CARPATHIAN J. MATH.
Volume **38** (2022), No. 3,
Pages 619 - 630

Online version at https://semnul.com/carpathian/
Print Edition: ISSN 1584 - 2851; Online Edition: ISSN 1843 - 4401
DOI: https://doi.org/10.37193/CJM.2022.03.08

Dedicated to Prof. Emeritus Mihail Megan on the occasion of his 75^{th} anniversary

Operators with Brownian unitary dilations

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ABSTRACT. Certain bounded linear operators T on a complex Hilbert space $\mathcal{H}$ which have 2-isometric liftings S on another space $\mathcal{K} \supset \mathcal{H}$ are being investigated. We refer also to a more special type of liftings, as well as to those which additionally meet the condition $S^*S\mathcal{H} \subset \mathcal{H}$. Furthermore we describe other types of dilations for T, which are close to 2-isometries. Among these we refer to expansive (concave) operators and also to Brownian unitary dilations. Different matrix representations for such operators are obtained, where matrix entries involve contractive operators.

1. INTRODUCTION AND PRELIMINARIES

In this paper we denote by $\mathcal{B}(\mathcal{H}, \mathcal{H}')$ the Banach space of all bounded linear operators acting between two complex Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$ and $\mathcal{B}(\mathcal{H}) = \mathcal{B}(\mathcal{H}, \mathcal{H})$ is considered a C^*-algebra with the identity operator $I = I_{\mathcal{H}}$. For $T \in \mathcal{B}(\mathcal{H}, \mathcal{H}')$, $\mathcal{R}(T) \subset \mathcal{H}'$ and $\mathcal{N}(T) \subset \mathcal{H}$ stand for the range and the kernel of T, while $T^* \in \mathcal{B}(\mathcal{H}', \mathcal{H})$ means the adjoint operator of T. If $\mathcal{M}$ is a subspace of $\mathcal{H}$ we write $\overline{\mathcal{M}}$ for the closure of $\mathcal{M}$ in $\mathcal{H}$. When $\mathcal{M}$ is closed we denote by $P_{\mathcal{M}} \in \mathcal{B}(\mathcal{H})$ the orthogonal projection with $\mathcal{R}(P_{\mathcal{M}}) = \mathcal{M}$, and by $P_{\mathcal{H},\mathcal{M}} \in \mathcal{B}(\mathcal{H}, \mathcal{M})$ the projection of $\mathcal{H}$ onto $\mathcal{M}$. The (closed) subspace $\mathcal{M}$ is invariant (resp. reducing) for $T \in \mathcal{B}(\mathcal{H})$ if $TP_{\mathcal{M}} = P_{\mathcal{M}}TP_{\mathcal{M}}$ (resp. $TP_{\mathcal{M}} = P_{\mathcal{M}}T$). When $\mathcal{M}$ is invariant for T, the operator $T_{\mathcal{M}} = T|_{\mathcal{M}} \in \mathcal{B}(\mathcal{M})$ is the *restriction* of T to $\mathcal{M}$, while T is an *extension* for $T_{\mathcal{M}}$. In this case $\mathcal{K} \ominus \mathcal{M}$ is an invariant subspace for T^*.

Let $\mathcal{K}, \mathcal{K}'$ be Hilbert spaces which contain $\mathcal{H}$ respectively $\mathcal{H}'$ as closed subspaces. An operator $S \in \mathcal{B}(\mathcal{K}, \mathcal{K}')$ is a *lifting* for $T \in \mathcal{B}(\mathcal{H}, \mathcal{H}')$ if $P_{\mathcal{K}',\mathcal{H}'}S = TP_{\mathcal{K},\mathcal{H}}$. When this occurs one has $S(\mathcal{K} \ominus \mathcal{H}) \subset \mathcal{K}' \ominus \mathcal{H}'$. Equivalently, S is a lifting for T if and only if $S^*J_{\mathcal{H}',\mathcal{K}'} = J_{\mathcal{H},\mathcal{K}}T^*$ where $J_{\mathcal{H},\mathcal{K}} = P^*_{\mathcal{K},\mathcal{H}}$ is the embedding mapping of $\mathcal{H}$ into $\mathcal{K}$, and similarly $J_{\mathcal{H}',\mathcal{K}'} = P^*_{\mathcal{K}',\mathcal{H}'}$. It is obvious that if we take $\mathcal{K}' = \mathcal{K}$ and $\mathcal{H}' = \mathcal{H}$, the relation $S^*J_{\mathcal{H},\mathcal{K}} = J_{\mathcal{H},\mathcal{K}}T^*$ exactly means that S^* is an extension for T^*, and in this case $S(\mathcal{K} \ominus \mathcal{H}) \subset \mathcal{K} \ominus \mathcal{H}$. More generally, we say that $S \in \mathcal{B}(\mathcal{K})$ is a *dilation* of $T \in \mathcal{B}(\mathcal{H})$ if $T^n = P_{\mathcal{K},\mathcal{H}}S^nJ_{\mathcal{H},\mathcal{K}}$ for every integer $n \geq 0$. When this happens we also say that T is a *compression* of S.

An operator $A \in \mathcal{B}(\mathcal{H})$ is said to be positive (in notation $A \geq 0$) if $\langle Ah, h\rangle \geq 0$ for any $h \in \mathcal{H}$, where $\langle \cdot, \cdot \rangle$ denotes the scalar product in any Hilbert space. When $A \geq 0$ we write $A^{1/2}$ for its square root. According to the terminology of [19] we say that an operator $T \in \mathcal{B}(\mathcal{H})$ is an *A-contraction* for a positive operator $A \in \mathcal{B}(\mathcal{H})$ if $T^*AT \leq A$ and $A \neq 0$. In this case $T\mathcal{N}(A) \subset \mathcal{N}(A)$ and $T^*\overline{\mathcal{R}(A)} \subset \overline{\mathcal{R}(A)}$. Also we say that T is an *A-isometry* if $T^*AT = A$. Clearly, T is a *contraction* if $T^*T \leq I$ and T is an *isometry* if $T^*T = I$. Also, T is *unitary* if T and T^* are isometries, and T is *expansive* if $T^*T \geq I$, or in other words $\Delta_T := T^*T - I \geq 0$.

Received: 30.01.2022. In revised form: 21.04.2022. Accepted: 28.04.2022
2020 *Mathematics Subject Classification.* 47A05, 47A15, 47A20, 47A63.
Key words and phrases. *2-isometric lifting, Brownian unitary operator, A-contraction.*

An operator $T \in \mathcal{B}(\mathcal{H})$ is *concave* if it is a Δ_T-contraction and T is a *2-isometry* if it is a Δ_T-isometry. In both cases T is expansive and $\mathcal{N}(\Delta_T)$ is invariant for T, while $V = T|_{\mathcal{N}(\Delta_T)}$ is an isometry.

A 2-isometry T on $\mathcal{H}$ is called *Brownian unitary* if $U = T^*|_{\overline{\mathcal{R}(\Delta_T)}}$ is unitary, and $E = \delta^{-1}P_{\mathcal{N}(\Delta_T)}T|_{\overline{\mathcal{R}(\Delta_T)}}$ is an isometry with $\mathcal{R}(E) = \mathcal{N}(V^*)$, V as above, while $\delta = \|\Delta_T^{1/2}\|$.

Obviously, the class of 2-isometries contains the isometries, while the unitary operators are considered to be Brownian unitaries with $\delta = 0$. These latter operators are essential in the dilation theory of contractions initiated by Bela Sz.-Nagy and Ciprian Foiaş and developed by many authors (see [11, 22]). On the other hand, different classes of operators close to 2-isometries and more general to A-contractions have been studied intensively lately. We are referring here only to some articles like [1, 2, 3, 4, 5, 9, 10, 12, 13, 16, 17, 18, 19].

In this paper we continue the study of operators T with 2-isometric liftings, which was started and developed in [6, 7, 8, 14, 15, 20, 21]. So, in Section 2 we refer to general 2-isometric liftings and show that they can be obtained by some expansive (even concave) liftings. Also, we see that 2-isometric liftings for T can be also induced by dilations of T which have triangulations with contraction entries, which suggests a relationship with the isometric liftings of contractions. We characterize the operators T with such a more particular triangulation, by 2-isometric liftings S with $S^*S\mathcal{H} \subset \mathcal{H}$ and having the covariance operator Δ_S a scalar multiple of an orthogonal projection.

In Section 3 we study an extension $\widetilde{T}$ of an operator T that has a Brownian unitary dilation B. We show that $\widetilde{T}^*$ is an A-contraction, where A is related to $\mathcal{N}(\Delta_B)$ and we describe the triangulation of $\widetilde{T}$ under the decomposition $\overline{\mathcal{R}(A)} \oplus \mathcal{N}(A)$ in the terms of B. As an application we characterize the operators T with 2-isometric liftings S satisfying $S^*S\mathcal{H} \subset \mathcal{H}$ by using a Brownian unitary extension B of S. Also, we prove that these operators have an extension with a more particular matrix structure, namely having as entries contractions and even coisometries. The cases when $\mathcal{R}(A)$ is closed and some compressions of T are similar to contractions are also considered.

