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Dedicated to Prof. Emeritus Mihail Megan on the occasion of his 75^{th} *anniversary*

Lie symmetries of the nonlinear Fokker-Planck equation based on weighted Tsallis entropy

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ABSTRACT. We determine the nonlinear Fokker-Planck equation in one dimension, based on a weighted Tsallis entropy and we derive its associated Lie symmetries. The corresponding Lyapunov functions and Bregman divergences are also found, together with a formula linking the drift function, the diffusion function and a diffusion constant. We solve the MaxEnt problem associated to the weighted Tsallis entropy.

1. INTRODUCTION

The nonlinear Fokker-Planck equation (NFPE) is one of the fundamental equations in Statistical Mechanics. It governs phenomena which may be modeled by the time evolution of the probability density function of the velocity of a particle, moving under the influence of both deterministic forces and random forces.

As it happened with other remarkable notions from Physics, it was adapted for many other scientific domains, through an epistemiologic transfer pattern evolving from "Mechanism" to "Physicalism" (see [18] for a philosophical detour).

In this paper, we make a creative review of the Lie symmetries associated to the NFPE and apply it to a general statistical model, based on the weighted Tsallis entropy. We act inside the "geometrization paradigm", looking for hidden differential geometric information from invariants associated to the NFPE. In order to describe our recipe, it is convenient first to look at the ingredients.

1.1. Historical comments.

In the second half of the 19-th Century, a newborn branch of Physics, the Statistical Mechanics, brought into attention new phenomena which were not susceptible to be addressed with the classical Newtonian notions and techniques. In particular, the growing need of modeling the evolution of stochastic systems led to the (linear) Fokker-Planck equation (LFPE), which appeared at the beginning of the 20-th Century in the papers of Fokker, Planck and Smoluchowski. Two decades later, Kolmogorov re-discovered the LFPE, hence its other name: the "Kolmogorov forward equation". For more details, see the monographs [7, 33, 44, 55].

The first references to (some particular cases of) the *nonlinear* Fokker-Planck equation (NFPE) (a.k.a. the McKean-Vlasov equation and Vlasov-Fokker-Planck equation) are usually attributed to Vlasov, in 1938 ([60]) and McKean, around 1966 - 1969 ([37], [38] apud [20]). The first root is contested, in the favor of Jeans, which eventually discovered it in 1915 ([30] apud [26]). Concerning the second root, one might consider also, in our opinion,

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some earlier papers of Fuller and McKean, in the '50s ([16,17,36]). For the general framework concerning the NFPE, the first source and the ultimate resource is the monograph of T.D. Frank [20]. A list of recent research papers includes e.g. [3,12,14,21,34,39,45,54,64].

Lie symmetries were defined by S. Lie and F. Klein, in the second half of the 19-th Century, as the first attempt to geometrize the differential equations and the partial differential equations. Even if the main goal of the classification failed (at the beginning), the newborn theory developed into a fundamental cornerstone of the 20-th Century Mathematics and Physics. The book of Olver [43] and the surveys [42,63] offer the panorama of this vast field. We need here only to recall the contributions to the study of the Lie symmetries associated to the linear FPE and the NFPE, related to our paper: [1,9–11,56,62].

Entropy is too general a subject to be treated superficially here, even if only historically. We recall only the papers related to the NFPE associated to the Tsallis, Kaniadakis, Sharma-Taneja-Mittal entropies [56,62] with references therein and the surveys [27,59].

The information measures consider only the probability mass function or probability density function. In 1968, Belis and Guiasu highlighted the importance of integrating the quantitative, objective and probabilistic concept of information with the qualitative, subjective and non-stochastic concept of utility. They introduced ([4,22]) the concept of **weighted entropy** to construct a shift-dependent information measure with properties similar to Shannon entropy. Subsequently, the weighted entropy was characterized axiomatically ([22,23]). Some recent contributions about the weighted entropies include [2,8,24,27,31,35,47,48,57,58], to quote but a few.

The maximum entropy (MaxEnt) problem is an important optimization challenge, asking for a specific distribution of probability p^{ME}, which maximizes a given entropy functional and is subject to some given constraints. Since its first statement in 1957, by E.T. Jaynes, the subject developed into a mainstream topic, as described in the monographs [25,32]. For each new family of entropy functional, the MaxEnt problem was given (by a quite standard method) a specific solution (see, for example, [15,19,41,46–52,62]).

1.2. **The content of the paper.** In Section §2, we remind the equivalent forms of the NFPE, closely following [43,56,61] for notations and conventions. The Lie symmetries of the NFPE are defined as prolongations vector fields on a three-dimensional domain and they span a Lie algebra. When non-trivial, this Lie algebra contains geometric information about the symmetries of the solutions of the NFPE.

In Section §3, we review the general procedure of weighting a given entropy functional and we give some examples (Tsallis, Kaniadakis). The Bregman divergence associated to two PDFs is considered, and we show how the weighting procedure may apply to it, also. For a time-dependent PDF, a fixed potential energy function and a fixed entropy, we remind the classical associated notions of: energy average function, of Lyapunov functional and of the current density.

The Section §4 contains the main results of the paper. We determine the variation of the Lyapunov function associated to a w-weighted Tsallis entropy. Using the associated current density, we obtain the NFPE for the w-weighted Tsallis entropy. We establish formulas linking the drift function $\mathfrak{d}$ and the diffusion function $\mathfrak{D}$ to the so-called "drift" D, in order to avoid possible confusions of terms. The time dependency of the Lyapunov functional is studied. Finally, we prove a result stating that the associated Bregman divergence may be interpreted as a "distance", measured through Lyapunov differences.

Applications are given in Section §5, concerning the Lie symmetries of the NFPE associated to the w-weighted Tsallis entropy. Using the formulas proven in Section §4, we obtain Theorem 5.4, which is slightly more general: in fact, we derive the Lie symmetries for a NFPE with "arbitrary" coefficient functions. For some specific particular cases, we point out the Lie (sub)algebras determining these symmetries. Our analysis focus on the finite dimensional case and is not intended to be exhaustive.

In Section §6 we make a short digression into the Optimization Theory of PDFs, and we solve the MaxEnt problem associated to the w-weighted Tsallis entropy.

1.3. **Conventions.** All the integrals are supposed to be correctly defined. Partial derivatives are supposed to commute with the integral. All the analytic and the geometric objects are supposed to be differentiable, even if, in some cases, a weaker assumption would suffice (for example continuity or integrability).

2. NONLINEAR FOKKER-PLANCK EQUATIONS IN ONE DIMENSION

In what follows, we adopt the general framework from [20], including most of the notions and notations.

Consider a time-dependent probability density function (PDF) $p = p(x,t)$ defined on $\mathbb{R}^2$; a drift $\mathfrak{d} = \mathfrak{d}(x,t,p)$ and a diffusion coefficient $\mathfrak{D} = \mathfrak{D}(x,t,p)$ defined on $U \times \mathbb{R}$, where U is an open subset of $\mathbb{R}^2$. We have $\int_{-\infty}^{\infty} p(x,t)dx = 1$, $p(x,t) \geq 0$ and we suppose $\mathfrak{D}$ to be everywhere non-negative. The associated general NFPE in one (spatial) dimension is

$$\frac{\partial}{\partial t}p(x,t) = -\frac{\partial}{\partial x}[\mathfrak{d}(x,t,p)p(x,t)] + \frac{\partial^2}{\partial x^2}[\mathfrak{D}(x,t,p)p(x,t)] \tag{2.1}$$

or, equivalently, with the obvious notations,

$$\begin{aligned} p_t = {} & (-\mathfrak{d} - p\mathfrak{d}_p + 2\mathfrak{D}_x + 2p\mathfrak{D}_{xp})p_x + \\ & +(\mathfrak{D} + p\mathfrak{D}_p)p_{xx} + (2\mathfrak{D}_p + p\mathfrak{D}_{pp})(p_x)^2 + (\mathfrak{D}_{xx} - \mathfrak{d}_x)p. \end{aligned} \tag{2.2}$$

Its condensed form is

$$\Delta p(x,t) = 0,$$

where we denoted the nonlinear Fokker-Planck operator

$$\begin{aligned} \Delta = {} & \frac{\partial}{\partial t} + (\mathfrak{d} + \mathfrak{d}_p I - 2\mathfrak{D}_x - 2\mathfrak{D}_{xp} I)\frac{\partial}{\partial x} - \\ & -(\mathfrak{D} + \mathfrak{D}_p I)\frac{\partial^2}{\partial x^2} - (2\mathfrak{D}_p + \mathfrak{D}_{pp} I)(\frac{\partial}{\partial x})^2 + (\mathfrak{d}_x - \mathfrak{D}_{xx})I \ . \end{aligned}$$

Here I is the identity operator, i.e. $I(p) = p$.

