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Dedicated to Prof. Emeritus Mihail Megan on the occasion of his 75^{th} anniversary

Periodic solutions for certain Hamiltonian systems in arbitrary dimension and global parametrization of some manifolds

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ABSTRACT. It has been recently shown that the limit cycle situation is not valid for Hamiltonian systems in dimension two, under appropriate conditions. The applications concern global parametrizations of closed curves in the plane and optimal design problems. Here, we discuss a partial extension of this result, for certain Hamiltonian-type systems in higher dimension.

1. INTRODUCTION

In this paper, we consider the so-called iterated Hamiltonian systems, introduced in [19]. They play an important role in shape optimization [1], [11], [12], [13], or in mathematical programming [21], [22].

In dimension two or three they become the simplest Hamiltonian systems [15], [18]. If $D \subset R^2$ is a bounded domain and $j \in C^1(\overline{D})$ is the Hamiltonian, then we consider the classical ODE system

$$x'(t) = -\frac{\partial j}{\partial y}(x(t), y(t)), t \in I, \tag{1.1}$$

$$y'(t) = \frac{\partial j}{\partial x}(x(t), y(t)), t \in I, \tag{1.2}$$

$$x(0) = x_0, y(0) = y_0, \tag{1.3}$$

where $(x_0, y_0) \in D$ is given and I denotes the maximal existence interval around the origin, ensured by the Peano theorem. We assume that $j(x_0, y_0) = 0$ and

$$\nabla j(x_0, y_0) \neq 0, \tag{1.4}$$

otherwise the solution of (1.1)-(1.3) is constant and doesn't leave the initial condition.

The Hamiltonian remains constant on the trajectory, $j(x(t), y(t)) = 0,\ t \in I$.

Notice that, although the right-hand side is just continuous, the system (1.1)-(1.3) has the uniqueness property. In fact, the system (1.1)-(1.2) can be reduced to one ODE, by using the implicit function theorem applied to $j(x, y) = 0$ around (x_0, y_0) and due to (1.4). Such arguments have been applied in [4] and have been extended to arbitrary dimension in [19], to obtain the implicit parametrization theorem.

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It is clear that, in dimension two, the Hamiltonian system (1.1)-(1.3) gives a local parametrization of the implicitly defined curve $j(x, y) = 0$, around the initial point (x_0, y_0) and under hypothesis (1.4).

We denote the corresponding curve by

$$J = \{(x, y) \in D;\ j(x, y) = 0\}. \tag{1.5}$$

The Poincaré-Bendixson theorem [6], [17], assumes that (1.1), (1.2) has no equilibrium points in the ω-limit set. Here, we assume

$$|\nabla j(x, y)| > 0,\ \forall (x, y) \in J, \tag{1.6}$$

where we take into account that $(x(t), y(t)) \in J$ for any $t \in I$, by the properties of Hamiltonian systems. It turns out that, for Hamiltonian systems, under hypothesis (1.5), (1.6), the limit cycle situation is not possible, that is the solution of (1.1)-(1.3) has to be periodic. In fact, (1.1)-(1.3) gives a global parametrization of the curve G, in dimension two, if we also assume $|j(x, y)| > 0$ on ∂D, [20].

This geometric property has important applications in shape and topology optimization problems [11], [12], [13], [20].

Here, we extend the periodicity argument to curves in arbitrary dimension, defined via Hamiltonian systems, as discussed in [20]. The question of extending the Poincaré-Bendixson theorem to higher dimension is an open one. This partial extension result, valid for certain ODE systems, has an interest also due to its geometric significance. However, we underline that it is limitted to one dimensional manifolds (curves) in Euclidean spaces.

In the next section, we discuss in detail the three dimensional case, where we also comment an example concerning surfaces. The general finite dimensional case is investigated in the last section, for curves.

2. Dimension three

Let $D \subset R^3$ be a bounded domain and $F, G \in C^1(\overline{D})$ be two given functions. We assume the independence condition

$$\frac{D(F, G)}{D(y, z)} \neq 0,\ \text{in } (x_0, y_0, z_0), \tag{2.7}$$

some given point in D. By (2.7), we get that $\nabla F(x_0, y_0, z_0) \times \nabla G(x_0, y_0, z_0) \neq 0$ and $\nabla F(x_0, y_0, z_0) \neq 0$, $\nabla G(x_0, y_0, z_0) \neq 0$. The choice of (y, z) in (2.7) is just to fix the setting and we also assume that

$$\frac{\partial G}{\partial z}(x_0, y_0, z_0) \neq 0. \tag{2.8}$$

Another natural condition to be imposed here is

$$F(x_0, y_0, z_0) = G(x_0, y_0, z_0) = 0 \tag{2.9}$$

and the implicit equations

$$F(x, y, z) = G(x, y, z) = 0 \tag{2.10}$$

define locally two surfaces around (x_0, y_0, z_0), due to (2.8), (2.9). We denote by C their intersection, a curve in $D \subset R^3$, passing through (x_0, y_0, z_0). For G, we can apply the implicit function theorem and find a neighborhood V of (x_0, y_0) and $g \in C^1(V)$ such that

$$G(x, y, g(x, y)) = 0, \ \forall (x, y) \in V. \tag{2.11}$$

For F, a similar setting can be developped and we shall indicate later the corresponding arguments.