2. Operators with liftings close to 2-isometries

We will further investigate the operators with 2-isometric liftings, by means of some intermediate liftings, extensions or dilations which lead to 2-isometries. In this regard we show first of all that the 2-isometric liftings can be obtained by intermediate expansive or A-contractive liftings.

Theorem 2.1. *For $T \in \mathcal{B}(\mathcal{H})$ non-contractive the following statements are equivalent:*

(i) *T has a 2-isometric lifting;*

(ii) *T has a lifting $\widehat{T} \in \mathcal{B}(\widehat{\mathcal{H}})$ such that $\widehat{T}$ is an A-contraction for a positive operator A on $\widehat{\mathcal{H}}$ with $A \geq \Delta_{\widehat{T}}$;*

(iii) *T has an expansive lifting $\widetilde{T} \in \mathcal{B}(\widetilde{\mathcal{H}})$ which under a decomposition $\widetilde{\mathcal{H}} = \mathcal{H}_0 \oplus \mathcal{H}_1$ has a triangulation of the form*

$$\widetilde{T} = \begin{pmatrix} V & X \\ 0 & Z \end{pmatrix}, \tag{2.1}$$

where V is an isometry on $\mathcal{H}_0$ and Z is an A_1-contraction on $\mathcal{H}_1$ with $A_1 \geq X^*X + \Delta_Z$;

(iv) *T has a concave lifting.*

Proof. The implication (i)⇒(iv) is trivial. Assume that T has a concave lifting $\widetilde{T}$ on a Hilbert space $\widetilde{\mathcal{H}} \supset \mathcal{H}$. Then $\Delta_{\widetilde{T}} = \widetilde{T}^*\widetilde{T} - I \geq 0$ i.e. $\widetilde{T}$ is expansive, and $\widetilde{T}^*\Delta_{\widetilde{T}}\widetilde{T} \leq \Delta_{\widetilde{T}}$

i.e. $\widetilde{T}$ is a $\Delta_{\widetilde{T}}$-contraction. So $\mathcal{N}(\Delta_{\widetilde{T}})$ is an invariant subspace for $\widetilde{T}$, hence $\widetilde{T}$ has a matrix representation (2.1) under the decomposition $\widetilde{\mathcal{H}} = \mathcal{N}(\Delta_{\widetilde{T}}) \oplus \overline{\mathcal{R}(\Delta_{\widetilde{T}})}$ with $V = \widetilde{T}|_{\mathcal{N}(\Delta_{\widetilde{T}})}$ an isometry. Also, since $\Delta_{\widetilde{T}} \geq 0$ we have $V^*X = 0$, and using this fact we get that $\Delta_{\widetilde{T}} = 0 \oplus \Delta_0$, where $\Delta_0 = X^*X + \Delta_Z \geq 0$ on $\overline{\mathcal{R}(\Delta_{\widetilde{T}})}$. In addition, the above inequality ensures that $Z^*\Delta_0 Z \leq \Delta_0$ and $\Delta_0 \neq 0$ (T being non-contractive) i.e. Z is a Δ_0-contraction. Thus the entries V, X and Z of $\widetilde{T}$ have the required properties in (2.1), hence (iv) implies (iii).

Now suppose that T has an expansive lifting $\widetilde{T}$ of the form (2.1) under a decomposition $\widetilde{\mathcal{H}} = \mathcal{H}_0 \oplus \mathcal{H}_1$. Since V is an isometry on $\mathcal{H}_0$ and $\Delta_{\widetilde{T}} \geq 0$ one has $V^*X = 0$ and Z is an A_1-contraction on $\mathcal{H}_1$ with $A_1 \geq X^*X + \Delta_Z =: \Delta_1 \geq 0$, we obtain that $\widetilde{T}$ is an A-contraction where $A = 0 \oplus A_1$ on $\mathcal{H}_0 \oplus \mathcal{H}_1$. Also, the previous inequality for A_1 leads to $A \geq 0 \oplus \Delta_1 = \Delta_{\widetilde{T}}$, hence the lifting $\widetilde{T}$ of T has the required property in (ii). We conclude that (iii) implies (ii).

Finally, let's assume that $\widehat{T}$ and A on $\widehat{\mathcal{H}} \supset \mathcal{H}$ are as in the statement (ii). Let $\mathcal{H}' = \ell^2_+(\overline{\mathcal{R}(A - \Delta_{\widehat{T}})})$ and $\widehat{S} \in \mathcal{B}(\mathcal{H}' \oplus \widehat{\mathcal{H}})$ be the operator with the block matrix

$$\widehat{S} = \begin{pmatrix} S_+ & (A - \Delta_{\widehat{T}})^{1/2} \\ 0 & \widehat{T} \end{pmatrix},$$

where S_+ is the forward shift on $\mathcal{H}'$ with $\mathcal{N}(S_+^*) = \overline{\mathcal{R}(A - \Delta_{\widehat{T}})}$. Then $\Delta_{\widehat{S}} = 0 \oplus A$ on $\mathcal{K} = \mathcal{H}' \oplus \widehat{\mathcal{H}}$ and

$$\widehat{S}^*\Delta_{\widehat{S}}\widehat{S} = 0 \oplus \widehat{T}^*A\widehat{T} \leq 0 \oplus A = \Delta_{\widehat{S}},$$

hence $\widehat{S}$ is a concave operator. Since $\widehat{S}$ has a 2-isometric lifting S (see [7, 8]) and $\widehat{S}$ is a lifting for T, it follows that S is also a 2-isometric lifting for T. Thus (ii) implies (i). □

Remark 2.1. In the implication (ii) ⇒(i) we can get by [8, Theorem 4.1] a 2-isometric lifting $\widehat{S}$ for $\widehat{T}$ with $\widehat{S}^*\widehat{S}\widehat{\mathcal{H}} \subset \widehat{\mathcal{H}}$. Moreover, for the expansive lifting $\widetilde{T}$ from (iii) of T we get by [8, Theorem 3.7] a 2-isometric lifting $\widetilde{S}$ on $\widetilde{\mathcal{K}} \supset \widetilde{\mathcal{H}}$ with $\widetilde{\mathcal{K}} \ominus \widetilde{\mathcal{H}} \subset \mathcal{N}(\Delta_{\widetilde{S}})$. But $\mathcal{H}$ is neither invariant for $\widehat{S}^*\widehat{S}$, nor for $\widetilde{S}^*\widetilde{S}$, in general, when we consider $\widehat{S}$ and $\widetilde{S}$ as liftings for T.

However, if $\widehat{T}$ is a concave lifting for T as in (iv) and $\widetilde{T}$ is an extension for T as in (2.1) with the properties from (iii), while $\widetilde{S}$ and $\widehat{S}$ are as above, then for $S_0 = \widetilde{S}|_{\widetilde{\mathcal{H}}^\perp \oplus \mathcal{H}}$ and $S_1 = \widehat{S}|_{\widehat{\mathcal{H}}^\perp \oplus \mathcal{H}}$ we have $\widetilde{\mathcal{H}}^\perp \subset \mathcal{N}(\Delta_{S_0})$, respectively $S_1^*S_1\mathcal{H} \subset \mathcal{H}$. Obviously, S_0 and S_1 are liftings for T, and S_0 also satisfies the condition $S_0^*S_0\mathcal{H} \subset \mathcal{H}$. Such 2-isometric liftings were studied in [7, 8, 14, 15, 20, 21]. But this special case can be now presented as a consequence of the above theorem.

Corollary 2.1. *For $T \in \mathcal{B}(\mathcal{H})$ non-contractive the following statements are equivalent:*

(i) *T has a 2-isometric lifting S with $S^*S\mathcal{H} \subset \mathcal{H}$;*
(ii) *T is an A_0-contraction for an operator $A_0 \geq \Delta_T$;*
(iii) *T has an expansive lifting $\widetilde{T}$ of the form* (2.1) *on $\widetilde{\mathcal{H}} = \mathcal{H}_0 \oplus \mathcal{H}_1$, with $\widetilde{T}^*\widetilde{T}\mathcal{H} \subset \mathcal{H}$, $V = \widetilde{T}|_{\mathcal{H}_0}$ an isometry, $Z = P_{\mathcal{H}_1}\widetilde{T}|_{\mathcal{H}_1}$ an A_1-contraction for an operator $A_1 \geq X^*X + \Delta_Z$, such that $\widetilde{A}_1\mathcal{H} \subset \mathcal{H}$ where $\widetilde{A}_1 = 0 \oplus A_1$ on $\mathcal{H}_0 \oplus \mathcal{H}_1$ and $X = P_{\mathcal{H}_0}\widetilde{T}|_{\mathcal{H}_1}$;*
(iv) *T has a concave lifting $\widehat{T}$ with $\widehat{T}^*\widehat{T}\mathcal{H} \subset \mathcal{H}$.*

Proof. The implications (i)⇒(iv) and (iv)⇒(iii) are obvious, if we take $\widehat{T} = S$, respectively $\widetilde{T} = \widehat{T}$ and $A_1 = \Delta_{\widetilde{T}}|_{\mathcal{H}_1} = X^*X + \Delta_Z$.