The third equivalent form of the NFPE (named the continuity equation) is

$$\frac{\partial}{\partial t}p + \frac{\partial}{\partial x}J = 0, \tag{2.3}$$

where the current function $J = J(x,t,p)$ is defined by

$$J(x,t,p) := \mathfrak{d}(x,t,p)p(x,t) - \frac{\partial}{\partial x}[\mathfrak{D}(x,t,p)p(x,t)].$$

This formula may be rewritten, in an equivalent simplified form, as

$$J = (\mathfrak{d} - \mathfrak{D}_x)p - (\mathfrak{D}_p p + \mathfrak{D})p_x. \tag{2.4}$$

In particular, if $\mathfrak{d} = \mathfrak{d}(x,t)$ and $\mathfrak{D} = \mathfrak{D}(x,t)$, we obtain the linear Fokker-Planck operator

$$\Delta = \frac{\partial}{\partial t} + (\mathfrak{d} - 2\mathfrak{D}_x)\frac{\partial}{\partial x} - \mathfrak{D}\frac{\partial^2}{\partial x^2} + (\mathfrak{d}_x - \mathfrak{D}_{xx})I,$$

the associated linear Fokker-Planck equation

$$p_t = (-\mathfrak{d} + 2\mathfrak{D}_x)p_x + \mathfrak{D}p_{xx} + (\mathfrak{D}_{xx} - \mathfrak{d}_x)p$$

and the continuity equation (2.3) for the "linear" current

$$J(x,t) := \mathfrak{d}(x,t)p(x,t) - \frac{\partial}{\partial x}\Big(\mathfrak{D}(x,t)p(x,t)\Big).$$

For the linear FPE, there exists a dual interpretation, via the Ito's lemma, namely the (associated) stochastic differential equation

$$dx = \mathfrak{d}(x,t)dt + \sqrt{2\mathfrak{D}(x,t)}\, dW_t,$$

where W_t is a Wiener process. In the sequel, we shall not deepen this duality.

Consider a linear differential operator ([43])

$$L = \xi(x,t,p)\partial_x + \eta(x,t,p)\partial_t + \phi(x,t,p)\partial_p,$$

where η, ξ and ϕ are differentiable functions on $U \times \mathbb{R}$. We say L is a Lie symmetry operator for nonlinear Fokker-Planck operator Δ if there exists a differentiable function $R = R(x,t,p)$ on $U \times \mathbb{R}$, such that

$$[L, \Delta] = R(x,t,p)\Delta. \tag{2.5}$$

Denote $\mathfrak{S}$ the set of all the Lie symmetry operators for Δ. It is known that a Lie symmetry operator of Δ maps solutions of (2.1) into solutions of (2.1) and that $\mathfrak{S}$ is a (possible infinite dimensional) Lie algebra, governing the symmetries of the solutions of the NFPE.

In the particular case of the LFPE, the Lie symmetries were computed in [1, 9, 10]. In [56, 61], the Lie symmetries were computed for a NFPE which arose from the Sharma-Taneja-Mittal entropy. In the sections §4, §5 and §6 of the present paper, we shall determine the Lie symmetries of some special NFPE, namely those with current functions derived from the Lyapunov operators associated to the weighted Tsallis entropy.

The next section contains a short digression on the needed entropy notions and results.

3. Entropies and their associated Lyapunov values and currents

Consider an arbitrary PDF ρ (without a priori time-dependency) and $\varphi = \varphi(x)$ a differentiable function. The associated (normalized) entropy is the number

$$H[\rho] = -\int_{\mathbb{R}} \rho(x)\varphi(\rho(x))dx. \tag{3.6}$$

Let $w = w(x)$ be a positive differentiable "weighting" function. Then the corresponding w-weighted entropy is

$$H^w[\rho] = -\int_{\mathbb{R}} w(x)\rho(x)\varphi(\rho(x))dx. \tag{3.7}$$

In general, several additional constraints are imposed on both φ and w; in what follows, we neglect them, in order not to break the main line of discourse.

Remark 3.1. (i) For a detailed and more general setting concerning entropy, see our paper [27].

(ii) The notation $H[\rho]$ is redundant, but we use it in order to emphasize the functional dependence on ρ of the entropy. Similar notations will be used for other functionals too.

(iii) The function $\varphi(x) := log(x)$ gives the Boltzmann–Gibbs–Shannon (BGS) entropy.

(iv) For any fixed $q \in \mathbb{R}\backslash\{1\}$,

$$\varphi_{\{q\}}(x) := \frac{x^{1-q}-1}{1-q} \tag{3.8}$$

provides a Tsallis entropy; when $q \to 1$ we recover the BGS entropy. Therefore, by convention, we may consider $q \in \mathbb{R}$. The function $\varphi_{\{q\}}$ is called sometimes the Tsallis q-logarithm and is denoted also by $log^T_{\{q\}}$. Its inverse is the Tsallis q-exponential

$$exp^T_{\{q\}}(x) := [1+(1-q)x]_+^{\frac{1}{1-q}}.$$

(v) For any fixed $k \in [-1,1]\backslash\{0\}$,

$$\varphi_{\{k\}}(x) := \frac{x^k - x^{-k}}{2k} \tag{3.9}$$

provides a Kaniadakis entropy (a.k.a. k-deformed entropy); when $k \to 0$ we recover the BGS entropy. Therefore, by convention, we may consider $k \in [-1,1]$. The function $\varphi_{\{k\}}$ is called sometimes the Kaniadakis k-logarithm and is denoted also by $log^K_{\{k\}}$. Its inverse is the Kaniadakis k-exponential

$$exp^K_{\{k\}}(x) := [kx + \sqrt{1+k^2x^2}]^{\frac{1}{k}}.$$

In the next two sections, we shall exemplify the weighting procedure, by applying it only to the previous Tsallis entropy.

Remark 3.2. Consider now two PDFs ρ_1 and ρ_2 , and a convex differentiable function $f : \mathbb{R} \to \mathbb{R}$.

(i) The associated Bregman divergence is defined by

$$D_f(\rho_1 \parallel \rho_2) := \int_{\mathbb{R}} \{f(\rho_1(x)) - f(\rho_2(x)) - (\rho_1(x) - \rho_2(x))f'(\rho_2(x))\}dx. \tag{3.10}$$

Divergences act as distances in the space of PDFs, measuring how far a PDF differs from another.

(ii) Similarly, we can define the w- weighted Bregman divergence $D^w_f(\rho_1 \parallel \rho_2)$.

(iii) Formula (3.10) suggests that $D_f(\rho_1 \parallel \rho_2)$ may be considered a z-weighted (BGS) entropy by itself, written

$$H^z[\rho_1] = -\int_{\mathbb{R}} z(x)\rho_1(x)log(\rho_1(x))dx,$$

where

$$z(x) := -\frac{f(\rho_1(x)) - f(\rho_2(x)) - (\rho_1(x) - \rho_2(x))f'(\rho_2(x))}{\rho_1(x) \cdot log(\rho_1(x))}.$$

Usually, the weighting function is independent on the PDF it is associated with. However, there exists situations in which one can use the weighting function with this more general property.