Denote by $\overline{\theta}(x, y, z) = \nabla F(x, y, z) \times \nabla G(x, y, z)$, a vector field defined in D. In the non critical points, the vectors $\nabla F(x, y, z)$, $\nabla G(x, y, z)$ are orthogonal to the surfaces $F(x, y, z) = 0$, $G(x, y, z) = 0$, respectively. Consequently, $\overline{\theta}(x, y, z)$ is tangent to the curve $C \subset D$. In [18], the following system:

$$(x'(t), y'(t), z'(t)) = \overline{\theta}(x(t), y(t), z(t)), \ t \in I, \tag{2.12}$$

$$(x(0), y(0), z(0)) = (x_0, y_0, z_0), \tag{2.13}$$

has been introduced and provides a local parametrization of C around (x_0, y_0, z_0) and I is the local (maximal) existence interval. In fact (2.12), (2.13) is a special case of the general iterated Hamiltonian systems discussed in [19], in the setting of the implicit parametrization theorem.

Assume now that

$$|F(x, y, z)| + |G(x, y, z)| > 0 \text{ on } \partial D, \tag{2.14}$$

that is at least one of the two quantities is not null on ∂D. As a consequence of (2.14), we get $C \cap \partial D = \emptyset$. Moreover, by standard structure results in the theory of ODE's (see Barbu [2], Thm. 9) and since D is bounded, we infer by (2.14) that the system (2.12), (2.13) has the global existence property, i.e. $I = (-\infty, +\infty)$.

The uniqueness is a consequence of the Hamiltonian structure, although the right-hand side in (2.12) is just continuous, see [19].

Although $I = (-\infty, +\infty)$, the parametrization of C may have a local character (for instance, if $\overline{\theta}$ has an equilibrium point on C).

By using (2.11), we can reduce the three dimensional system (2.12), (2.13) to a two dimensional system:

$$\begin{aligned} x'(t) = {} & F_y(x(t), y(t), g(x(t), y(t)))G_z(x(t), y(t), g(x(t), y(t))) - \\ & - F_z(x(t), y(t), g(x(t), y(t)))G_y(x(t), y(t), g(x(t), y(t))), \end{aligned} \tag{2.15}$$

$$\begin{aligned} y'(t) = {} & F_z(x(t), y(t), g(x(t), y(t)))G_x(x(t), y(t), g(x(t), y(t))) - \\ & - F_x(x(t), y(t), g(x(t), y(t)))G_z(x(t), y(t), g(x(t), y(t))), \end{aligned} \tag{2.16}$$

$$(x(0), y(0)) = (x_0, y_0). \tag{2.17}$$

In (2.15)-(2.17), we have made the implicit assumption that the representation (2.11) remains valid along C. Examples of this type can be obtained easily as in Ex.2 in [22], with $G(x, y, z) = \frac{2}{\sqrt{3}} z - x^2 - y^2$ and $F(x, y, z) = (x^2 + y^2 + z^2 + 3)^2 - 64(y^2 + z^2)$ (the intersection between a torus and a paraboloid), etc. This global type assumption is necessary here since we shall show that the parametrization of C via (2.12), (2.13) is global. An example related to applications in mechanics is given in [8].

The implicit function theorem is equivalent with the inverse function theorem [[7], §3.3] and, for this result, there are known characterizations of the global existence, the Hadamard-Caccioppoli theorem ([9], Thm. 2.8), or Palais [16], Corollary 4.3. The literature is very rich in effective conditions ensuring the global property and we quote here just [5], [3], [7] and their references.

Here, F_x, F_y, F_z, etc. denote partial derivatives and the system (2.15)-(2.17) inherits the global existence property and the uniqueness property from (2.12),(2.13). It is a rewriting of the first two equations in (2.12).

We also strengthen the hypothesis (2.7):

$$\frac{D(F,G)}{D(y,z)} \neq 0 \text{ on } \overline{C} \subset D. \tag{2.18}$$

Then, the system (2.15)-(2.17) has no equilibrium points and the Poincaré-Bendixson theorem in R^2 (see [17], Thm. 21, p. 247) yields that (2.15)-(2.17) is either periodic or it is a limit cycle (assuming that F, G are in $C^2(\overline{D})$). Moreover, the corresponding ω-limit set is a closed trajectory of (2.15)-(2.17).