Now let us assume that the assertion (iii) is true. We represent the lifting $\widetilde{T}$ of T and the operator $\widetilde{A}_1$ (from (iii)) with $\widetilde{A}_1\mathcal{H} \subset \mathcal{H}$ on $\widetilde{\mathcal{H}} = \mathcal{H}^\perp \oplus \mathcal{H}$, in the form

$$\widetilde{T} = \begin{pmatrix} Y_0 & Y_1 \\ 0 & T \end{pmatrix}, \quad \widetilde{A}_1 = A_2 \oplus A_0.$$

Since Z is an A_1-contraction in (2.1) and $\widetilde{A}_1 = 0 \oplus A_1$ on $\widetilde{\mathcal{H}} = \mathcal{H}_0 \oplus \mathcal{H}_1$ we infer (using (2.1)) that $\widetilde{T}^*\widetilde{A}_1\widetilde{T} \leq \widetilde{A}_1$. Expressing this relation in the terms of the above representations for $\widetilde{T}$ and $\widetilde{A}_1$ on $\widetilde{\mathcal{H}} = \mathcal{H}^\perp \oplus \mathcal{H}$ we get that $T^*A_0T \leq A_0$ and $A_0 \geq 0$ because $\widetilde{A}_1 \geq 0$. But $A_0 \neq 0$ as we will see below, so T is an A_0-contraction.

Next we use that $\widetilde{T}^*\widetilde{T}\mathcal{H} \subset \mathcal{H}$ (by (iii)), which means $Y_0^*Y_1 = 0$ in the above matrix of $\widetilde{T}$. Hence $\Delta_{\widetilde{T}} = \Delta_{Y_0} \oplus (Y_1^*Y_1 + \Delta_T)$ on $\widetilde{\mathcal{H}} = \mathcal{H}^\perp \oplus \mathcal{H}$, and $\Delta_{\widetilde{T}} \leq \widetilde{A}_1$ because $\Delta_{\widetilde{T}}|_{\mathcal{H}_1} = X^*X + \Delta_Z \leq A_1$ (by (iii)). We obtain that

$$\Delta_T \leq Y_1^*Y_1 + \Delta_T = \Delta_{\widetilde{T}}|_{\mathcal{H}} \leq \widetilde{A}_1|_{\mathcal{H}} = A_0,$$

and as T is not a contraction we have $A_0 \neq 0$, which completes the assertion (ii). Thus (iii) implies (ii), while (ii) implies (i) by [8, Theorem 4.1]. □

A direct consequence of Theorem 2.1 and of the last assertion in Remark 2.1 is the following

Corollary 2.2. *Let $T \in \mathcal{B}(\mathcal{H})$ having an expansive lifting (or extension) $\widetilde{T}$ on $\widetilde{\mathcal{H}} \supset \mathcal{H}$, such that $\widetilde{T}$ has a triangulation* (2.1) *with V an isometry and Z similar to a contraction. Then T has a 2-isometric lifting (respectively, a 2-isometric lifting S with $\widetilde{\mathcal{H}}^\perp \subset \mathcal{N}(\Delta_S)$).*

Another characterization for the operators with 2-isometric liftings can be obtained using more general dilations than Brownian unitary dilations. Recall that by the famous result of Agler-Stankus from [2, Theorem 5.80] every 2-isometry has a Brownian unitary extension which retains the covariance. So each operator with 2-isometric lifting has a Brownian unitary dilation and the converse is also true. But an intermediate dilation appears in this setting, which can be easily used in applications and to provide examples.

Theorem 2.2. *For $T \in \mathcal{B}(\mathcal{H})$ non-contractive the following statements are equivalent:*

(i) *T has a 2-isometric lifting;*

(ii) *T has a dilation $\widehat{T}$ on $\widehat{\mathcal{H}} \supset \mathcal{H}$ which under a decomposition $\widehat{\mathcal{H}} = \mathcal{H}_0 \oplus \mathcal{H}_1$ has a triangulation of the form*

$$\widehat{T} = \begin{pmatrix} C_0 & \delta C_1 \\ 0 & C \end{pmatrix}, \tag{2.2}$$

where $\delta > 0$ is a scalar and C, C_j $(j = 0, 1)$ are contractions, such that there exist a Hilbert space $\mathcal{E}$, an isometry $J_0 : \mathcal{D}_{C_0} \to \mathcal{E}$ and a contraction $J_1 : \mathcal{D}_{C_1} \to \mathcal{E}$ satisfying the condition

$$D_{C_0}J_0^*J_1D_{C_1} + C_0^*C_1 = 0. \tag{2.3}$$

Proof. Assume that T has a 2-isometric lifting S on $\mathcal{K} = \mathcal{H}' \oplus \mathcal{H}$ and let B on $\widetilde{\mathcal{K}} = \mathcal{K} \oplus \mathcal{K}'$ be a Brownian unitary extension of S. Then B has triangulations of the form

$$B = \begin{pmatrix} S & \star \\ 0 & \star \end{pmatrix} = \begin{pmatrix} W & \star \\ 0 & \widetilde{T} \end{pmatrix} = \begin{pmatrix} V & \delta E \\ 0 & U \end{pmatrix} \tag{2.4}$$

respectively under the decompositions

$$\widetilde{\mathcal{K}} = \mathcal{K} \oplus \mathcal{K}' = \mathcal{H}' \oplus (\mathcal{H} \oplus \mathcal{K}') = \mathcal{N}(\Delta_B) \oplus \mathcal{R}(\Delta_B),$$

where the lifting S of T has on $\mathcal{K} = \mathcal{H}' \oplus \mathcal{H}$ the representation

$$S = \begin{pmatrix} W & \star \\ 0 & T \end{pmatrix}.$$

Clearly $W = S|_{\mathcal{H}'}$ is a 2-isometry, $\widetilde{T} = P_{\mathcal{M}} B|_{\mathcal{M}}$ from (2.4) is an extension for T on $\mathcal{M} = \mathcal{H} \oplus \mathcal{K}'$, $V = B|_{\mathcal{N}(\Delta_B)}$ and $E : \mathcal{R}(\Delta_B) \to \mathcal{N}(\Delta_B)$ are isometries with $\mathcal{R}(E) = \mathcal{N}(V^*)$, U is unitary on $\mathcal{R}(\Delta_B)$ and $\delta = \|\Delta_B\|^{1/2} = \|\Delta_S\|^{1/2} > 0$ (T being non-contractive).

Since B is a lifting for $\widetilde{T}$ and $\widetilde{T}$ is an extension for T it follows that B is a dilation for T, which has the form (2.2) by the last triangulation in (2.4). Here the condition (2.3) is given by $V^*E = 0$ (quoted above) and $\mathcal{E} = \{0\}$. So (i) implies (ii).

Conversely, we suppose that T has a dilation $\widehat{T}$ as in (2.2) on $\widehat{\mathcal{H}} \supset \mathcal{H}$, with C, C_j contractions satisfying the condition (2.3) for $j = 0, 1$ (as in (ii)). Since C is a contraction it has an isometric lifting. Then by [15, Theorem 2.5] (or by Theorem 2.3 below) it follows that $\widehat{T}$ has a 2-isometric lifting $\widehat{S}$ on $\widehat{\mathcal{K}} \supset \widehat{\mathcal{H}}$. As $\widehat{T}$ is a dilation for T, it has a matrix representation of the form

$$\widehat{T} = \begin{pmatrix} \star & \star & \star \\ 0 & T & \star \\ 0 & 0 & \star \end{pmatrix}$$

under a decomposition $\widehat{\mathcal{H}} = \mathcal{K}_0 \oplus \mathcal{H} \oplus \mathcal{K}_1$. Since $\widehat{S}$ is a lifting for $\widehat{T}$ it follows that $\widehat{S}$ is also a dilation for T, therefore $\widehat{S}$ has relative to T a similar representation as $\widehat{T}$ of above, under the decomposition $\widehat{\mathcal{K}} = [(\widehat{\mathcal{K}} \ominus \widehat{\mathcal{H}}) \oplus \mathcal{K}_0] \oplus \mathcal{H} \oplus \mathcal{K}_1$. Hence $S_0 = \widehat{S}|_{\widehat{\mathcal{K}} \ominus \mathcal{K}_1}$ will be a 2-isometric lifting for T, which proves that (ii) implies (i). □

Remark that the condition (2.3) is more general than $C_0^* C_1 = 0$. In fact, this condition shows that there exist an isometric lifting $V_0 \in \mathcal{B}(\mathcal{E} \oplus \mathcal{H}_0)$ for C_0 and a contractive lifting $\widetilde{C}_1 \in \mathcal{B}(\mathcal{H}_1, \mathcal{E} \oplus \mathcal{H}_0)$ for C_1 such that $V_0^* \widetilde{C}_1 = 0$, for some Hilbert space $\mathcal{E}$.

