$, where, for $B \in P_{cp}(X)$, by the orbit of B, denoted by $\mathcal{O}(B)$, we mean the set $B \cup \underset{n\in\mathbb{N},\, \omega_1,..,\omega_n\in I}{\cup} (f_{\omega_1} \circ f_{\omega_2} \circ ... \circ f_{\omega_n})(B)$ and we adopt the notation $\mathcal{O}(\{x\}) \overset{\text{not}}{=} \mathcal{O}(x)$ for every $x \in X$.

We denote by $\mathcal{S} = ((X, d), (f_i)_{i\in I})$ such a system.

As for the case of IFSs, one can associate to an orbital iterated function system $\mathcal{S}$ its fractal operator. It turns out (see [10]) that the fractal operator associated to an orbital function system is a weakly Picard operator with respect to the Hausdorff-Pompeiu metric, every of its fixed points being called an attractor of the system.

If $\mathcal{S} = ((X, d), (f_i)_{i\in I})$ is an orbital iterated function system, $K \in P_{cp}(X)$ and $x \in X$, we shall use the following notations:

$$\lim_{n\to\infty} F_S^{[n]}(K) \overset{\text{not}}{=} A_K$$

and

$$A_{\{x\}} \overset{\text{not}}{=} A_x.$$

F. Orbital fuzzy iterated function systems

An orbital fuzzy iterated function system consists of:

i) an orbital iterated function system $((X,d),(f_i)_{i\in I})$;

ii) an admissible system of grey level maps $(\rho_i)_{i\in I}$ i.e. $\rho_i(0)=0$, ρ_i is nondecreasing and right continuous for every $i\in I$ and there exists $j\in I$ such that $\rho_j(1)=1$.

We denote by $\mathcal{S}_Z=((X,d),(f_i)_{i\in I},(\rho_i)_{i\in I})$ such a system.

One can associate to such a system $\mathcal{S}_Z=((X,d),(f_i)_{i\in I},(\rho_i)_{i\in I})$ the function $Z:\mathcal{F}_X^{**}\to\mathcal{F}_X^{**}$, given by

$$Z(u)=\underset{i\in I}{\vee}\rho_i(f_i(u)),$$

for all $u\in\mathcal{F}_X^{**}$, which is called the fuzzy Hutchinson-Barnsley operator associated to $\mathcal{S}_Z$. Note that, in view of Proposition 2.12 from [13], Z is well defined.

We shall use the following notations:

$$\{u\in\mathcal{F}_X^{**}\mid \text{for each } x\in[u]^* \text{ there exist}$$

$$w_x,y_x\in X \text{ such that } x,y_x\in\mathcal{O}(w_x) \text{ and } u(y_x)=1\}\overset{\text{not}}{=}\mathcal{F}_{\mathcal{S}}^{**}$$

and

$$\{u\in\mathcal{F}_{\mathcal{S}}^{**}\mid u \text{ is upper semicontinuous}\}\overset{\text{not}}{=}\mathcal{F}_{\mathcal{S}}^{*}.$$

Note that

$$\delta_x\in\mathcal{F}_{\mathcal{S}}^{*},$$

for every $x\in X$.

For $u\in\mathcal{F}_{\mathcal{S}}^{*}$ and $x\in[u]^*$ (hence there exist $w_x,y_x\in X$ such that $x,y_x\in\mathcal{O}(w_x)$ and $u(y_x)=1$), we shall use the following notations:

$$\lim_{n\to\infty}Z^{[n]}(u)\overset{\text{not}}{=}\mathbf{u}_u\overset{\text{Lemma 3.1 from [12] \& Remark 1.3}}{\in}\mathcal{F}_{\mathcal{S}}^{*},$$

$$\lim_{n\to\infty}Z^{[n]}(u^x)\overset{\text{not}}{=}\mathbf{u}_x\overset{\text{Lemma 3.1 from [12] \& Remark 1.3}}{\in}\mathcal{F}_{\mathcal{S}}^{*}$$

and

$$w_u\overset{\text{not}}{=}\underset{x\in[u]^*}{\vee}\mathbf{u}_x,$$

where $u^x\in\mathcal{F}_{\mathcal{S}}^{*}$ is described by

$$u^x(y)=\begin{cases} u(y), \text{ if } y\in\overline{\mathcal{O}(w_x)}\\ 0, \text{ otherwise}\end{cases},$$

for every $y\in X$. Note that the existence of the above limits is based on Remark 1.5 and that, according to Proposition 2.1, $\mathbf{u}_x$ is well defined.