Similarly, one can use any other entropy functional, instead the (BGS) one.

Consider now a *time-dependent* PDF $p = p(x,t)$, as in Section §2, and a function $V = V(x)$, modeling the potential energy of the system. Formula (3.6) provides us with a function $H[p] = H[p](t)$. We denote the time-dependent energy average function

$$U[p](t) := \int_{\mathbb{R}} V(x)p(x,t)dx. \tag{3.11}$$

Let D be a positive real constant (which contains information about the diffusion of the system). We define the function

$$\mathcal{L}_H[p] := U[p] - D \cdot H[p]. \tag{3.12}$$

If $\mathcal{L}_H[p]$ is non-positive, it will be called the Lyapunov function associated to the entropy function $H[p]$ and to D. The correspondence $p \to \mathcal{L}_H[p]$ is called the Lyapunov functional. We define the current density associated to $\mathcal{L}_H[p]$ the function $J = J(x,t)$, given by

$$J(x,t) := -p(x,t)\,\frac{\partial}{\partial x}\Big(\frac{\delta \mathcal{L}_H[p]}{\delta p}\Big)(x,t). \tag{3.13}$$

Similar formulas occur for weighted entropies also.

In the Sections 4 and 5, our object of study will be the NFPE associated to J, via the continuity equation (2.3).

We finished the preliminary part of the paper. We have at our disposal all the necessary notions and notations, in order to apply them in the particular case of the weighted Tsallis entropy.

4. Generalized statistical mechanics based on weighted Tsallis entropy

Consider: $p = p(x,t)$ a time-dependent PDF; $w = w(x,t)$ a time-dependent weighting function; $V = V(x)$ the potential energy function; D a positive constant; q a fixed real number, $q \neq 1$ and $H_q^{wT}[p]$ the associated w-weighted Tsallis entropy function, based on (3.8). Denote by $\mathcal{L}_q^{wT}[p]$ the Lyapunov function, calculated by formulas (3.11) and (3.12). Denote J_q^{wT} the associated current density, given by formula (3.13).

4.1. **The NFPE based on weighted Tsallis entropy.**

Theorem 4.1. *With the previous notations, the following relation holds*

$$\frac{\delta \mathcal{L}_q^{wT}[p]}{\delta p}(x,t) = V(x) + D\frac{1}{1-q}w(x,t)[(2-q)(p(x,t))^{1-q} - 1].$$

Proof. By definition,

$$\frac{\delta \mathcal{L}_q^{wT}[p]}{\delta p}(x,t) = \frac{\delta}{\delta p}\left(\int_{\mathbb{R}} V(x)p(x,t)dx + D\int_{\mathbb{R}} w(x,t)p(x,t)\frac{(p(x,t))^{1-q}-1}{1-q}dx\right) =$$

$$= V(x) + Dw(x,t)\left(\frac{(p(x,t))^{1-q}-1}{1-q} + (p(x,t))^{1-q}\right).$$

We rewrite, in simplified form, as

$$\frac{\delta \mathcal{L}_q^{wT}[p]}{\delta p} = V + Dw(\frac{p^{1-q}-1}{1-q} + p^{1-q}) =$$

$$= V + Dw\frac{(2-q)p^{1-q} - 1}{1-q} =$$
$$= V + D \cdot \frac{1}{1-q} w \cdot [(2-q)p^{1-q} - 1].$$

□

Theorem 4.2. *With the previous notations, the following relation holds*

$$\frac{\partial}{\partial x}\left(\frac{\delta \mathcal{L}_q^{wT}[p]}{\delta p}\right) = -h + D(2-q)wp^{-q}\frac{\partial p}{\partial x} + \frac{D}{1-q}[(2-q)p^{1-q} - 1]\frac{\partial w}{\partial x},$$

where $h(x) := -\frac{\partial}{\partial x}V(x)$ *is a drift force.*

Proof. We have

$$\frac{\partial}{\partial x}\left(\frac{\delta \mathcal{L}_q^{wT}[p]}{\delta p}\right) = \frac{\partial}{\partial x}V + D\frac{1}{1-q}[(2-q)p^{1-q} - 1]\frac{\partial w}{\partial x} +$$
$$+ Dw(2-q)p^{-q}\frac{\partial p}{\partial q} = -h + D(2-q)wp^{-q}\frac{\partial p}{\partial x} + \frac{D}{1-q}[(2-q)p^{1-q} - 1]\frac{\partial w}{\partial x}.$$

□

Remark 4.3. (i) We calculate $J_q^{wT} = J_q^{wT}(x,t)$, as

$$J_q^{wT} = ph - D(2-q)wp^{1-q}\frac{\partial p}{\partial x} - \frac{D}{1-q}[(2-q)p^{2-q} - p]\frac{\partial w}{\partial x}. \tag{4.14}$$

The continuity equation (2.3) leads to the NFPE for the general w-weighted Tsallis entropy H_q^{wT}

$$\frac{\partial}{\partial t}p = \frac{\partial}{\partial x}\left\{-ph + D(2-q)wp^{1-q}\frac{\partial p}{\partial x} + \frac{D}{1-q}[(2-q)p^{2-q} - p]\frac{\partial w}{\partial x}\right\}. \tag{4.15}$$

We write it in the equivalent simplified form

$$p_t + A \cdot p_x + B \cdot (p_x)^2 + E \cdot p_{xx} + G = 0, \tag{4.16}$$

where $A = A(x,t,p)$, $B = B(x,t,p)$, $E = E(x,t,p)$, $G = G(x,t,p)$ are given by:

$$A = h - D(2-q)w_x p^{1-q} - \frac{D}{1-q}\Big((2-q)^2 \cdot p^{1-q} - 1\Big)w_x, \tag{4.17}$$

$$B = -D(2-q)(1-q) \cdot p^{-q} \cdot w,$$

$$E = -D(2-q) \cdot p^{1-q} \cdot w,$$

$$G = h_x \cdot p - \frac{D}{1-q}\Big((2-q) \cdot p^{2-q} - p\Big) \cdot w_{xx}.$$

In particular, for the classical Tsallis entropy and for $h(x) = -x$, we have $w = 1$ and the previous coefficient functions become

$$A = h \ , \ B = -D(2-q)(1-q) \cdot p^{-q} \ , \ E = -D(2-q) \cdot p^{1-q} \ , \ G = -p. \tag{4.18}$$

Thus, we recovered the formula (25) in [56], for the NFPE based on the classical Tsallis entropy.

We consider now the (BGS) entropy, by particularizing $h(x) = -x$, $w = 1$ and $q = 1$. We obtain

$$A(x) = -x \ , \ B = 0 \ , \ E = -D \ , \ G(p) = -p \quad . \tag{4.19}$$

Thus, we recovered the formula (24) in [56], for the linear FPE based on the classical (BGS) entropy.

(ii) From (4.14) and (2.4) we get an implicit equation involving $\mathfrak{d}$, $\mathfrak{D}$ and D. We can explicitly obtain $\mathfrak{d}$ depending on $\mathfrak{D}$ and D, as

$$\begin{aligned} \mathfrak{d} = \mathfrak{D}_x + h - \frac{D}{1-q}\Big[(2-q)p^{1-q} - 1\Big]w_x + \\ + \mathfrak{D}p_x p^{-1} + \mathfrak{D}_p p_x - (2-q)D \cdot w \cdot p^{-q} p_x. \end{aligned} \tag{4.20}$$

Analogously, we can explicitly obtain D depending on $\mathfrak{D}$ and $\mathfrak{d}$.

In order to explicitly obtain $\mathfrak{D}$ depending on D and $\mathfrak{d}$, we must solve a linear PDE of first order.

In some papers, the constant D is called the diffusion coefficient. From (4.20), we see that D is, in fact, only a part of the diffusion function $\mathfrak{D}$.