Notice that Theorem 2.2 is an effective generalization of [21, Theorem 2.1] where we characterized the operators T on $\mathcal{H}$ that have 2-isometric liftings S with $S^*S\mathcal{H} \subset \mathcal{H}$, in terms of an extension for T of the form (2.2). We retrieve this result in the Theorem 3.6 below.

In the general case, Theorem 2.2 shows that the operators T with 2-isometric liftings are exactly the compressions of operators with triangulations (2.2), which satisfy the condition (2.3). But this means that one can get some extensions for T that have liftings of the form (2.2), as we will see in the next section. We now describe by means of 2-isometric liftings the operators of the form (2.2).

Theorem 2.3. *For $T \in \mathcal{B}(\mathcal{H})$ the following statements are equivalent:*

(i) *T has a 2-isometric lifting S with $S^*S\mathcal{H} \subset \mathcal{H}$ and $\Delta_S = \sigma^2 P$ with P an orthogonal projection and a scalar $\sigma > 0$;*

(ii) *T has a triangulation* (2.2) *under a decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ with $C_0 = T|_{\mathcal{H}_0}$, $C^* = T^*|_{\mathcal{H}_1}$ and $C_1 = \delta^{-1} P_{\mathcal{H}_0} T|_{\mathcal{H}_1}$ contractions for some scalar $\delta > 0$, such that C_0 and C_1 satisfy the condition* (2.3).

Proof. Let T, S, P and δ be as in (i). In what follows we may assume, without loos of generality, that T is not a contraction. Let $W = S|_{\mathcal{H}'}$ with $\mathcal{H}' = \mathcal{K} \ominus \mathcal{H}$. Then as $\Delta_S \mathcal{H} \subset \mathcal{H}$ (by (i)) it follows that $\Delta_S = 0 \oplus (\Delta_W|_{\mathcal{H}'}) \oplus (\Delta_S|_{\mathcal{H}})$ under the decomposition $\mathcal{K} = \mathcal{N}(\Delta_W) \oplus \mathcal{R}(\Delta_W) \oplus \mathcal{H}$. We remark from this representation of Δ_S that

$$\mathcal{N}(\Delta_S) = \mathcal{N}(\Delta_W) \oplus \mathcal{N}(\Delta_S|_{\mathcal{H}}), \quad \mathcal{R}(\Delta_S) = \Delta_W \mathcal{H}' \oplus \Delta_S \mathcal{H} = \mathcal{R}(\Delta_W) \oplus (\mathcal{H} \cap \mathcal{R}(\Delta_S)).$$

Since $\mathcal{R}(\Delta_S)$ is closed (by (i)) we obtain that $\mathcal{R}(\Delta_W)$ is closed, too.

Now we use the fact that S is a Δ_S-isometry i.e. $S^*\Delta_S S = \Delta_S$. This ensures that $\mathcal{N}(\Delta_S)$ is invariant for S, which implies that $\mathcal{H}_0 = \mathcal{N}(\Delta_S|_{\mathcal{H}})$ is invariant for T, because if $h \in \mathcal{H}$ and $\Delta_S h = 0$ then

$$Th = P_{\mathcal{H}} Sh \in P_{\mathcal{H}}\mathcal{N}(\Delta_S) = \mathcal{N}(\Delta_S|_{\mathcal{H}}),$$

therefore $T\mathcal{H}_0 \subset \mathcal{H}_0$. Thus it follows that on $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ with $\mathcal{H}_1 = \Delta_S\mathcal{H}$, T has a triangulation of the form (2.2) with the entries $C_0 = T|_{\mathcal{H}_0}$, $C = P_{\mathcal{H}_1}T|_{\mathcal{H}_1}$ and $\widetilde{C}_1 = P_{\mathcal{H}_0}S|_{\mathcal{H}_1}$. Putting $D = P_{\mathcal{N}(\Delta_S)}S|_{\mathcal{H}_1} = \delta C'$ for a contraction C' and a scalar $\delta \geq \|D\|$, we have

$$D = \delta \begin{bmatrix} C_2 & C_1 \end{bmatrix}^{\mathrm{tr}} : \mathcal{H}_1 \to \mathcal{N}(\Delta_W) \oplus \mathcal{H}_0,$$

with C_j contractions ($j = 1, 2$) and $\widetilde{C}_1 = \delta C_1$.

Notice that since $\mathcal{H}_0 \subset \mathcal{N}(\Delta_S)$ and $S|_{\mathcal{N}(\Delta_S)}$ is an isometry, we have that $C_0 = T|_{\mathcal{H}_0} = P_{\mathcal{H}_0}S|_{\mathcal{H}_0}$ is a contraction. On the other hand, as S is a Δ_S-isometry and $\Delta_S = \sigma^2 P$ with $P = P_{\mathcal{R}(\Delta_S)}$ (by (i)) it follows that $S^*PS = P$. Also, one has the relation $PS = PSP$, because $S\mathcal{N}(P) \subset \mathcal{N}(P) = \mathcal{N}(\Delta_S)$. But as S is a P-isometry, there exists an isometry V_1 on $\mathcal{R}(P) = \mathcal{R}(\Delta_S)$ such that $PS = V_1 P$, which yields $S^*|_{\mathcal{R}(\Delta_S)} = PV_1^*$. Then for the operator C from (2.2) we have $C^* = T^*|_{\mathcal{H}_1} = S^*|_{\mathcal{H}_1} = PV_1^*|_{\mathcal{H}_1}$, therefore $C = P_{\mathcal{H}_1}V_1|_{\mathcal{H}_1}$ is a contraction. This also implies that $\mathcal{H}_1 \neq \{0\}$ (by our assumption that T is not a contraction), and also that $\widetilde{C}_1 \neq 0$, so $\delta > 0$ in (2.2). To end the proof of (ii) it remains to show the condition (2.3) for C_0, C_1.

For this (using the above notation) we represent the isometry $V = S|_{\mathcal{N}(\Delta_S)}$ on $\mathcal{N}(\Delta_S) = \mathcal{N}(\Delta_W) \oplus \mathcal{H}_0$ and the operator $D : \mathcal{H}_1 \to \mathcal{N}(\Delta_W) \oplus \mathcal{H}_0$ in the form

$$V = \begin{pmatrix} V_0 & C_0' \\ 0 & C_0 \end{pmatrix}, \quad D = \delta \begin{pmatrix} C_2 \\ C_1 \end{pmatrix},$$

where $V_0 = V|_{\mathcal{N}(\Delta_W)}$ is an isometry and $C_0' : \mathcal{H}_0 \to \mathcal{N}(\Delta_W)$ is a contraction such that $C_0'^* C_0' + C_0^* C_0 = I$ (V being an isometry). So there exists an isometry $J_0 : \mathcal{D}_{C_0} \to \mathcal{N}(\Delta_W)$ satisfying the relation $J_0 D_{C_0} = C_0'$. On the other hand, since $\delta^{-1}D = C'$ is a contraction we have $C_2^*C_2 + C_1^*C_1 \leq I$ i.e. $C_2^*C_2 \leq D_{C_1}^2$. Hence there exists a contraction J_1 from $\mathcal{D}_{C_1}$ into $\mathcal{N}(\Delta_W)$ such that $C_2 = J_1 D_{C_1}$. Finally, since S is expansive and V is an isometry we need to have $V^* P_{\mathcal{N}(\Delta_S)} S|_{\mathcal{R}(\Delta_S)} = 0$, which implies $V^*D = 0$ and later that

$$D_{C_0} J_0^* J_1 D_{C_1} + C_0^* C_1 = C_0'^* C_2 + C_0^* C_1 = 0.$$

Therefore C_0, C_1 satisfy the condition (2.3), and we proved that (i) implies (ii).