Remark 1.3. (see Proposition 2.11 and Lemma 3.3 from [12]). In the above framework, we have

$$Z(\mathcal{F}_X^{*})\subseteq\mathcal{F}_X^{*},\ Z(\mathcal{F}_{\mathcal{S}}^{**})\subseteq\mathcal{F}_{\mathcal{S}}^{**} \text{ and } Z(\mathcal{F}_{\mathcal{S}}^{*})\subseteq\mathcal{F}_{\mathcal{S}}^{*}.$$

Remark 1.4. (see Claim 3.5 from the proof of Theorem 3.1 from [12]). In the above framework, $\mathbf{Z}:\mathcal{F}_{\mathcal{S}}^{*}\to\mathcal{F}_{\mathcal{S}}^{*}$, given by $\mathbf{Z}(u)=Z(u)$ for every $u\in\mathcal{F}_{\mathcal{S}}^{*}$, is continuous.

Remark 1.5. (see Theorem 3.1 from [12]). In the above framework, $\mathbf{Z}$ is weakly Picard. Its fixed points are called fuzzy fractals generated by the orbital fuzzy iterated function system $\mathcal{S}_Z$.

Remark 1.6. (see Lemma 3.4 from [12]). In the above framework, for each family $(u_j)_{j\in J}$ of elements from $\mathcal{F}_X^{**}$, where J is infinite, we have

$$\underset{j\in J}{\vee} Z(u_j) = \max_{j\in J} Z(u_j),$$

provided that:

i) there exists $K \in P_{cp}(X)$ such that supp$u_j \subseteq K$ for all $j \in J$;

ii) $\underset{j\in J}{\vee} u_j = \max\limits_{j\in J} u_j$;

iii) $\underset{j\in J}{\vee} u_j \in \mathcal{F}_X^{*}$.

Remark 1.7. (see Lemma 3.5 from [12]). In the above framework, for every families $(u_j)_{j\in J}$ and $(v_j)_{j\in J}$ of elements from $\mathcal{F}_X^{**}$, where J is infinite, we have

$$d_\infty(\underset{j\in J}{\vee} u_j, \underset{j\in J}{\vee} v_j) \leq \sup_{j\in J} d_\infty(u_j, v_j),$$

provided that:

i) there exists $K \in P_{cp}(X)$ such that supp$u_j \subseteq K$ and supp$v_j \subseteq K$ for all $j \in J$;

ii) $\underset{j\in J}{\vee} u_j = \max\limits_{j\in J} u_j$ and $\underset{j\in J}{\vee} v_j = \max\limits_{j\in J} v_j$.

Remark 1.8. In the above framework,

$$\text{supp}\mathbf{u}_x \subseteq A_x,$$

for every $u \in \mathcal{F}_{\mathcal{S}}^{*}$ and $x \in [u]^{*}$.

Indeed, it results from the following relations:

$$\text{supp}\mathbf{u}_x \overset{\text{Remark 1.4}}{=} \text{supp}Z(\mathbf{u}_x) \overset{\text{Lemma 3.7 from [12]}}{\subseteq} F_{\mathcal{S}}(\text{supp}\mathbf{u}_x).$$

It also can be derived from Theorem 2.4.2 from [4].

Remark 1.9. (see Lemma 4.5 from [11]). In the above framework, for every $x_1, x_2 \in X$, we have

$$A_{x_1} = A_{x_2}$$

provided that $\overline{\mathcal{O}(x_1)} \cap \overline{\mathcal{O}(x_1)} \neq \emptyset$.

Remark 1.10. (see Remark 2.1 from [12]). In the above framework,

$$\overline{\mathcal{O}(x)} = \mathcal{O}(x) \cup A_x,$$

for every $x \in X$.

Remark 1.11. In the above framework,

$$w_u \in \mathcal{F}_X^{**},$$

for every $u \in \mathcal{F}_{\mathcal{S}}^{*}$.