In our paper, the NFPE was introduced and studied from two complementary directions. The first one was via $\mathfrak{d}$ and $\mathfrak{D}$, in Section §2; this NFPE is more general, as it is associated to an arbitrary entropy. The second one was via D, in this section; this NFPE is more particular, as it is based on the weighted Tsallis entropy only. Formula (4.20) clarifies the relationship between these two approaches.

(iii) Consider now a particular case and suppose the following sufficient condition for (4.20) holds, expressed by the system of equations

$$\begin{cases} \mathfrak{d} - \mathfrak{D}_x = h - \dfrac{D}{1-q}\Big[(2-q)p^{1-q} - 1\Big]w_x \\ p \cdot \mathfrak{D}_p + \mathfrak{D} = (2-q)D \cdot w \cdot p^{1-q} \end{cases}.$$

We integrate the second equation and we obtain

$$\mathfrak{D}(x,t,p) = D \cdot w(x,t) \cdot p^{1-q} + \frac{\mathfrak{c}(x,t)}{p(x,t)}, \tag{4.21}$$

where $\mathfrak{c}$ is an arbitrary function, which ensures the positivity of $\mathfrak{D}$.

We calculate $\mathfrak{D}_x$ and we replace in the first equation of the system. We get

$$\mathfrak{d} = \mathfrak{D}_x + h - \frac{D}{1-q}\Big[(2-q)p^{1-q} - 1\Big]w_x,$$

that is

$$\mathfrak{d} = h + D \cdot w_x \cdot p^{1-q} + \tag{4.22}$$

$$+\Big((1-q)D\cdot w\cdot p^{-q}-\mathfrak{c}\cdot p^{-2}\Big)\cdot p_x+$$
$$+\,\mathfrak{c}_x\cdot p^{-1}-\frac{D}{1-q}\Big((2-q)p^{1-q}-1\Big)w_x\,.$$

(iv) Suppose the weighted function w is static, i.e. $w=w(x)$. Then, the stationary state $p^{st}=p^{st}(x)$ can be obtained by imposing the condition of the current-free $J(x,t)=0$, so there exists a real constant C such that

$$V(x)+\frac{1}{1-q}Dw(x)[(2-q)(p^{st}(x))^{1-q}-1]=C.$$

We get the equivalent two formulas

$$V(x)+Dw(x)[(2-q)\frac{(p^{st}(x))^{1-q}-1}{1-q}+1]=C$$

and

$$V(x)+Dw(x)[(2-q)\log_{\{q\}}^{T}(p^{st}(x))+1]=C.$$

By multiplying with $p^{st}(x)$ and by integrating over x, we get

$$\int_{\mathbb{R}}V(x)p^{st}(x)dx+D(2-q)\int_{\mathbb{R}}w(x)p^{st}(x)\log_{\{q\}}^{T}(p^{st}(x))dx+D\int_{\mathbb{R}}w(x)p^{st}(x)dx=C,$$

that is

$$U[p^{st}]-D(2-q)H_q^{wT}[p^{st}]+DE_{p^{st}}[w]=C,$$

where we denoted

$$E_{p^{st}}[w]:=\int_{\mathbb{R}}w(x)p^{st}(x)dx.$$

We get the equivalent formula

$$H_q^{wT}[p^{st}]=\frac{1}{D(2-q)}\left\{-C+U[p^{st}]+DE_{p^{st}}(w)\right\}. \tag{4.23}$$

4.2. **Time-dependency of the Lyapunov function.** We study the time dependency of the Lyapunov functional $\mathcal{L}_q^{wT}$ during the time evolution of $p(x,t)$, according to the NFPE. Differentiating in (3.12) w.r.t. t, using (2.3) and (4.14), we get

$$\frac{d\mathcal{L}_q^{wT}[p]}{dt}(t)=\int_{\mathbb{R}}\frac{\partial p}{\partial t}(x,t)\cdot\frac{\delta}{\delta p}\Big\{U[p]-DH_q^{wT}[p]\Big\}(x,t)dx=$$
$$=\int_{\mathbb{R}}-\frac{\partial J}{\partial x}(x,t)\cdot\frac{\delta}{\delta p}\Big\{U[p]-DH_{\{q\}}^{wT}[p]\Big\}(x,t)dx=$$
$$=\int_{\mathbb{R}}\frac{\partial}{\partial x}\Big\{-p(x,t)h(x)+D(2-q)w(x,t)(p(x,t))^{1-q}\frac{\partial p}{\partial x}(x,t)+$$
$$+\frac{D}{1-q}[(2-q)(p(x,t))^{2-q}-p(x,t)]\frac{\partial w}{\partial x}(x,t)\Big\}\cdot$$
$$\cdot\Big\{V(x)+\frac{Dw(x,t)}{1-q}[(2-q)(p(x,t))^{1-q}-1]\Big\}\,dx.$$

We integrate by parts and we obtain

$$\frac{d\mathcal{L}_q^{wT}[p]}{dt}(t) = -\int_{\mathbb{R}} \Big\{ -p(x,t)h(x) + D(2-q)w(x,t)(p(x,t))^{1-q}\frac{\partial p}{\partial x}(x,t)+$$
$$+\frac{D}{1-q}[(2-q)(p(x,t))^{2-q} - p(x,t)]\frac{\partial w}{\partial x}(x,t)\Big\}\cdot$$
$$\cdot\frac{\partial}{\partial x}\left\{V(x) + \frac{D}{1-q}w(x,t)[(2-q)(p(x,t))^{1-q} - 1]\right\}dx =$$
$$= -\int_{\mathbb{R}} p(x,t)\Big\{ -h(x) + D(2-q)w(x,t)(p(x,t))^{-q}\frac{\partial p}{\partial x}(x,t)+$$
$$+\frac{D}{1-q}[(2-q)(p(x,t))^{1-q} - 1]\frac{\partial w}{\partial x}(x,t)\Big\}\cdot$$
$$\cdot\Big\{\frac{\partial V}{\partial x}(x) + \frac{D}{1-q}[(2-q)(p(x,t))^{1-q} - 1]\frac{\partial w}{\partial x}(x,t)+$$
$$+Dw(x,t)(2-q)(p(x,t))^{-q}\frac{\partial p}{\partial x}(x,t)\Big\}dx =$$
$$= -\int_{\mathbb{R}} p(x,t)\Big\{ -h(x) + Dw(x,t)(2-q)(p(x,t))^{-q}\frac{\partial p}{\partial x}(x,t)+$$
$$+\frac{D}{1-q}[(2-q)(p(x,t))^{1-q} - 1]\frac{\partial w}{\partial x}(x,t)\Big\}^2 dx.$$

Therefore,

$$\frac{d\mathcal{L}_q^{wT}(t)}{dt} \leq 0\,.$$

4.3. **Relation with Bregman divergence.** Let $p = p(x,t)$ be a time-dependent PDF and let q be a real constant, $q \neq 1$. Let $w = w(x)$ be a time-independent weighting function. We consider the convex differentiable function $f : \mathbb{R} \to \mathbb{R}$, $f(z) := z\log^T_{\{q\}}(z)$.