Conversely, let us assume that T has a triangulation as in (ii) on $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$. Let V_1 be the (minimal) isometric lifting on $\mathcal{K}_1 = \mathcal{H}_1 \oplus \mathcal{H}_2$ of the contraction $C = P_{\mathcal{H}_1}T|_{\mathcal{H}_1}$, where $\mathcal{H}_2 = \ell^2_+(\mathcal{D}_C)$, while $\mathcal{D}_C = \overline{\Delta_C \mathcal{H}_1}$ is the defect space of C. Consider the space $\mathcal{H}_{-1} = \ell^2_+(\mathcal{E} \oplus \mathcal{H}_2)$ where $\mathcal{E}$ is the Hilbert space quoted in (ii). Denote by S_+ the forward shift on $\mathcal{H}_{-1}$ and let $J : \mathcal{E} \to \mathcal{H}_{-1}$, $J_2 : \mathcal{H}_2 \to \mathcal{H}_{-1}$ be the embedding mappings. We define the isometries V_0 on $\mathcal{K}_0 = \mathcal{H}_{-1} \oplus \mathcal{H}_0$ and V_2 from $\mathcal{K}_1 = \mathcal{H}_1 \oplus \mathcal{H}_2$ into $\mathcal{K}_0 = \mathcal{H}_{-1} \oplus \mathcal{H}_0$ having, respectively, the block matrices

$$V_0 = \begin{pmatrix} S_+ & JJ_0 D_{C_0} \\ 0 & C_0 \end{pmatrix}, \quad V_2 = \begin{pmatrix} JJ_1 D_{C_1} & J_2 \\ C_1 & 0 \end{pmatrix}, \tag{2.5}$$

where the contractions C_0, C_1 and J_0, J_1 are from (2.2) and (2.3).

Now we define the operator S on $\mathcal{K} = \mathcal{K}_0 \oplus \mathcal{K}_1$ with the block matrix

$$S = \begin{pmatrix} V_0 & \delta V_2 \\ 0 & V_1 \end{pmatrix}, \tag{2.6}$$

with the scalar $\delta > 0$ from (2.2). It is easy to see (by using the condition (2.3)) that $V_0^* V_2 = 0$, which leads to the fact that $\Delta_S = 0 \oplus \delta^2 I$ on $\mathcal{K}_0 \oplus \mathcal{K}_1$. Thus we have $\Delta_S = \delta^2 P_{\mathcal{R}(\Delta_S)}$ and trivially $S^* \Delta_S S = \Delta_S$, that is S is a 2-isometry. Also, with the matrices from (2.2), (2.5) and (2.6) we obtain for S the representations

$$S = \begin{pmatrix} S_+ & C_2 & J_2 \\ 0 & T & 0 \\ 0 & C' & V' \end{pmatrix} \begin{bmatrix} \mathcal{H}_{-1} \\ \mathcal{H} \\ \mathcal{H}_2 \end{bmatrix} = \begin{pmatrix} S_+ & J_2 & C_2 \\ 0 & V' & C' \\ 0 & 0 & T \end{pmatrix} \begin{bmatrix} \mathcal{H}_{-1} \\ \mathcal{H}_2 \\ \mathcal{H} \end{bmatrix} = \begin{pmatrix} W & E \\ 0 & T \end{pmatrix} \begin{bmatrix} \mathcal{K} \ominus \mathcal{H} \\ \mathcal{H} \end{bmatrix}.$$

Here $C_2 = \begin{bmatrix} JJ_0 D_{C_0} & \delta JJ_1 D_{C_1} \end{bmatrix} : \mathcal{H}_0 \oplus \mathcal{H}_1 \to \mathcal{H}_{-1}$, $C' = \begin{bmatrix} 0 & J'D_C \end{bmatrix} : \mathcal{H}_0 \oplus \mathcal{H}_1 \to \mathcal{H}_2$ with $J' : \mathcal{D}_C \to \mathcal{H}_2$ the embedding mapping, while V' is the forward shift on $\mathcal{H}_2$. We firstly infer that S is a lifting for T and later, since $W^* E = 0$ (as $S_+^* JJ_i D_{C_i} = 0$, $J_2^* JJ_i D_{C_i} = 0$ for $i = 0, 1$ and $V'^* J' D_C = 0$), we conclude that $S^* S \mathcal{H} \subset \mathcal{H}$. Thus S has the required properties in (i), which proves that (ii) implies (i). □

Corollary 2.3. *If $T \in \mathcal{B}(\mathcal{H})$ satisfies the equivalent conditions of Theorem 2.3 then T has a 2-isometric lifting S with a triangulation of the form* (2.6)*, where the entries V_j ($j = 0, 1, 2$) are isometries and $\delta > 0$ is the scalar from the triangulation* (2.2) *of T.*

Remark that the liftings from (2.6) are more special than those mentioned in Theorem 2.3 (i). Such 2-isometries were considered in [12, 13].

Notice finally that the two properties of S from the assertion (i) before do not involve each other, in general. The condition $S^* S \mathcal{H} \subset \mathcal{H}$ ensures only that T has an extension of the form (2.2) satisfying (2.3), by [21, Theorem 2.1]. So Theorem 2.3 refers to a more special class of operators than those mentioned in Corollary 2.1. Let us also mention that other characterizations for the operators T from Theorem 2.3 were obtained in [21, Theorem 2.2].

3. Extensions of operators with Brownian unitary dilations

We continue the study of operators described in Theorem 2.1 and Theorem 2.2. Each such operator T has a Brownian unitary dilation obtained as an extension of a 2-isometric lifting of T. Using such dilations we describe some extensions for T, which lead to 2-isometric liftings for T. Let's start with the following result.

Theorem 3.4. *Let $T \in \mathcal{B}(\mathcal{H})$ be non-contractive and having a 2-isometric lifting S on $\mathcal{K} = \mathcal{H}' \oplus \mathcal{H}$. Let B on $\widetilde{\mathcal{K}} = \mathcal{K} \oplus \mathcal{K}'$ be a Brownian unitary extension of S, and let $A = P_{\mathcal{M}} P_{\mathcal{N}(\Delta_B)}|_{\mathcal{M}}$ where $\mathcal{M} = \mathcal{H} \oplus \mathcal{K}'$. Then $A \neq 0$ and the following statements hold:*

(i) *$\widetilde{T} = P_{\mathcal{M}} B|_{\mathcal{M}}$ is an extension for T which under the decomposition $\mathcal{M} = \overline{\mathcal{R}(A)} \oplus \mathcal{N}(A)$ has the triangulation*

$$\widetilde{T} = \begin{pmatrix} B_0 & B_1 \\ 0 & V_1^* \end{pmatrix}, \quad B_0 = P_{\overline{\mathcal{R}(A)}} B|_{\overline{\mathcal{R}(A)}}, \quad B_1 = P_{\overline{\mathcal{R}(A)}} B|_{\mathcal{N}(A)}, \quad V_1 = B^*|_{\mathcal{N}(A)}, \tag{3.7}$$

such that $\widetilde{T}^*$ is an A-contraction, B_0^* is an A_0-contraction with $A_0 = A|_{\overline{\mathcal{R}(A)}}$, and V_1 is an isometry.

(ii) *$\overline{\mathcal{R}(A)} = \overline{P_{\mathcal{M}} \mathcal{N}(\Delta_B)} \supset \mathcal{M} \cap \mathcal{N}(\Delta_B)$, $\mathcal{N}(A) = \mathcal{M} \cap \mathcal{R}(\Delta_B)$ and*

$$\mathcal{R}(\Delta_W) \oplus \overline{\mathcal{R}(A)} = [\mathcal{N}(\Delta_B) \ominus \mathcal{N}(\Delta_W)] \oplus [\mathcal{R}(\Delta_B) \ominus \mathcal{N}(A)], \tag{3.8}$$

where $W = S|_{\mathcal{H}'}$. In addition, $\mathcal{R}(A)$ is closed if and only if $\mathcal{R}(\Delta_W)$ is closed; in this case B_0 is similar to a contraction.

Proof. Let T, S and B be as above. Then B has triangulations of the form

$$B = \begin{pmatrix} S & \star \\ 0 & \star \end{pmatrix} = \begin{pmatrix} W & \star \\ 0 & \widetilde{T} \end{pmatrix} = \begin{pmatrix} V & \delta E \\ 0 & U \end{pmatrix}, \tag{3.9}$$

respectively under the decompositions

$$\widetilde{\mathcal{K}} = \mathcal{K} \oplus \mathcal{K}' = \mathcal{H}' \oplus \mathcal{M} = \mathcal{N}(\Delta_B) \oplus \mathcal{R}(\Delta_B),$$

where the lifting S of T has on $\mathcal{K} = \mathcal{H}' \oplus \mathcal{H}$ the triangulation

$$S = \begin{pmatrix} W & \star \\ 0 & T \end{pmatrix}.$$

Here $W = S|_{\mathcal{H}'}$ is a 2-isometry, $\widetilde{T} = P_{\mathcal{M}}B|_{\mathcal{M}}$ is an extension for T on $\mathcal{M} = \mathcal{H} \oplus \mathcal{K}'$, $V = B|_{\mathcal{N}(\Delta_B)}$ and $E : \mathcal{R}(\Delta_B) \to \mathcal{N}(\Delta_B)$ are isometries with $\mathcal{R}(E) = \mathcal{N}(V^*)$, while U is unitary on $\mathcal{R}(\Delta_B)$.