Indeed, on the one hand, as $\mathbf{u}_x \in \mathcal{F}_X^{*}$ is normal for every $x \in [u]^{*}$, we deduce that w_u is normal. On the other hand, since supp$\mathbf{u}_x \overset{\text{Remark 1.8}}{\subseteq} A_x \overset{\text{Proposition 5 from [14]}}{\subseteq} A_{\text{supp}u} \in P_{cp}(X)$ for every $x \in [u]^{*}$, we conclude that suppw_u =supp $\underset{x\in[u]^{*}}{\vee} \mathbf{u}_x \subseteq A_{\text{supp}u} \in P_{cp}(X)$, so supp$w_u \in P_{cp}(X)$.

Remark 1.12. In the above framework,

$$\text{supp}u^x \subseteq \text{supp}u,$$

for every $u \in \mathcal{F}_{\mathcal{S}}^{*}$ and $x \in [u]^{*}$.

Remark 1.13. (see Claim 3.2 from the proof of Theorem 3.1 from [12]). In the above framework, we have:

a)

$$Z^{[n]}(u) = \bigvee_{x \in [u]^*} Z^{[n]}(u^x),$$

for every $n \in \mathbb{N}$ and every $u \in \mathcal{F}_{\mathcal{S}}^*$.

b)

$$d_\infty(Z^{[n]}(u), \mathbf{u}_u) \leq \frac{C^n}{1-C} diam(F_S(\text{supp}u) \cup \text{supp}u),$$

for all $n \in \mathbb{N}$ and $u \in \mathcal{F}_{\mathcal{S}}^*$.

2. Main results

Proposition 2.1. *Let $\mathcal{S}_Z = ((X,d),(f_i)_{i\in I},(\rho_i)_{i\in I})$ be an orbital fuzzy iterated function system, $u \in \mathcal{F}_{\mathcal{S}}^*$ and $x \in [u]^*$ (hence there exist $w_x, y_x \in X$ such that $x, y_x \in \mathcal{O}(w_x)$ and $u(y_x) = 1$). Then*

$$\lim_{n\to\infty} Z^{[n]}(\delta_s) = \mathbf{u}_x,$$

for every $s \in \overline{\mathcal{O}(w_x)}$. In particular

$$\lim_{n\to\infty} Z^{[n]}(\delta_x) = \mathbf{u}_x.$$

Proof. Because $((\overline{\mathcal{O}(w_x)}, d), (\widetilde{f_i})_{i\in I}, (\rho_i)_{i\in I})$, where $\widetilde{f_i} : \overline{\mathcal{O}(w_x)} \to \overline{\mathcal{O}(w_x)}$ is given by $\widetilde{f_i}(y) = f_i(y)$ for every $y \in \overline{\mathcal{O}(w_x)}$, has a unique fuzzy fractal (as it is an iterated fuzzy function system) and $\delta_{s|X\smallsetminus\overline{\mathcal{O}(w_x)}} = u^x_{|X\smallsetminus\overline{\mathcal{O}(w_x)}} = Z^{[n]}(\delta_{s|X\smallsetminus\overline{\mathcal{O}(w_x)}}) = Z^{[n]}(u^x_{|X\smallsetminus\overline{\mathcal{O}(w_x)}}) = 0$ for every $s \in \overline{\mathcal{O}(w_x)}$ and every $n \in \mathbb{N}$, we deduce that

$$\lim_{n\to\infty} Z^{[n]}(\delta_s) = \lim_{n\to\infty} Z^{[n]}(u^x) = \mathbf{u}_x,$$

for every $s \in \overline{\mathcal{O}(w_x)}$. □

Proposition 2.2. *Let $\mathcal{S}_Z = ((X,d),(f_i)_{i\in I},(\rho_i)_{i\in I})$ be an orbital fuzzy iterated function system and $u \in \mathcal{F}_{\mathcal{S}}^*$. Then*

$$\mathbf{u}_y = \mathbf{u}_x,$$

for every $x \in [u]^*$ and every $y \in [\mathbf{u}_x]^*$.

Proof. Let us consider $x \in [u]^*$ and $y \in [\mathbf{u}_x]^*$. As $u \in \mathcal{F}_{\mathcal{S}}^*$, there exist $w_x, y_x \in X$ such that

$$x, y_x \in \mathcal{O}(w_x) \tag{2.1}$$

and $u(y_x) = 1$.