Consider $p^{ME} = p^{ME}(x)$ the Tsallis maximum entropy PDF and define

$$r(x) := p^{ME}(x) = exp^T_{\{q\}}\left[-\frac{\gamma+\beta V(x)}{w(x)(2-q)} - \frac{1}{2-q}\right].$$

We have $f'(z) = (2-q)\log^T_{\{q\}}(z) + 1$ and we obtain, successively,

$$f'(p^{ME}(x)) = (2-q)\left[-\frac{\gamma+\beta V(x)}{w(x)(2-q)} - \frac{1}{2-q}\right] + 1 = -\frac{\gamma+\beta V(x)}{w(x)}\,.$$

The w-weighted Bregman divergence writes

$$D_q^w(p \parallel p^{ME})(t) = \int_{\mathbb{R}} w(x)\Big[p(x,t)\log^T_{\{q\}}(p(x,t)) - p^{ME}(x)\log^T_{\{q\}}(p^{ME}(x)) -$$
$$-\Big(p(x,t) - p^{ME}(x)\Big)\left(-\frac{\gamma+\beta V(x)}{w(x)}\right)\Big]dx.$$

We get successively

$$D_q^w(p \parallel p^{ME})(t) = \int_{\mathbb{R}} w(x)p(x,t)\log^T_{\{q\}}(p(x,t))dx - \int_{\mathbb{R}} w(x)p^{ME}(x)\log^T_{\{q\}}(p^{ME}(x))dx+$$
$$+\int_{\mathbb{R}}\Big(\gamma+\beta V(x)\Big)\Big(p(x,t) - p^{ME}(x)\Big)dx,$$

$$D_q^w(p \parallel p^{ME}) = \beta\cdot U[p] - H_q^{wT}[p] - \beta\cdot U[p^{ME}] + H_q^{wT}[p^{ME}].$$

We proved the

Theorem 4.3. *With the previous notations and conventions, suppose, in addition, that $\beta > 0$. Then we have*

$$D_q^w(p \parallel p^{ME}) = \beta \cdot \Big(\tilde{\mathcal{L}}_q^{wT}[p] - \tilde{\mathcal{L}}_q^{wT}[p^{ME}]\Big),$$

where the Lyapunov functional $\tilde{\mathcal{L}}_q^{wT}$ is constructed via (3.12), with $D := \frac{1}{\beta}$.

Remark 4.4. (i) We already remarked that the divergence acts as a distance on the space of the PDFs. The previous theorem enlightens more: it express this distance in terms of differences of two values of some Lyapunov functional.

(ii) Special cases: a) For $w = 1$ we obtain the Tsallis entropy based approach. b) The case $q \longrightarrow 1$ corresponds to the weighted Shannon entropy approach. c) For $w = 1$ and $q \longrightarrow 1$ we get the classical Shannon entropy case.

5. The Lie symmetries of the NFPE based on the weighted Tsallis entropy

We determine the Lie symmetries of the NFPE (4.16), associated to the w-weighted Tsallis entropy, using the algorithm described in [43]. For the moment, we consider *arbitrary* functions A, B, E and G, depending on the variables x, t and p. Only after determining the final system of equations we shall replace these functions with their particular values from (4.17).

We look for vector fields of the form

$$X = \eta(x,t,p)\partial_t + \xi(x,t,p)\partial_x + \phi(x,t,p)\partial_p\,, \tag{5.24}$$

where η, ξ and ϕ are differentiable functions on $U \times \mathbb{R}$, as in (2.5). We consider the second prolongation of p, as

$$p^{(2)} = (p; p_x, p_t; p_{xx}, p_{xt}, p_{tt})\,,$$

and we write the NFPE (4.16) as $F(x,t,p^{(2)}) = 0$, where

$$\begin{aligned} F(x,t,p^{(2)}) = p_t(x,t) + A(x,t,p)p_x(x,t) + B(x,t,p)(p_x)^2(x,t) + \\ + E(x,t,p)p_{xx}(x,t) + G(x,t,p). \end{aligned} \tag{5.25}$$

We shall neglect the variables and we simplify the formulas:

$$F = p_t + A \cdot p_x + B \cdot (p_x)^2 + E \cdot p_{xx} + G.$$

We remark that F has maximal rank everywhere.

We consider the second prolongation of the vector field X, as

$$pr^{(2)}X = \eta \cdot \partial_t + \xi \cdot \partial_x + \phi \cdot \partial_p + \Phi^x \frac{\partial}{\partial_{p_x}} + \Phi^t \frac{\partial}{\partial_{p_t}} +$$

$$+\Phi^{xx} \frac{\partial}{\partial_{p_{xx}}} + \Phi^{xt} \frac{\partial}{\partial_{p_{xt}}} + \Phi^{tt} \frac{\partial}{\partial_{p_{tt}}},$$

where

$$\Phi^x = \phi_x + (\phi_p - \xi_x)p_x - \eta_x p_t - \xi_p p_x^2 - \eta_p p_x p_t,$$

$$\Phi^t = \phi_t + (\phi_p - \eta_t)p_t - \xi_t p_x - \eta_p p_t^2 - \xi_p p_x p_t,$$

$$\Phi^{xx} = \phi_{xx} + (2\phi_{xp} - \xi_{xx})p_x - \eta_{xx}p_t + (\phi_{pp} - 2\xi_{xp})p_x^2 -$$

$$-2\eta_{xp}p_x p_t - \xi_{pp}p_x^3 - \eta_{pp}p_x^2 p_t + (\phi_p - 2\xi_x)p_{xx} -$$

$$-2\eta_x p_{xt} - 3\xi_p p_x p_{xx} - \eta_p p_t p_{xx} - 2\eta_p p_x p_{xt},$$

$$\Phi^{xt} = \phi_{xt} + (\phi_{pt} - \xi_{xt})p_x - \xi_{pt}p_x^2 + (\phi_{px} - \eta_{xt})p_t - \eta_{px}(p_t)^2 -$$

$$-\xi_t p_{xx} + (\phi_{pp} - \xi_{px} - \eta_{pt})p_x p_t + (\phi_p - \xi_x - \eta_t)p_{xt} - \eta_x p_{tt} - \xi_p p_t p_{xx} -$$

$$-2\eta_p p_t p_{xt} - 2\xi_p p_x p_{xt} - \eta_p p_x p_{tt} - \xi_{pp}(p_x)^2 p_t - \eta_{pp}p_x(p_t)^2,$$

$$\Phi^{tt} = \phi_{tt} + (2\phi_{pt} - \eta_{tt})p_t - \xi_{tt}p_x + (\phi_{pp} - 2\eta_{pt})p_t^2 -$$

$$-2\xi_{pt}p_x p_t - \eta_{pp}p_t^3 - \xi_{pp}p_t^2 p_x + (\phi_p - 2\eta_t)p_{tt} -$$

$$-2\xi_t p_{xt} - 3\eta_p p_t p_{tt} - \xi_p p_x p_{tt} - 2\xi_p p_t p_{xt} \quad .$$

The unknown functions η, ξ, ϕ are solutions of the equation

$$pr^{(2)}X\Big(F(x,t,p^{(2)})\Big) = 0.$$

With the previous notations, this equation writes

$$\xi \cdot \Big[A_x p_x + B_x(p_x)^2 + E_x p_{xx} + G_x\Big] + \tag{5.26}$$

$$+\eta \cdot \Big[A_t p_x + B_t(p_x)^2 + E_t p_{xx} + G_t\Big] +$$

$$+\phi \cdot \Big[A_p p_x + B_p(p_x)^2 + E_p p_{xx} + G_p\Big] +$$

$$+\Phi^x \cdot \Big(A + 2Bp_x\Big) + \Phi^t + E\Phi^{xx} = 0.$$

In the formulas giving Φ^x, Φ^t and Φ^{xx}, we replace

$$p_t = -A \cdot p_x - B \cdot (p_x)^2 - E \cdot p_{xx} - G.$$

Back in (5.26), we obtain

$$\xi \cdot \Big\{A_x p_x + B_x(p_x)^2 + E_x p_{xx} + G_x\Big\} +$$

$$\eta \cdot \Big\{A_t p_x + B_t(p_x)^2 + E_t p_{xx} + G_t\Big\} +$$

$$\phi \cdot \Big\{A_p p_x + B_p(p_x)^2 + E_p p_{xx} + G_p\Big\} +$$

$$A\phi_x + A(\phi_p - \xi_x)p_x + A^2\eta_x p_x + AB\eta_x(p_x)^2 + AE\eta_x p_{xx} + AG\eta_x -$$

$$-A\xi_p(p_x)^2 + A^2\eta_p(p_x)^2 + AB\eta_p(p_x)^3 + AE\eta_p p_x p_{xx} + AG\eta_p p_x +$$

$$+2B\phi_x p_x + 2B(\phi_p - \xi_x)(p_x)^2 + 2AB\eta_x(p_x)^2 + 2B^2\eta_x(p_x)^3 + 2BE\eta_x p_x p_{xx} +$$