Using the last representation of B in (3.9) as well as that $V^*E = 0$ we obtain

$$BP_{\mathcal{N}(\Delta_B)}B^* = VV^* \oplus 0 \leq P_{\mathcal{N}(\Delta_B)}.$$

Since $P_{\mathcal{N}(\Delta_B)} \neq 0$ (as we see below) it follows that B^* is a $P_{\mathcal{N}(\Delta_B)}$-contraction. Now representing $P_{\mathcal{N}(\Delta_B)}$ on $\widetilde{\mathcal{K}} = \mathcal{H}' \oplus \mathcal{M}$ in the form

$$P_{\mathcal{N}(\Delta_B)} = \begin{pmatrix} \star & \star \\ \star & A \end{pmatrix}, \quad A = P_{\mathcal{M}}P_{\mathcal{N}(\Delta_B)}|_{\mathcal{M}},$$

we get by the above inequality that

$$BP_{\mathcal{N}(\Delta_B)}B^* = \begin{pmatrix} \star & \star \\ \star & \widetilde{T}A\widetilde{T}^* \end{pmatrix} \leq \begin{pmatrix} \star & \star \\ \star & A \end{pmatrix}. \tag{3.10}$$

Hence $\widetilde{T}A\widetilde{T}^* \leq A$.

Let us note that $A \neq 0$ (so $P_{\mathcal{N}(\Delta_B)} \neq 0$). Indeed, if $A = 0$ we have $P_{\mathcal{N}(\Delta_B)}\mathcal{M} = \{0\}$, so $\mathcal{H} \subset \mathcal{M} \subset \mathcal{R}(\Delta_B)$. This gives by (3.9) that $T = P_{\mathcal{H}}B|_{\mathcal{H}} = P_{\mathcal{H}}U|_{\mathcal{H}}$, therefore T is a contraction, which contradicts the hypothesis. So $A \neq 0$ and as $A \geq 0$ from (3.10) it follows that $\widetilde{T}^*$ is an A-contraction. Then $\overline{\mathcal{R}(A)}$ is an invariant subspace for $\widetilde{T}$, hence $\widetilde{T}$ has the triangulation (3.7) under $\mathcal{M} = \overline{\mathcal{R}(A)} \oplus \mathcal{N}(A)$, with the entries B_0, B_1 and V_1^* inferred from the second matrix of B in (3.9).

Now from the definition of A we see that $\mathcal{N}(A) = \mathcal{M} \cap \mathcal{R}(\Delta_B)$ and

$$U^*\mathcal{N}(A) = B^*\mathcal{N}(A) = \widetilde{T}^*\mathcal{N}(A) \subset \mathcal{N}(A),$$

so $V_1 = B^*|_{\mathcal{N}(A)} = U^*|_{\mathcal{N}(A)}$ is an isometry. Also, using the triangulation (3.7) of $\widetilde{T}$ as well as the representation $A = A_0 \oplus 0$ with $A_0 = A|_{\overline{\mathcal{R}(A)}} \neq 0$, we infer from the inequality $\widetilde{T}A\widetilde{T}^* \leq A$ that $B_0A_0B_0^* \leq A_0$, that is B_0 is an A_0-contraction. The assertion (i) is proved.

Next we notice that because $W = S|_{\mathcal{H}'}$ is a 2-isometry, $\mathcal{N}(\Delta_W)$ is invariant for $W = S|_{\mathcal{H}'} = B|_{\mathcal{H}'}$ and also for B. So $B|_{\mathcal{N}(\Delta_W)}$ is an isometry, hence $\mathcal{N}(\Delta_W) \subset \mathcal{N}(\Delta_B)$. This and the fact that $\mathcal{N}(A) \subset \mathcal{R}(\Delta_B)$ give for $\widetilde{\mathcal{K}}$ the decompositions

$$\begin{aligned} \widetilde{\mathcal{K}} &= \mathcal{N}(\Delta_W) \oplus \overline{\mathcal{R}(\Delta_W)} \oplus \overline{\mathcal{R}(A)} \oplus \mathcal{N}(A) \\ &= \mathcal{N}(\Delta_W) \oplus [\mathcal{N}(\Delta_B) \ominus \mathcal{N}(\Delta_W)] \oplus [\mathcal{R}(\Delta_B) \ominus \mathcal{N}(A)] \oplus \mathcal{N}(A) \end{aligned}$$

whence one obtains the relation (3.8).

Clearly, from the definition of A we have $\mathcal{R}(A) \subset P_{\mathcal{M}}\mathcal{N}(\Delta_B)$. Conversely, if $k = P_{\mathcal{M}}k_0$ with $k_0 \in \mathcal{N}(\Delta_B)$ then for every $k_1 \in \mathcal{N}(A) = \mathcal{M} \cap \mathcal{R}(\Delta_B)$ we have $(k, k_1) = (k_0, k_1) = 0$, therefore $k \in \overline{\mathcal{R}(A)}$. So $P_{\mathcal{M}}\mathcal{N}(\Delta_B) \subset \overline{\mathcal{R}(A)}$ and with the converse inclusion of above we get finally $\overline{\mathcal{R}(A)} = \overline{P_{\mathcal{M}}\mathcal{N}(\Delta_B)}$. Obviously, $\mathcal{M} \cap \mathcal{N}(\Delta_B) \subset \overline{\mathcal{R}(A)}$.

For the last assertion in (ii) we infer from [10, Remark 2.12-2] that $\mathcal{R}(A)$ is closed (in $\widetilde{\mathcal{K}}$) if and only if $\mathcal{R}(\Delta_B) + \mathcal{M}$ is closed, or equivalently $\mathcal{N}(\Delta_B) + \mathcal{H}' = \mathcal{N}(\Delta_B) + \overline{\mathcal{R}(\Delta_W)}$ is closed. But for the 2-isometry $W = B|_{\mathcal{H}'}$ we have

$$\Delta_W = P_{\mathcal{H}'}\Delta_B|_{\mathcal{H}'} = \delta^2 P_{\mathcal{H}'}P_{\mathcal{R}(\Delta_B)}|_{\mathcal{H}'}$$

where $\delta = \|\Delta_B\|^{1/2}$. Then by the same remark in [10] we can assert that $\mathcal{R}(\Delta_W)$ is closed (in $\widetilde{\mathcal{K}}$) if and only if $\mathcal{R}(\Delta_B) + \mathcal{M}$ is closed. Thus we conclude that $\mathcal{R}(A)$ and $\mathcal{R}(\Delta_W)$ are simultaneously closed. Clearly, in this case the operator $A_0 = A|_{\mathcal{R}(A)}$ is invertible in $\mathcal{B}(\mathcal{R}(A))$, and as B_0^* is an A_0-contraction it follows that B_0 is similar to a contraction. The assertion (ii) is proved. □

Some arguments in this proof lead to the next improved version of the result mentioned in [21, Corollary 3.3]. Among other things, we will see that the inclusion $\mathcal{M} \cap \mathcal{N}(\Delta_B) \subset \overline{\mathcal{R}(A)}$ from Theorem 3.4 (ii) may be strict, in general.

Theorem 3.5. *Let $T \in \mathcal{B}(\mathcal{H})$ having a 2-isometric lifting S on $\mathcal{K} \supset \mathcal{H}$ and let B be a Brownian unitary extension for S on $\widetilde{\mathcal{K}} \supset \mathcal{K}$ with $\|\Delta_B\| = \|\Delta_S\| > 0$. The following assertions are equivalent:*

(i) $S^*S\mathcal{H} \subset \mathcal{H}$;
(ii) $\mathcal{R}(\Delta_{B|_{\mathcal{K}\ominus\mathcal{H}}}) \subset \mathcal{R}(\Delta_B)$;
(iii) $\mathcal{M} \cap \mathcal{N}(\Delta_B) = \overline{\mathcal{R}(A)}$, *where* $A = P_{\mathcal{M}}P_{\mathcal{N}(\Delta_B)}|_{\mathcal{M}}$ *and* $\mathcal{M} = \mathcal{H} \oplus (\widetilde{\mathcal{K}} \ominus \mathcal{K})$.