In addition

$$y \in supp\mathbf{u}_x \overset{\text{Remark 1.8}}{\subseteq} A_x \overset{\text{(2.1) \& Remark 1.9}}{=} A_{w_x} \overset{\text{Remark 1.10}}{\subseteq} \overline{\mathcal{O}(w_x)}. \tag{2.2}$$

Therefore

$$\mathbf{u}_x \overset{\text{(2.2) \& Proposition 2.1}}{=} \lim_{n\to\infty} Z^{[n]}(\delta_y) \overset{\text{Proposition 2.1}}{=} \mathbf{u}_y.$$

□

Proposition 2.3. *Let $\mathcal{S}_Z = ((X,d),(f_i)_{i\in I},(\rho_i)_{i\in I})$ be an orbital fuzzy iterated function system and $u \in \mathcal{F}_{\mathcal{S}}^*$. Then the function $U : [u]^* \to \mathcal{F}_X^*$, given by*

$$U(x) = \mathbf{u}_x,$$

for every $x \in [u]^*$, is continuous.

Proof. We are going to prove that U is sequentially continuous.

We consider $(x_n)_{n\in\mathbb{N}} \subseteq [u]^*$ and $x \in [u]^*$ such that $\lim_{n\to\infty} x_n = x$ and we will prove that $\lim_{n\to\infty} U(x_n) = U(x)$, i.e. $\lim_{n\to\infty} \mathbf{u}_{x_n} = \mathbf{u}_x$.

We have

$$\text{supp}\delta_x \overset{\text{Remark 1.2}}{=} \{x\} \subseteq K \text{ and } \text{supp}\delta_{x_n} \overset{\text{Remark 1.2}}{=} \{x_n\} \subseteq K, \tag{2.3}$$

for every $n \in \mathbb{N}$, where

$$\{x_n \mid n \in \mathbb{N}\} \cup \{x\} \overset{\text{not}}{=} K \in P_{cp}(X).$$

Since

$$\lim_{n\to\infty} d_\infty(\delta_{x_n}, \delta_x) = \lim_{n\to\infty} \sup_{\alpha\in(0,1]} h([\delta_{x_n}]^\alpha, [\delta_x]^\alpha) =$$

$$= \lim_{n\to\infty} h(\{x_n\}, \{x\}) = \lim_{n\to\infty} d(x_n, x) = 0,$$

via Remark 1.4, we infer that

$$\lim_{n\to\infty} d_\infty(Z^{[m]}(\delta_{x_n}), Z^{[m]}(\delta_x)) = 0, \tag{2.4}$$

for every $m \in \mathbb{N}$.

The equality

$$\lim_{n\to\infty} Z^{[n]}(\delta_s) \overset{\text{Proposition 2.1}}{=} \mathbf{u}_s,$$

which is valid for every $s \in [u]^*$, leads to the conclusion that

$$d_\infty(\mathbf{u}_s, Z^{[n]}(\delta_s)) \overset{\text{Remark 1.2 \& Remark 1.13, b)}}{\leq} \frac{C^n}{1-C} diam(F_S(\{s\}) \cup \{s\}), \tag{2.5}$$

for every $s \in [u]^*$ and every $n \in \mathbb{N}$.

Note that

$$d_\infty(\mathbf{u}_{x_n}, \mathbf{u}_x) \leq$$

$$\leq d_\infty(\mathbf{u}_{x_n}, Z^{[m]}(\delta_{x_n})) + d_\infty(Z^{[m]}(\delta_{x_n}), Z^{[m]}(\delta_x)) + d_\infty(Z^{[m]}(\delta_x), \mathbf{u}_x) \overset{(2.5)}{\leq}$$

$$\leq 2\frac{C^m}{1-C} diam(F_S(K) \cup K) + d_\infty(Z^{[m]}(\delta_{x_n}), Z^{[m]}(\delta_x)), \tag{2.6}$$

for every $m, n \in \mathbb{N}$.

Let us consider a fixed $\varepsilon > 0$, but arbitrarily chosen.

As $\lim_{m\to\infty} 2\frac{C^m}{1-C} diam(F_S(K) \cup K) = 0$, there exists $m_0 \in \mathbb{N}$ such that $2\frac{C^{m_0}}{1-C} diam(F_S(K) \cup K) < \frac{\varepsilon}{2}$ and, via (2.6), we obtain

$$d_\infty(\mathbf{u}_{x_n}, \mathbf{u}_x) \leq \frac{\varepsilon}{2} + d_\infty(Z^{[m_0]}(\delta_{x_n}), Z^{[m_0]}(\delta_x)), \tag{2.7}$$

for every $n \in \mathbb{N}$.