$$+2BG\eta_x p_x - 2B\xi_p(p_x)^3 + 2AB\eta_p(p_x)^3 + 2B^2\eta_p(p_x)^4 + 2BE\eta_p(p_x)^2 p_{xx} + 2BG\eta_p(p_x)^2 +$$

$$+\phi_t - (\phi_p - \eta_t)Ap_x - (\phi_p - \eta_t)B(p_x)^2 - (\phi_p - \eta_t)Ep_{xx} - (\phi_p - \eta_t)G - \xi_t p_x -$$

$$-\eta_p \cdot \Big\{ A^2(p_x)^2 + B^2(p_x)^4 + 2AB(p_x)^3 + E^2(p_{xx})^2 + G^2 +$$

$$+2EGp_{xx} + 2AEp_x p_{xx} + 2AGp_x + 2BE(p_x)^2 p_{xx} + 2BG(p_x)^2 \Big\} +$$

$$+A\xi_p(p_x)^2 + B\xi_p(p_x)^3 + E\xi_p p_x p_{xx} + G\xi_p p_x +$$

$$+E \cdot \Big\{ \phi_{xx} + (2\phi_{xp} - \xi_{xx})p_x + A\eta_{xx}p_x + B\eta_{xx}(p_x)^2 + E\eta_{xx}p_{xx} + G\eta_{xx} +$$

$$+(\phi_{pp} - 2\xi_{xp})(p_x)^2 + 2A\eta_{xp}(p_x)^2 + 2B\eta_{xp}(p_x)^3 + 2E\eta_{xp}p_x p_{xx} + 2G\eta_{xp}p_x - \xi_{pp}(p_x)^3 +$$

$$+A\eta_{pp}(p_x)^3 + B\eta_{pp}(p_x)^4 + E\eta_{pp}(p_x)^2 p_{xx} + G\eta_{pp}(p_x)^2 + (\phi_p - 2\xi_x)p_{xx} - 2\eta_x p_{xt} -$$

$$-3\xi_p p_x p_{xx} + A\eta_p p_x p_{xx} + B\eta_p(p_x)^2 p_{xx} + E\eta_p(p_{xx})^2 + G\eta_p p_{xx} - 2\eta_p p_x p_{xt} \Big\} = 0 \; .$$

We consider that the left side is a formal polynomial in 1, p_x, $(p_x)^2$, p_{xx}, $(p_x)^3$, $p_x p_{xx}$, $(p_x)^4$, $(p_x)^2 p_{xx}$, $(p_{xx})^2$, p_{xt}, $p_x p_{xt}$ (in this order), which is identically null. The respective coefficients vanish and we obtain the following system of PDEs

$$\xi \cdot G_x + \eta \cdot G_t + \phi \cdot G_p + A \cdot \phi_x + AG \cdot \eta_x + \phi_t - G^2 \cdot \eta_p + E \cdot \phi_{xx} + EG\eta_{xx} - G \cdot (\phi_p - \eta_t) = 0 \; , \tag{5.27}$$

$$\xi \cdot A_x + \eta \cdot A_t + \phi \cdot A_p + A \cdot (\eta_t - \xi_x) + A^2 \cdot \eta_x + 2B \cdot \phi_x + 2BG \cdot \eta_x - AG \cdot \eta_p +$$

$$+E \cdot (2\phi_{xp} - \xi_{xx}) + AE \cdot \eta_{xx} + 2EG \cdot \eta_{xp} + G \cdot \xi_p - \xi_t = 0,$$

$$\xi \cdot B_x + \eta \cdot B_t + \phi \cdot B_p + 3AB \cdot \eta_x + B \cdot (\phi_p + \eta_t - 2\xi_x) +$$

$$+BE \cdot \eta_{xx} + E \cdot (\phi_{pp} - 2\xi_{xp}) + 2AE \cdot \eta_{xp} + EG \cdot \eta_{pp} = 0,$$

$$\xi \cdot E_x + \eta \cdot E_t + \phi \cdot E_p + AE \cdot \eta_x - EG \cdot \eta_p + E^2 \cdot \eta_{xx} + E \cdot (\eta_t - 2\xi_x) = 0,$$

$$AB \cdot \eta_p + 2B^2 \cdot \eta_x - B \cdot \xi_p + 2BE \cdot \eta_{xp} - E \cdot \xi_{pp} + AE \cdot \eta_{pp} = 0,$$

$$2BE \cdot \eta_x + 2E^2 \cdot \eta_{xp} - 2E \cdot \xi_p = 0,$$

$$B^2 \cdot \eta_p + BE \cdot \eta_{pp} = 0,$$

$$E^2 \cdot \eta_{pp} + BE \cdot \eta_p = 0,$$

$$E^2 \cdot \eta_p - E^2 \cdot \eta_p = 0,$$

$$E \cdot \eta_x = 0,$$

$$E \cdot \eta_p = 0.$$

Suppose E is nowhere vanishing. From the last two equations, we obtain $\eta = \eta(t)$, and the previous three equations become tautologic. From the 6-th equation we deduce $\xi = \xi(x,t)$. Now, the 5-th equation becomes tautologic. The system (5.27) restrains to the first four equations only, which may be rewritten as:

$$\xi \cdot G_x + \eta \cdot G_t + \phi \cdot G_p + A \cdot \phi_x + \phi_t + E \cdot \phi_{xx} - G \cdot (\phi_p - \eta_t) = 0\ , \tag{5.28}$$

$$\xi \cdot A_x + \eta \cdot A_t + \phi \cdot A_p + A \cdot (\eta_t - \xi_x) + 2B \cdot \phi_x + E \cdot (2\phi_{xp} - \xi_{xx}) - \xi_t = 0,$$

$$\xi \cdot B_x + \eta \cdot B_t + \phi \cdot B_p + B \cdot (\phi_p + \eta_t - 2\xi_x) + E \cdot \phi_{pp} = 0,$$

$$\xi \cdot E_x + \eta \cdot E_t + \phi \cdot E_p + E \cdot (\eta_t - 2\xi_x) = 0.$$

Theorem 5.4. *(i) With the previous notations, consider the NFPE (4.16), with arbitrary coefficient functions A, B, E, G. Then the Lie symmetries form the trivial Lie algebra, spanned by the null vector field.*

(ii) If the functions A, B, E, G are time-independent, then the Lie symmetries form a Lie algebra spanned by the vector field

$$\mathcal{X}_1 = \partial_t. \tag{5.29}$$

(iii) If the functions B, E, G are x-independent and $A_x = -1$, then the Lie symmetries form a Lie algebra spanned by the vector field

$$\mathcal{X}_2 = e^{-t}\partial_x. \tag{5.30}$$

(iv) If the functions A, B, E, G are time-independent, the functions B, E, G are x-independent, and $A_x = -1$, then the Lie symmetries form a Lie algebra spanned by the vector fields

$$\mathcal{X}_1 = \partial_t \quad , \quad \mathcal{X}_2 = e^{-t}\partial_x. \tag{5.31}$$

(v) If $A = -x$, $B = c_1 p^{2\alpha-1}$, $E = c_2 p^{2\alpha}$, $G = c_3 p$, with α, c_1, c_2, c_3 arbitrary real constants, then the Lie symmetries form a Lie algebra spanned by the vector fields

$$\mathcal{X}_1 = \partial_t \quad , \quad \mathcal{X}_2 = e^{-t}\partial_x \quad , \quad \mathcal{X}_3 = \alpha x \partial_x + p\partial_p. \tag{5.32}$$

Remark 5.5. (i) It is important to stress that the Lie symmetries in formulas (5.29)- (5.32) do not depend on the w-weighted Tsallis entropy features. For example, the Lie symmetries in (5.31) were discovered for the NFPE based on the Sharma-Taneja-Mittal entropy (cf. [56]), which is more general that the Tsallis entropy.