If this is the case then A is an orthogonal projection and

$$\mathcal{N}(\Delta_B) = \mathcal{N}(\Delta_{B|_{\mathcal{K}\ominus\mathcal{H}}}) \oplus \mathcal{R}(A), \quad \mathcal{R}(\Delta_B) = \mathcal{R}(\Delta_{B|_{\mathcal{K}\ominus\mathcal{H}}}) \oplus \mathcal{N}(A).$$

Proof. Preserving the notation from the previous proof we have $W := B|_{\mathcal{H}'} = S|_{\mathcal{H}'}$ where $\mathcal{H}' = \mathcal{K} \ominus \mathcal{H}$. Assume that the condition (iii) is verified. Then every $k \in \mathcal{N}(\Delta_B)$ can be written as $k = P_{\mathcal{H}'}k \oplus P_{\mathcal{M}}k$, and $P_{\mathcal{M}}k \in \overline{\mathcal{R}(A)} \subset \mathcal{N}(\Delta_B)$. So $P_{\mathcal{H}'}k \in \mathcal{N}(\Delta_B)$ which gives $\Delta_W P_{\mathcal{H}'}k = P_{\mathcal{H}'}\Delta_B P_{\mathcal{H}'}k = 0$ i.e. $P_{\mathcal{H}'}k \in \mathcal{N}(\Delta_W)$. Thus it follows that $\mathcal{N}(\Delta_B) = \mathcal{N}(\Delta_W) \oplus \overline{\mathcal{R}(A)}$ and this implies $\mathcal{R}(\Delta_{B|_{\mathcal{H}'}}) \subset \mathcal{R}(\Delta_B)$ i.e. the condition of (ii). We conclude that (iii) implies (ii).

Next we assume the condition from (ii) to be satisfied. This firstly yields $\mathcal{R}(\Delta_W) = \mathcal{H}' \cap \mathcal{R}(\Delta_B)$, so $\mathcal{R}(\Delta_W)$ is closed. Now by (3.8) we have $\mathcal{R}(\Delta_W) \oplus \mathcal{N}(A) \subset \mathcal{R}(\Delta_B)$. Let $k \in \mathcal{R}(\Delta_B)$ such that k is orthogonal on $\mathcal{R}(\Delta_W) \oplus \mathcal{N}(A)$. So by (3.8) we get $k \in \mathcal{N}(\Delta_W) \oplus \overline{\mathcal{R}(A)}$. Since k is orthogonal on $\mathcal{N}(\Delta_B)$, k is also orthogonal on $\mathcal{N}(\Delta_W) \subset \mathcal{N}(\Delta_B)$, hence $k \in \overline{\mathcal{R}(A)}$. Then $Ak = P_{\mathcal{M}}P_{\mathcal{N}(\Delta_B)}k = 0$, so $k = 0$ because A is injective on $\overline{\mathcal{R}(A)}$. Given the choice of k we conclude that $\mathcal{R}(\Delta_B) = \mathcal{R}(\Delta_W) \oplus \mathcal{N}(A)$ and $\mathcal{N}(\Delta_B) = \mathcal{N}(\Delta_W) \oplus \overline{\mathcal{R}(A)}$.

Now $S^*S|_{\mathcal{H}'} = P_{\mathcal{K}}B^*B|_{\mathcal{H}'}$, $\mathcal{H}'$ being invariant for S and B, and because B is Brownian unitary we have $\Delta_B = \delta^2 P_{\mathcal{R}(\Delta_B)}$, where $\delta^2 = \|\Delta_B\| = \|\Delta_S\| > 0$. Thus we obtain

$$\begin{aligned} S^*S\mathcal{H}' &= S^*S(\mathcal{N}(\Delta_W) \oplus \mathcal{R}(\Delta_W)) \subset \mathcal{N}(\Delta_W) + P_{\mathcal{K}}B^*B\mathcal{R}(\Delta_W) \\ &\subset \mathcal{N}(\Delta_W) \oplus \mathcal{R}(\Delta_W) + P_{\mathcal{K}}\Delta_B\mathcal{R}(\Delta_W) = \mathcal{H}' + \delta^2 P_{\mathcal{K}}\mathcal{R}(\Delta_W) = \mathcal{H}', \end{aligned}$$

taking into account that $\mathcal{R}(\Delta_W) \subset \mathcal{H}' \cap \mathcal{R}(\Delta_B)$ and $\mathcal{K} = \mathcal{H}' \oplus \mathcal{H}$. Hence $S^*S\mathcal{H} \subset \mathcal{H}$ i.e. the condition (i). In addition, we obtain that $A = I \oplus 0$ on $\mathcal{M} = [\mathcal{N}(\Delta_B) \ominus \mathcal{N}(\Delta_W)] \oplus [\mathcal{R}(\Delta_B) \ominus \mathcal{R}(\Delta_W)]$, that is A is an orthogonal projection. We have shown that (ii) implies (i), while (i) implies (ii) by the proof of [21, Theorem 2.1], because $\|\Delta_B\| = \|\Delta_S\|$.

Finally, we saw above that in hypothesis (ii) we have $\mathcal{N}(\Delta_B) = \mathcal{N}(\Delta_W) \oplus \overline{\mathcal{R}(A)}$, therefore $\overline{\mathcal{R}(A)} \subset \mathcal{M} \cap \mathcal{N}(\Delta_B)$. Since the converse inclusion is also valid (see Theorem 3.4 (ii)), we obtain the condition of (iii). Hence (ii) implies (iii). □

The special 2-isometric liftings discussed in this theorem are expressed by their Brownian unitary extensions. But they can be also described in terms of triangulation (3.7), which has a particular shape in this case. Thus we add another statement equivalent to those of Corollary 2.1.

Theorem 3.6. *An operator $T \in \mathcal{B}(\mathcal{H})$ has a 2-isometric lifting S on $\mathcal{K} \supset \mathcal{H}$ with $S^*S\mathcal{H} \subset \mathcal{H}$ if and only if T has an extension $\widetilde{T}$ on $\mathcal{M} \supset \mathcal{H}$ which under a decomposition $\mathcal{M} = \mathcal{M}_0 \oplus \mathcal{M}_1$ has a triangulation of the form* (2.2) *with $C_0 = \widetilde{T}|_{\mathcal{M}_0}$ and $C_1 = \delta^{-1} P_{\mathcal{M}_0}\widetilde{T}|_{\mathcal{M}_1}$ contractions for a scalar $\delta > 0$ which satisfy the condition* (2.3)*, and with $C = P_{\mathcal{M}_1}\widetilde{T}|_{\mathcal{M}_1}$ a coisometry.*

Proof. Assume that T on $\mathcal{H}$ and S on $\mathcal{K} = \mathcal{H}' \oplus \mathcal{H}$ are as above such that $S^*S\mathcal{H} \subset \mathcal{H}$. Then T has the extension $\widetilde{T}$ of the form (3.7) on $\mathcal{M} = \mathcal{R}(A) \oplus \mathcal{N}(A)$, induced by a Brownian unitary extension B of S on a space $\widetilde{\mathcal{K}} = \mathcal{K} \oplus \mathcal{K}' = \mathcal{H}' \oplus \mathcal{M}$. Since $\mathcal{R}(A) \subset \mathcal{N}(\Delta_B)$ by Theorem 3.5, in the matrix (3.7) we obtain that B_0 is a contraction, $B_1 = \delta C_1$ with a contraction C_1 and $\delta = \|\Delta_B\|^{1/2} > 0$, while $C = V_1^*$ is a coisometry.

Now by Theorem 3.5 we have $\mathcal{N}(\Delta_B) = \mathcal{N}(\Delta_W) \oplus \mathcal{R}(A)$ and $\mathcal{R}(\Delta_B) = \mathcal{R}(\Delta_W) \oplus \mathcal{N}(A)$ where $W = B|_{\mathcal{H}'}$, while $\mathcal{R}(A)$ and $\mathcal{R}(\Delta_W)$ are closed. So the isometries $V = B|_{\mathcal{N}(\Delta_B)}$ and $E = \delta^{-1} P_{\mathcal{N}(\Delta_B)} B|_{\mathcal{R}(\Delta_B)}$ from the canonical triangulation (3.9) of B have the block matrices of the form

$$V = \begin{pmatrix} V_0 & J_0 D_{B_0} \\ 0 & B_0 \end{pmatrix} \begin{bmatrix} \mathcal{N}(\Delta_W) \\ \mathcal{R}(A) \end{bmatrix}, \quad E = \begin{pmatrix} W_0 & J_1 D_{C_1} \\ 0 & C_1 \end{pmatrix} : \begin{bmatrix} \mathcal{R}(\Delta_W) \\ \oplus \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{N}(\Delta_W) \\ \oplus \\ \mathcal{R}(A) \end{bmatrix}$$

with $V_0, W_0, J_0 : \mathcal{D}_{B_0} \to \mathcal{N}(\Delta_W)$ and $J_1 : \mathcal{D}_{C_1} \to \mathcal{N}(\Delta_W)$ isometries. Since $V^*E = 0$ and using these representations for V and E it follows that $D_{B_0} J_0^* J_1 D_{C_1} + B_0^* C_1 = 0$, that is the condition (2.3) for the contractions B_0 and C_1 from the triangulation (3.7) of the extension $\widetilde{T}$ for T. An implication of the proposition is proved.