Since $\lim_{n\to\infty} d_\infty(Z^{[m_0]}(\delta_{x_n}), Z^{[m_0]}(\delta_x)) \overset{(2.4)}{=} 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that

$$d_\infty(Z^{[m_0]}(\delta_{x_n}), Z^{[m_0]}(\delta_x)) < \frac{\varepsilon}{2}, \tag{2.8}$$

for every $n \in \mathbb{N}$, $n \geq n_\varepsilon$.

Using (2.7) and (2.8), we get

$$d_\infty(\mathbf{u}_{x_n}, \mathbf{u}_x) < \varepsilon,$$

for every $n \in \mathbb{N}$, $n \geq n_\varepsilon$, which proves that $\lim_{n\to\infty} \mathbf{u}_{x_n} = \mathbf{u}_x$. □

Proposition 2.4. *Let $\mathcal{S}_Z = ((X,d),(f_i)_{i\in I},(\rho_i)_{i\in I})$ be an orbital fuzzy iterated function system and $u \in \mathcal{F}^*_{\mathcal{S}}$. Then*

$$[\bigvee_{x\in[u]^*} \mathbf{u}_x]^\alpha = \bigcup_{x\in[u]^*} [\mathbf{u}_x]^\alpha,$$

for every $\alpha \in (0,1]$ and $u \in \mathcal{F}^*_{\mathcal{S}}$.

Proof. Let us consider a fixed $\alpha \in (0,1]$, but arbitrarily chosen.

First we prove the inclusion

$$[\bigvee_{x\in[u]^*} \mathbf{u}_x]^\alpha \subseteq \bigcup_{x\in[u]^*} [\mathbf{u}_x]^\alpha. \tag{2.9}$$

For $y \in [\bigvee_{x\in[u]^*} \mathbf{u}_x]^\alpha$ we have

$$\sup_{x\in[u]^*} \mathbf{u}_x(y) \geq \alpha \tag{2.10}$$

and consequently there exists $x \in [u]^*$ such that $\mathbf{u}_x(y) > 0$, i.e. $y \in [\mathbf{u}_x]^*$. Moreover Proposition 2.2 ensures us that $\mathbf{u}_x(y) = \mathbf{u}_y(y)$ for every $x \in [u]^*$ such that $\mathbf{u}_x(y) > 0$ and thus, via (2.10), we conclude that $\mathbf{u}_x(y) \geq \alpha$ (so $y \in [\mathbf{u}_x]^\alpha$) for every $x \in [u]^*$ such that $\mathbf{u}_x(y) > 0$. Consequently $y \in \bigcup_{x\in[u]^*} [\mathbf{u}_x]^\alpha$ and the justification of (2.9) is finished.

Now we prove the inclusion

$$\bigcup_{x\in[u]^*} [\mathbf{u}_x]^\alpha \subseteq [\bigvee_{x\in[u]^*} \mathbf{u}_x]^\alpha. \tag{2.11}$$

For $y \in \bigcup_{x\in[u]^*} [\mathbf{u}_x]^\alpha$ there exists $x_y \in [u]^*$ such that $y \in [\mathbf{u}_{x_y}]^\alpha$, i.e. $\mathbf{u}_{x_y}(y) \geq \alpha$. Hence $\sup_{x\in[u]^*} \mathbf{u}_x(y) \geq \mathbf{u}_{x_y}(y) \geq \alpha$, i.e. $y \in [\bigvee_{x\in[u]^*} \mathbf{u}_x]^\alpha$, and the justification of (2.11) is finalized.

In view of (2.9) and (2.11) the proof is completed. □

As a by-product of the above Proposition's proof we have the following:

Remark 2.14. Let $\mathcal{S}_Z = ((X,d),(f_i)_{i\in I},(\rho_i)_{i\in I})$ be an orbital fuzzy iterated function system. Then

$$\bigvee_{x\in[u]^*} \mathbf{u}_x = \max_{x\in[u]^*} \mathbf{u}_x,$$

for every $u \in \mathcal{F}^*_{\mathcal{S}}$.