(ii) The Lie algebras spanned by (5.29) and by (5.30), respectively, are isomorphic with the (one-dimensional) Lie algebra $\mathfrak{g}_1$. (We use the notations from Mubarakzyanov's classifications of low-dimensional Lie algebras [40].) This corresponds to the (local) Lie group of translations of the real line.

We remark that

$$[\mathcal{X}_1, \mathcal{X}_2] = -\mathcal{X}_2,$$

hence the Lie algebra spanned by (5.31) is the (non-commutative 2-dimensional) algebra $\mathfrak{g}_{2,1}$. It corresponds to the (local) Lie group of affine transformations of the real line.

We calculate that

$$[\mathcal{X}_1, \mathcal{X}_2] = -\mathcal{X}_2 \;\;,\;\; [\mathcal{X}_1, \mathcal{X}_3] = 0 \;\;,\;\; [\mathcal{X}_2, \mathcal{X}_3] = \alpha \mathcal{X}_2,$$

hence the Lie algebra spanned by (5.32) is a non-commutative 3-dimensional algebra. This is (isomorphic with) the decomposable Bianchi III Lie algebra $\mathfrak{g}_{2,1} \oplus \mathfrak{g}_1$.

(iii) With the previous notations, consider the NFPE associated to the w-weighted Tsallis entropy. Then the Lie symmetries form a Lie algebra spanned by the vector fields

$$X = \xi \partial_x + \eta \partial_t + \phi \partial_p,$$

with $\eta = \eta(t)$, $\xi = \xi(x,t)$ and η, ξ, ϕ satisfy the PDEs system (5.28) and A, B, E, G are given in formula (4.17). In general, this Lie algebra is trivial.

(iv) In particular, for the classical Tsallis entropy (with $q \neq 1$), we have $w = 1$ and we recover the Lie symmetries derived in [56], Subsection 4.1. b), namely the Lie algebra spanned by the vector fields:

$$\mathcal{X}_1 = \partial_t \;\;,\;\; \mathcal{X}_2 = e^{-t}\partial_x \;\;,\;\; \mathcal{X}_3 = \frac{1-q}{2} x\partial_x + p\partial_p \;\;,\;\; \mathcal{X}_4 = e^{(q-3)t}\Big(x\partial_x - \partial_t - p\partial_p\Big).$$

We remark that their non-vanishing Lie brackets are only

$$[\mathcal{X}_1, \mathcal{X}_2] = -\mathcal{X}_2 \;\;,\;\; [\mathcal{X}_2, \mathcal{X}_3] = \frac{1-q}{2}\mathcal{X}_2 \;\;,\;\; [\mathcal{X}_1, \mathcal{X}_4] = (q-3)\mathcal{X}_4.$$

We conclude that the Lie algebra spanned by these four vector fields is: the decomposable (3-dimensional) Lie algebra $\mathfrak{g}_{2,1} \oplus \mathfrak{g}_1$, if $q = 3$; the decomposable (4-dimensional) Lie algebra $\mathfrak{g}_{3,3} \oplus \mathfrak{g}_1$, if $q \neq 3$.

There exist the following 2-dimensional Lie sub-algebras: commutative $sp\{\mathcal{X}_1, \mathcal{X}_3\}$, $sp\{\mathcal{X}_2, \mathcal{X}_4\}$ and $sp\{\mathcal{X}_3, \mathcal{X}_4\}$; non-commutative $sp\{\mathcal{X}_1, \mathcal{X}_2\}$, $sp\{\mathcal{X}_1, \mathcal{X}_4\}$ and $sp\{\mathcal{X}_2, \mathcal{X}_3\}$.

For $q \neq 3$, there exist the following 3-dimensional Lie sub-algebras:

- $sp\{\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3\}$, $sp\{\mathcal{X}_1, \mathcal{X}_3, \mathcal{X}_4\}$ and $sp\{\mathcal{X}_2, \mathcal{X}_3, \mathcal{X}_4\}$, which are isomorphic with the decomposable Bianchi III Lie algebra $\mathfrak{g}_{2,1} \oplus \mathfrak{g}_1$;

- $sp\{\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_4\}$, which is isomorphic with: the Bianchi V Lie algebra $\mathfrak{g}_{3,3}$, for $q = 2$; the Bianchi VI Lie algebra $\mathfrak{g}_{3,4}$, for $q \neq 2$ (remember that $q \neq 3$) (in particular, for $q = 4$ we get the Poincaré Lie algebra).

(v) If, moreover, $q \to 1$, we recover the Lie symmetries for the linear FPE derived in [56], Subsection 4.1. a) and in [9,10], namely the Lie algebra spanned by the vector fields:

$$\mathcal{X}_1 = \partial_t\,,\ \ \mathcal{X}_2 = e^{-t}\partial_x\,,\ \ \mathcal{X}_3 = e^{t}\Big(\partial_x - \frac{1}{D}xp\partial_p\Big)\,,\ \ \mathcal{X}_4 = e^{-2t}\Big(x\partial_x - \partial_t - p\partial_p\Big),$$

$$\mathcal{X}_5 = e^{2t}\Big(x\partial_x + \partial_t - \frac{1}{D}x^2 p\partial_p\Big)\,,\ \ \mathcal{X}_6 = p\partial_p\,,\ \ \mathcal{X}_7 = \tilde{p}\cdot\partial_p,$$

where $\tilde{p}$ is an arbitrary solution of the linear FPE.

We restrict the study to the Lie algebra spanned by the first six vector fields, in order to avoid the infinite-dimensional sub-algebras. The non-vanishing Lie brackets are only:

$$[\mathcal{X}_1,\mathcal{X}_2] = -\mathcal{X}_2,\quad [\mathcal{X}_1,\mathcal{X}_3] = \mathcal{X}_3,\quad [\mathcal{X}_1,\mathcal{X}_4] = -2\mathcal{X}_4,\quad [\mathcal{X}_1,\mathcal{X}_5] = 2\mathcal{X}_5,$$

$$[\mathcal{X}_2,\mathcal{X}_3] = -\frac{1}{D}\mathcal{X}_6\ ,\ [\mathcal{X}_2,\mathcal{X}_5] = 2\mathcal{X}_3\ ,\ [\mathcal{X}_3,\mathcal{X}_4] = 2\mathcal{X}_2\ ,\ [\mathcal{X}_4,\mathcal{X}_5] = -4\mathcal{X}_1 - 2\mathcal{X}_6\,.$$

There exist the following Lie sub-algebras:

(2D) commutative $sp\{\mathcal{X}_1,\mathcal{X}_6\}$, $sp\{\mathcal{X}_2,\mathcal{X}_4\}$, $sp\{\mathcal{X}_2,\mathcal{X}_6\}$, $sp\{\mathcal{X}_3,\mathcal{X}_5\}$, $sp\{\mathcal{X}_3,\mathcal{X}_6\}$, $sp\{\mathcal{X}_4,\mathcal{X}_6\}$, $sp\{\mathcal{X}_5,\mathcal{X}_6\}$; non-commutative $sp\{\mathcal{X}_1,\mathcal{X}_2\}$, $sp\{\mathcal{X}_1,\mathcal{X}_3\}$, $sp\{\mathcal{X}_1,\mathcal{X}_4\}$, $sp\{\mathcal{X}_1,\mathcal{X}_5\}$.

(3D) commutative $sp\{\mathcal{X}_2,\mathcal{X}_4,\mathcal{X}_6\}$, $sp\{\mathcal{X}_3,\mathcal{X}_5,\mathcal{X}_6\}$; non-commutative $sp\{\mathcal{X}_1,\mathcal{X}_2,\mathcal{X}_4\}$, $sp\{\mathcal{X}_1,\mathcal{X}_2,\mathcal{X}_6\}$, $sp\{\mathcal{X}_1,\mathcal{X}_3,\mathcal{X}_5\}$, $sp\{\mathcal{X}_1,\mathcal{X}_3,\mathcal{X}_6\}$, $sp\{\mathcal{X}_1,\mathcal{X}_4,\mathcal{X}_6\}$, $sp\{\mathcal{X}_1,\mathcal{X}_5,\mathcal{X}_6\}$,$sp\{\mathcal{X}_2,\mathcal{X}_3,\mathcal{X}_6\}$.