Conversely, let us assume that T has an extension $\widetilde{T}$ on $\mathcal{M} \supset \mathcal{H}$ as above. Then $\widetilde{T}$ has a 2-isometric lifting $\widetilde{S}$ on $\widetilde{\mathcal{K}} = \mathcal{M}^\perp \oplus \mathcal{M}$ such that $\widetilde{S}^*\widetilde{S}\mathcal{M} \subset \mathcal{M}$ (by Theorem 2.3). But $\mathcal{K}_0 = \mathcal{M}^\perp \oplus \mathcal{H}$ is invariant for $\widetilde{S}$, so $S_0 = \widetilde{S}|_{\mathcal{K}_0}$ is a 2-isometric lifting for T. Also, since $\widetilde{S}^*\widetilde{S}\mathcal{M}^\perp \subset \mathcal{M}^\perp$ we get $S_0^* S_0 \mathcal{M}^\perp = \widetilde{S}^*\widetilde{S}\mathcal{M}^\perp \subset \mathcal{M}^\perp$, that is $S_0^* S_0 \mathcal{H} \subset \mathcal{H}$. The converse assertion is proved. □

From the last part of this proof we see that $\Delta_{\widetilde{S}} = \delta^2 P_{\mathcal{R}(\Delta_{\widetilde{S}})}$ with $\delta > 0$ (by Theorem 2.3), but Δ_{S_0} has not this form, in general. However, as $\mathcal{R}(\Delta_{\widetilde{S}|_{\mathcal{M}^\perp}}) \subset \mathcal{R}(\Delta_{\widetilde{S}})$ and $\Delta_{\widetilde{S}|_{\mathcal{M}^\perp}} = \Delta_{S_0|_{\mathcal{M}^\perp}}$ we have $\Delta_{S_0|_{\mathcal{M}^\perp}} = \delta^2 P$ with an orthogonal projection P. This leads to the following

Corollary 3.4. *If $T \in \mathcal{B}(\mathcal{H})$ satisfies the equivalent assertions of Theorem 3.6 then T has a 2-isometric lifting S on $\mathcal{K} \supset \mathcal{H}$ such that $S^*S\mathcal{H} \subset \mathcal{H}$ and $\Delta_{S|_{\mathcal{K}\ominus\mathcal{H}}} = \delta^2 P$ for an orthogonal projection P and a scalar $\delta > 0$.*

Regarding the operators $\widetilde{T}$ and A from Theorem 3.4 we give some additional properties.

Proposition 3.1. *Let $T \in \mathcal{B}(\mathcal{H})$ having a Brownian unitary dilation B on $\widetilde{\mathcal{K}} = \mathcal{H}' \oplus \mathcal{H} \oplus \mathcal{K}'$ with $\|\Delta_B\| > 0$, and let $\widetilde{T} = P_{\mathcal{M}} B|_{\mathcal{M}}$ and $A = P_{\mathcal{M}} P_{\mathcal{N}(\Delta_B)}|_{\mathcal{M}}$ where $\mathcal{M} = \mathcal{H} \oplus \mathcal{K}'$. The following statements hold.*

(i) *If $\mathcal{N}(A) \neq \{0\}$ then $\widetilde{T}$ is a $P_{\mathcal{N}(A)}$-contraction. In this case, either $\mathcal{H} \subset \overline{\mathcal{R}(A)}$ and then $\widetilde{T}|_{\overline{\mathcal{R}(A)}}$ is an extension for T, or T is an A_1-contraction with $A_1 = P_{\mathcal{H}} P_{\mathcal{N}(A)}|_{\mathcal{H}}$ and $\mathcal{N}(A_1) = \mathcal{H} \cap \overline{\mathcal{R}(A)}$.*

(ii) *If $\mathcal{R}(A)$ is closed and $\mathcal{N}(A) \neq \{0\}$ then T (respectively $T|_{\mathcal{N}(A_1)}$) is similar to a contraction if $A_1 = 0$ (respectively if $\mathcal{N}(A_1) \neq \{0\}$).*

Proof. (i). Assume that $\mathcal{N}(A) \neq \{0\}$. Then using the block matrix (3.7) we get $\widetilde{T}^* P_{\mathcal{N}(A)} \widetilde{T} \leq P_{\mathcal{N}(A)}$, that is $\widetilde{T}$ is a $P_{\mathcal{N}(A)}$-contraction. Let $A_1 = P_{\mathcal{H}} P_{\mathcal{N}(A)}|_{\mathcal{H}}$. Clearly, $A_1 = 0$ if and only if $P_{\mathcal{N}(A)}\mathcal{H} = \{0\}$ i.e. $\mathcal{H} \subset \overline{\mathcal{R}(A)}$. In this case, as $\widetilde{T}$ is an extension of T and $\overline{\mathcal{R}(A)}$ is invariant for $\widetilde{T}$, it follows that $\widetilde{T}|_{\overline{\mathcal{R}(A)}}$ is an extension for T. If $A_1 \neq 0$ then using the triangulations of $\widetilde{T}$ and $P_{\mathcal{N}(A)}$ under the decomposition $\mathcal{M} = \mathcal{H} \oplus \mathcal{K}'$ we get relations of the form

$$\begin{pmatrix} T^* A_1 T & \star \\ \star & \star \end{pmatrix} = \widetilde{T}^* P_{\mathcal{N}(A)} \widetilde{T} \leq P_{\mathcal{N}(A)} = \begin{pmatrix} A_1 & \star \\ \star & \star \end{pmatrix},$$

whence one infers that $T^* A_1 T \leq A_1$, that is T is an A_1-contraction. In this case it is obvious that $\mathcal{N}(A_1) = \mathcal{H} \cap \overline{\mathcal{R}(A)}$.

(ii). Assume that $\mathcal{R}(A)$ is closed and $\mathcal{N}(A) \neq \{0\}$. If $A_1 \neq 0$ then T is an A_1-contraction (by (i)), so $\mathcal{N}(A_1)$ is invariant for T. In this case we have that $\mathcal{N}(A_1) = \mathcal{H} \cap \mathcal{R}(A)$, therefore $T|_{\mathcal{N}(A_1)} = \widetilde{T}|_{\mathcal{N}(A_1)} = B_0|_{\mathcal{N}(A_1)}$, where $B_0 = \widetilde{T}|_{\mathcal{R}(A)}$ as in (3.7). Since B_0 is similar to a contraction (by Theorem 3.4 (ii)) it follows that $T|_{\mathcal{N}(A_1)}$ is similar to a contraction.

In the case when $A_1 = 0$ we have $\mathcal{H} \subset \mathcal{R}(A)$, so $T = B_0|_{\mathcal{H}}$ and (as above) T will be similar to a contraction. □

Remark 3.2. If $T, \widetilde{T}$ and A are as in Theorem 3.4 then the A-contraction $\widetilde{T}^*$ is a lifting for T^* having a triangulation

$$\widetilde{T}^* = \begin{pmatrix} V_1 & B_1^* \\ 0 & B_0^* \end{pmatrix} \tag{3.11}$$

under $\mathcal{M} = \mathcal{N}(A) \oplus \overline{\mathcal{R}(A)}$, where V_1 is an isometry. But when $\mathcal{N}(A) \neq \{0\}$ it is not contained in $\mathcal{N}(\Delta_{\widetilde{T}^*})$, where $\Delta_{\widetilde{T}^*} = \widetilde{T}\widetilde{T}^* - I$ has the decomposition

$$\Delta_{\widetilde{T}^*} = \begin{pmatrix} 0 & V_1^* B_1^* \\ B_1 V_1 & B_1 B_1^* + B_0 B_0^* - I \end{pmatrix}.$$

In fact we have $B_1 V_1 k \neq 0$ for $0 \neq k \in \mathcal{N}(A)$. Indeed, for such k we obtain from the proof of Theorem 3.4 the relations

$$B_1 V_1 k = P_{\overline{\mathcal{R}(A)}} B B^* k = P_{\overline{\mathcal{R}(A)}} \Delta_{B^*} k = \delta E U^* k.$$

Here for the last equality we used the triangulation of the Brownian unitary B from (3.9) with E an isometry and U unitary. Thus $B_1 V_1 k \neq 0$ for $k \neq 0$, which shows that $\mathcal{N}(A) \not\subset \mathcal{N}(\Delta_{\widetilde{T}^*})$.

Remark 3.3. Even under the condition $S^* S \mathcal{H} \subset \mathcal{H}$ (as in Theorem 3.6) it can be seen that $B_0^* B_1 \neq 0$ in (3.11), considering that $A \neq 0$ (by Theorem 3.4). In this case B_0 is a contraction (as we noted earlier), so $\widetilde{T}$ has a triangulation of the form (2.2), where B_0 and B_1 satisfy the condition (2.3), more general than $B_0^* B_1 = 0$.

Acknowledgments. This work was supported by a Hasso Plattner Excellence Research Grant (LBUS-HPI-ERG-2020-07), financed by the Knowledge Transfer Center of the Lucian Blaga University of Sibiu.

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