Proposition 2.5. *Let $\mathcal{S}_Z = ((X,d),(f_i)_{i\in I},(\rho_i)_{i\in I})$ be an orbital fuzzy iterated function system and $u \in \mathcal{F}^*_{\mathcal{S}}$. Then*

$$\{\mathbf{u}_x \mid x \in [u]^*\} = \{\mathbf{u}_x \mid x \in [u]^1\}.$$

In particular

$$w_u = \bigvee_{x\in[u]^*} \mathbf{u}_x = \bigvee_{x\in[u]^1} \mathbf{u}_x.$$

Proof. Note that

$$\{\mathbf{u}_x \mid x \in [u]^*\} \subseteq \{\mathbf{u}_x \mid x \in [u]^1\}.$$

Indeed, for $x \in [u]^*$ there exist $w_x, y_x \in X$ such that $x, y_x \in \mathcal{O}(w_x)$ and $u(y_x) = 1$. Then, based on Proposition 2.1, we have

$$\mathbf{u}_x = \lim_{n\to\infty} Z^{[n]}(\delta_{y_x}) = \mathbf{u}_{y_x},$$

so, as $u(y_x) = 1$, i.e. $y_x \in [u]^1$, we infer that $\mathbf{u}_x = \mathbf{u}_{y_x} \in \{\mathbf{u}_x \mid x \in [u]^1\}$.

As the inclusion

$$\{\mathbf{u}_x \mid x \in [u]^1\} \subseteq \{\mathbf{u}_x \mid x \in [u]^*\}$$

is obvious, the proof is completed. □

Proposition 2.6. *Let $\mathcal{S}_Z = ((X,d),(f_i)_{i\in I},(\rho_i)_{i\in I})$ be an orbital fuzzy iterated function system and $u \in \mathcal{F}^*_{\mathcal{S}}$. Then w_u is upper semicontinuous, so $w_u \in \mathcal{F}^*_X$.*

Proof. We have to show that

$$\overline{\lim_{n\to\infty}} w_u(x_n) \le w_u(x^*), \tag{2.12}$$

for every $(x_n)_{n\in\mathbb{N}} \subseteq X$ and $x^* \in X$ such that $\lim\limits_{n\to\infty} x_n = x^*$.

It suffices to consider the case when $\overline{\lim\limits_{n\to\infty}} w_u(x_n) \overset{\text{not}}{=} L > 0$ since otherwise (2.12) is obvious.

Let us consider a subsequence of $(x_n)_{n\in N}$ such that $\lim\limits_{k\to\infty} w_u(x_{n_k}) = L$.

For the sake of simplicity, we denote $(x_{n_k})_{k\in\mathbb{N}}$ by $(z_n)_{n\in\mathbb{N}}$.

So $\lim\limits_{n\to\infty} z_n = x^*$ and $\lim\limits_{n\to\infty} w_u(z_n) = L > 0$ and therefore we can suppose that

$$w_u(z_n) \ge \frac{L}{2}, \tag{2.13}$$

for every $n \in \mathbb{N}$.

Since

$$z_n \overset{(2.13)}{\in} [w_u]^{\frac{L}{2}} = [\underset{x\in[u]^*}{\vee} \mathbf{u}_x]^{\frac{L}{2}} \overset{\text{Proposition 2.4}}{=}$$

$$= \underset{x\in[u]^*}{\cup} [\mathbf{u}_x]^{\frac{L}{2}} \overset{\text{Proposition 2.5}}{=} \underset{x\in[u]^1}{\cup} [\mathbf{u}_x]^{\frac{L}{2}},$$

for every $n \in \mathbb{N}$, we can consider a sequence $(y_n)_{n\in\mathbb{N}} \subseteq [u]^1 \subseteq$supp$u$ such that

$$z_n \in [\mathbf{u}_{y_n}]^{\frac{L}{2}}, \tag{2.14}$$

for every $n \in \mathbb{N}$.

As supp$u \in P_{cp}(X)$, there exists a subsequence $(y_{n_k})_{k\in\mathbb{N}}$ of $(y_n)_{n\in\mathbb{N}}$ and $y \in X$ such that $\lim\limits_{k\to\infty} y_{n_k} = y$.

The upper semicontinuity of u implies

$$u(y) \ge \overline{\lim_{k\to\infty}} u(y_{n_k}) \overset{y_{n_k}\in[u]^1}{=} 1 > 0. \tag{2.15}$$

Claim

$$w_u(z_{n_k}) = \mathbf{u}_{y_{n_k}}(z_{n_k}),$$

for every $k \in \mathbb{N}$.