(4D) $sp\{\mathcal{X}_1,\mathcal{X}_2,\mathcal{X}_3,\mathcal{X}_6\}$, $sp\{\mathcal{X}_1,\mathcal{X}_2,\mathcal{X}_4,\mathcal{X}_6\}$, $sp\{\mathcal{X}_1,\mathcal{X}_3,\mathcal{X}_5,\mathcal{X}_6\}$.

(5D) $sp\{\mathcal{X}_1,\mathcal{X}_2,\mathcal{X}_3,\mathcal{X}_5,\mathcal{X}_6\}$.

(vi) Finding the complete solution of the general NFPE (4.16) seems an impossible task. Even when we restrict it to the case of w-weighted Tsallis entropy and we consider, in addition, formulas (4.17), the task does not become easier. Moreover, if we "remove" the weight (i.e. making $w := 1$), as for the classical Tsallis entropy, or even further, if we consider a linear FPE, the solution is not handy. The Lie symmetries offer a *qualitative* information about the NFPEs, through their group-invariant solutions (GISs). These are easier to tackle with, even if the information they reveal might be indirect and incomplete.

For the linear FPE, some GISs were studied in [5, 6]. For some NFPEs, some corresponding GISs were analyzed in [56]. These works consider only non-weighted entropies and one-dimensional GISs. As the following Corollary will show, some of their results remain true (under the same hypothesis) for weights depending on only one of the two variables.

We do not intend to deepen here this approach. We only point out two directions of further study. The first one is to remain in the weighted entropy context and to consider the relevance of *one-dimensional* GISs, following the line of research from [5,6,56].

The second one is to start a "higher dimensional" investigation, in the non-weighted entropy case (for the beginning). The previous sub-algebras ensure the existence of new 2D, 3D, 4D and 5D GISs; maybe some of them will prove useful in physical applications.

Corollary 5.1. *With the previous notations, consider the NFPE associated to the w-weighted Tsallis entropy.*

(i) If $w = w(x)$, then the corresponding Lie symmetries are of the form (5.29).

(ii) If $w = w(t)$, then the corresponding Lie symmetries are of the form (5.30).

(iii) Suppose $w = w(x,t)$ is arbitrary. Then the corresponding Lie symmetries are trivial.

6. A SHORT DIGRESSION INTO THE OPTIMIZATION THEORY OF PDFS

Let $V = V(x)$ be the potential energy function; $q \in (-\infty, 1) \bigcup (1, 2)$ and $U_0 > 0$ be fixed real numbers. We consider the following (MaxEnt) optimization problem:

$$\max H_q^T[p], \tag{6.33}$$

where $p = p(x)$ is a PDF satisfying

$$\int_{\mathbb{R}} V(x)p(x)dx = U_0,$$

and $H_q^T[p]$ is the associated Tsallis entropy function, based on (3.8). It is known ([62] and references therein) that the solution of the optimization problem (6.33) is

$$p^{ME}(x) = (2-q)^{\frac{1}{q-1}} \cdot exp_{\{q\}}^T\Big[-(\gamma + \beta V(x))\Big],$$

where β and γ represent the Lagrange multipliers associated to the optimization problem.

We calculate

$$p^{ME}(x) = \Big\{\frac{1}{2-q}\Big[1 + (1-q)(-(\gamma + \beta V(x)))\Big]\Big\}^{\frac{1}{1-q}} =$$

$$= \Big\{1 + (1-q)\Big[\frac{-(\gamma + \beta V(x))}{2-q} - \frac{1}{2-q}\Big]\Big\}^{\frac{1}{1-q}} = exp_{\{q\}}^T\Big[\frac{-(\gamma + \beta V(x))}{2-q} - \frac{1}{2-q}\Big].$$

We obtained an equivalent form for the MaxEnt problem (6.33), namely

$$p^{ME}(x) = exp_{\{q\}}^T\Big[\frac{-(\gamma + \beta V(x))}{2-q} - \frac{1}{2-q}\Big],$$

which is used, sometimes, in the literature, due to its more condensed form.

Consider now, in addition, a weighting function $w = w(x)$ and the following "weighted" optimization problem:

$$\max H_q^{wT}[p], \tag{6.34}$$

where $p = p(x)$ is a PDF satisfying

$$\int_{\mathbb{R}} V(x)p(x)dx = U_0$$

and $H_q^{wT}[p]$ is the associated w-weighted Tsallis entropy function, based on (3.8).

Theorem 6.5. *The solution of the optimization problem (6.34) is*

$$p_w^{ME}(x) = (2-q)^{\frac{1}{q-1}} \cdot exp_{\{q\}}^T\Big[-\frac{\gamma + \beta V(x)}{w(x)}\Big], \tag{6.35}$$

where β and γ represent the Lagrange multipliers associated to the optimization problem.

Proof. We adapt the standard procedure, see [13] §12.1. □

Remark 6.6. (i) A similar calculation leads to the equivalent form of (6.35)

$$p_w^{ME}(x) = exp_{\{q\}}^T \Big[\frac{-(\gamma + \beta V(x))}{(2-q)w(x)} - \frac{1}{2-q} \Big]. \tag{6.36}$$

(ii) We point out the "barrier" $q = 2$, which forbids (for greater values of the parameter q) the existence of PDFs with maximum entropy. Maybe it is only a "shadow" of a deeper property of the Tsallis entropies family. We mention here the review paper [59], with an interesting panorama on various q-dependent theoretical, experimental, observational and computational aspects.

(iii) The constants β and γ are determined by the constraints of the optimization problem.

(iv) Formally, the weighting procedure does not provide significant changes for the solution of the optimization problem. However, we must take into account that $exp_{\{q\}}^T$ is not a true exponential function, so its weighting factor cannot be easily separated from the part involving β, γ and V.

(v) With the previous notations, denote H^w the w-weighted Tsallis entropy $H_q^{wT}[p_w^{ME}]$, by

$$U^w := \int_{\mathbb{R}} V(x) \cdot p_w^{ME}(x) dx,$$

the mean force with respect to p_w^{ME}, by

$$E^w := \int_{\mathbb{R}} w(x) \cdot p_w^{ME}(x) dx,$$

the mean value of w with respect to p_w^{ME} and by

$$F^w := -\frac{\gamma + E^w}{\beta},$$

the w-weighted q-generalized free energy. A short calculation gives the w-weighted q-generalizations of the thermodynamic relations:

$$F^w = U^w + \frac{q-2}{\beta} H^w, \quad \frac{d}{d\beta}(\beta F^w) = U^w.$$

(All these entities depend on q, which we skipped in the previous formulas for the sake of simplicity. For physical interpretations, see [61,62].)

7. Concluding remarks

The main result of our paper is the determination of the NFPE associated to a w-weighted Tsallis entropy (formula (4.15) and, equivalently, (4.16) + (4.17)). Then we analyze the Lie symmetries of this equation (in the Corollary 5.1) and we compare them with those arising from the classical Tsallis entropy. In some particular cases, we determine some sub-algebras spanned by the respective vector fields, by identifying their isomorphism classes in the Bianchi classification (Remark 5.5. (iv),(v)).

As a by-product and as an extension, we prove the non-negativity of the associated Lyapunov function and a theorem expressing the Bregman divergence as a distance between constant level sets of the Lyapunov function (Theorem 4.3).

In the last section, we derive the solution for the MaxEnt problem associated to the w-weighted Tsallis entropy (Theorem 6.5) and a w-weighted q-generalization of the thermodynamic relations (Remark 6.6 (v)).

In a forthcoming paper, we shall make a similar study for the NFPE based on the weighted Kaniadakis entropy [53]. The new affine and conformal control tools from [28, 29] may be useful for applications of the Lie symmetries of the NFPE in the context of statistical manifolds.

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