Justification of the Claim. Let us consider a fixed $k \in \mathbb{N}$, but arbitrarily chosen. We have $w_u(z_{n_k}) = \underset{x\in[u]^*}{\vee} \mathbf{u}_x(z_{n_k}) \overset{y_{n_k}\in[u]^1}{\ge} \mathbf{u}_{y_{n_k}}(z_{n_k}) \overset{(2.14)}{\ge} \frac{L}{2} > 0$. Let us suppose, ad absurdum, that $w_u(z_{n_k}) > \mathbf{u}_{y_{n_k}}(z_{n_k})$ and let us consider $\beta \in \mathbb{R}$ such that

$$w_u(z_{n_k}) = \underset{x\in[u]^*}{\vee} \mathbf{u}_x(z_{n_k}) > \beta > \mathbf{u}_{y_{n_k}}(z_{n_k}). \tag{2.16}$$

Then there exists $s \in [u]^*$ such that

$$\mathbf{u}_s(z_{n_k}) > \beta > 0, \tag{2.17}$$

so, $z_{n_k} \in [\mathbf{u}_s]^*$ and, taking into account Proposition 2.2, we infer that

$$\mathbf{u}_s = \mathbf{u}_{z_{n_k}}. \tag{2.18}$$

Since $y_{n_k} \in [u]^1$ and $\mathbf{u}_{y_{n_k}}(z_{n_k}) \overset{(2.14)}{\ge} \frac{L}{2} > 0$, Proposition 2.2 assures us that

$$\mathbf{u}_{z_{n_k}} = \mathbf{u}_{y_{n_k}}. \tag{2.19}$$

The contradiction $\beta \overset{(2.17)}{<} \mathbf{u}_s(z_{n_k}) \overset{(2.18)\ \&\ (2.19)}{=} \mathbf{u}_{y_{n_k}}(z_{n_k}) \overset{(2.16)}{<} \beta$ ends the justification of the Claim.

Now let us consider a fixed $\varepsilon \in (0, \frac{L}{2})$, but arbitrarily chosen.

As $\lim\limits_{k\to\infty} w_u(z_{n_k}) = L$, there exists $k_\varepsilon \in \mathbb{N}$ such that $w_u(z_{n_k}) \overset{\text{Claim}}{=} \mathbf{u}_{y_{n_k}}(z_{n_k}) \geq L - \varepsilon$, i.e.

$$z_{n_k} \in [\mathbf{u}_{y_{n_k}}]^{L-\varepsilon}, \tag{2.20}$$

for every $k \in \mathbb{N}$, $k \geq k_\varepsilon$.

Since $\lim\limits_{k\to\infty} y_{n_k} = y$, via Proposition 2.3, we deduce that $\lim\limits_{k\to\infty} d_\infty(\mathbf{u}_{y_{n_k}}, \mathbf{u}_y) = 0$ and therefore

$$\lim_{k\to\infty} h([\mathbf{u}_{y_{n_k}}]^{L-\varepsilon}, [\mathbf{u}_y]^{L-\varepsilon}) = 0. \tag{2.21}$$

As $\lim\limits_{k\to\infty} z_{n_k} = x^*$, based on (2.20), (2.21) and Remark 1.1, we infer that $x^* \in [\mathbf{u}_y]^{L-\varepsilon}$, i.e.

$$\mathbf{u}_y(x^*) \geq L - \varepsilon > 0. \tag{2.22}$$

A similar argument with the one used in the justification of the Claim assures us that

$$\mathbf{u}_y(x^*) = w_u(x^*). \tag{2.23}$$

Indeed, we have $w_u(x^*) = \bigvee\limits_{x\in[u]^*} \mathbf{u}_x(x^*) \overset{(2.15)}{\geq} \mathbf{u}_y(x^*)$. Let us suppose, ad absurdum, that $w_u(x^*) > \mathbf{u}_y(x^*)$ and let us consider $\beta \in \mathbb{R}$ such that $w_u(x^*) = \bigvee\limits_{s\in[u]^*} \mathbf{u}_s(x^*) > \beta > \mathbf{u}_y(x^*)$. Then there exists $s \in [u]^*$ such that $\mathbf{u}_s(x^*) > \beta > 0$. We have $\mathbf{u}_s \overset{\text{Proposition 2.2}}{=} \mathbf{u}_{x^*} \overset{\text{Proposition 2.2}}{=} \mathbf{u}_y$ and therefore we get the contradiction $\beta < \mathbf{u}_s(x^*) = \mathbf{u}_y(x^*) < \beta$.

Thus, via (2.22) and (2.23) we infer that

$$w_u(x^*) \geq L - \varepsilon. \tag{2.24}$$

Relation (2.24) implies the validity of (2.12). □

Proposition 2.7. *Let $\mathcal{S}_Z = ((X,d), (f_i)_{i\in I}, (\rho_i)_{i\in I})$ be an orbital fuzzy iterated function system and $u \in \mathcal{F}^*_{\mathcal{S}}$. Then*

$$d_\infty(\mathbf{u}_u, w_u) = 0.$$

Proof. First let us note that, based on Lemma 3.4 from [12], Remark 2.14 and Proposition 2.6, a repeated use of Remark 1.6 ensures us that

$$\bigvee_{x\in[u]^*} Z^{[n]}(u^x) = \max_{x\in[u]^*} Z^{[n]}(u^x) \tag{2.25}$$

for every $n \in \mathbb{N}$.

We have

$$d_\infty(\mathbf{u}_u, w_u) \overset{\text{Remark 1.13, a)}}{\leq} d_\infty(\mathbf{u}_u, Z^{[n]}(u)) + d_\infty(\bigvee_{x\in[u]^*} Z^{[n]}(u^x), \bigvee_{x\in[u]^*} \mathbf{u}_x) \leq$$

$$\overset{\text{(2.25), Remark 1.7 \& Remark 2.14}}{\leq} d_\infty(\mathbf{u}_u, Z^{[n]}(u)) + \sup_{x\in[u]^*} d_\infty(Z^{[n]}(u^x), \mathbf{u}_x) \leq$$

$$\overset{\text{Remark 1.13, b)}}{\leq} d_\infty(\mathbf{u}_u, Z^{[n]}(u)) + \frac{C^n}{1-C} \sup_{x\in[u]^*} diam(F_S(\text{supp}u^x) \cup \text{supp}u^x) \leq$$

$$\overset{\text{Remark 1.12}}{\leq} d_\infty(\mathbf{u}_u, Z^{[n]}(u)) + \frac{C^n}{1-C} diam(F_S(\text{supp}u) \cup \text{supp}u), \tag{2.26}$$

for every $n \in \mathbb{N}$.

As

$$\lim_{n\to\infty} d_{\infty}(\mathbf{u}_u, Z^{[n]}(u)) = \lim_{n\to\infty} \frac{C^n}{1-C} diam(F_S(suppu) \cup suppu) = 0,$$

the conclusion follows by passing to limit, as n goes to ∞, in (2.26). □

Now we state what can be viewed as the main theorem of this paper.

Theorem 2.1. *Let $\mathcal{S}_Z = ((X,d),(f_i)_{i\in I},(\rho_i)_{i\in I})$ be an orbital fuzzy iterated function system and $u \in \mathcal{F}_{\mathcal{S}}^*$. Then*

$$\mathbf{u}_u = \bigvee_{x\in[u]^*} \mathbf{u}_x = \bigvee_{x\in[u]^1} \mathbf{u}_x = \max_{x\in[u]^*} \mathbf{u}_x = \max_{x\in[u]^1} \mathbf{u}_x.$$

Proof. Since $\mathbf{u}_u \in \mathcal{F}_X^*$, $w_u \overset{\text{Proposition 2.6}}{\in} \mathcal{F}_X^*$, $d_{\infty}(\mathbf{u}_u, w_u) \overset{\text{Proposition 2.7}}{=} 0$ and d_{∞} is a metric on $\mathcal{F}_X^*$ we conclude that $\mathbf{u}_u = w_u$, i.e., taking into account Proposition 2.5, we have

$$\mathbf{u}_u = \bigvee_{x\in[u]^*} \mathbf{u}_x = \bigvee_{x\in[u]^1} \mathbf{u}_x.$$

Moreover, by virtue of Remark 2.14 and Proposition 2.5, we have

$$\mathbf{u}_u = \bigvee_{x\in[u]^*} \mathbf{u}_x = \bigvee_{x\in[u]^1} \mathbf{u}_x = \max_{x\in[u]^*} \mathbf{u}_x = \max_{x\in[u]^1} \mathbf{u}_x.$$

□

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