Consider now the real mapping $F(x, y, g(x, y))$ defined again on V. Its gradient with respect to $(x, y) \in V$ is given by

$$\nabla_{x,y}F(x,y,g(x,y)) = [F_x(x,y,g(x,y)) - \frac{F_z(x,y,g(x,y))G_x(x,y,g(x,y))}{G_z(x,y,g(x,y))}, \quad F_y(x,y,g(x,y)) - \frac{F_z(x,y,g(x,y))G_y(x,y,g(x,y))}{G_z(x,y,g(x,y))}] \tag{2.19}$$

due to the classical derivation rule of the implicit function $g(x, y)$ defined in (2.11). Under hypothesis (2.18), we get that $\nabla_{x,y}F(x, y, g(x, y)) \neq 0$ on the solution of (2.15)-(2.17), i.e. on $\{(x, y) \in D;\ (x, y, z) \in C\} = \mathrm{pr}_3\overline{C}$. Notice that (2.19) makes sense on $\mathrm{pr}_3\overline{C}$ under the global hypothesis on (2.11).

We remark that $F(x(t), y(t), g(x(t), y(t))) = 0$ for any $t \in R$, due to the definition of C and to its representation via (2.12), (2.13). Therefore, the tangential derivative of $F(x, y, g(x, y))$ along $\mathrm{pr}_3\overline{C}$ is null in any point.

Assume that the trajectory $(x(t), y(t))$ is not periodic. We denote by Λ the corresponding ω-limit set, associated to (2.15)-(2.17) and by $(\hat{x}, \hat{y}) \in \Lambda$ some point on it. It is not a critical point of F in the sense of (2.19), due to (2.18).

Let $(\tilde{x}, \tilde{y})$ be another point on the trajectory Λ passing through $(\hat{x}, \hat{y})$ (see Hirsch et al. [6], Ch. 9.2). The tangential derivatives to $F(x, y, g(x, y))$ are null in both points, by the previous argument. We consider the normals to this second trajectory, in both $(\tilde{x}, \tilde{y})$ and $(\hat{x}, \hat{y})$.

The trajectory $(x(t), y(t))$ has to intersect at least one of the normals an infinity of times. Otherwise, $(x(t), y(t))$ cannot approximate all the points on the second trajectory and this contradicts the definition of Λ.

One can compute the derivative along this line as well. Assume that the normal through $(\hat{x}, \hat{y})$ is intersected an infinity of times by $(x(t), y(t))$.

It yields that $\nabla_{x,y}F(x, y, g(x, y)) = 0$ in $(\hat{x}, \hat{y})$ since $F(x(t), y(t), g(x(t), y(t))) = 0$, $\forall t \in R$. This contradicts the property $\nabla_{x,y}F(x, y, g(x, y)) \neq 0$ on $\mathrm{pr}_3\overline{C}$.

We obtain that $(x(t), y(t))$ is periodic and, consequently, the solution of (2.12), (2.13) is periodic too.

We have proved:

Proposition 2.1. *Under hypotheses $F, G \in C^2(\overline{D})$ and (2.14), (2.18), (2.11) global, the unique solution of (2.12), (2.13) is periodic.*

Remark 2.1. In dimension three, the case of a single implicit equation $F(x, y, z) = 0$ in D is of interest too. It defines a surface around any non critical point. The implicit parametrization theorem [15], ensures a local parametrization around non critical points, via two iterated Hamiltonian systems. In [15], in the case of the torus, a counterexample is indicated, to the possibility to get a global representation of the torus. Even if the solutions of both iterated Hamiltonian systems are periodic, the representation of the torus surface is partial, not global. Moreover, again in [15], another example related to the torus shows that in certain cases the global parametrization property may be valid. For applications in shape optimization [11], [12], [20], the global character of the representation is crucial. This question remains open in dimension three or higher.

We close this section with a numerical example in dimension three, computed by [14], related to the double-torus (also called the bitorus or pretzel). This is a closed surface of genus 2, defined for instance by the implicit equation:

$$F(x, y, z) = [x(x-1)^2(x-2) + y^2]^2 + z^2 - 0.01 = 0. \tag{2.20}$$

Other variants may be obtained starting from plane curves (for instance, of lemniscate type) $h(x, y) = 0$ by an "inflation" step $h(x, y)^2 + z^2 = \epsilon > 0$. Similarly, one can obtain the n-torus, a three dimensional closed surface of genus n.

A local parametrization around some initial point (x^0, y^0, z^0) can be constructed via the iterated Hamiltonian system [15], [21]:

$$\begin{aligned} x'(t) &= -F_y(x(t), y(t), z(t)), && t \in I_1, \\ y'(t) &= F_x(x(t), y(t), z(t)), && t \in I_1, \qquad (2.21) \\ z'(t) &= 0, && t \in I_1, \\ x(0) &= x^0,\ y(0) = y^0,\ z(0) = z^0; && \qquad (2.22) \end{aligned}$$

$$\begin{aligned} \dot{\varphi}(s, t) &= -F_z(\varphi(s, t), \psi(s, t), \xi(s, t)), && s \in I_2(t), \\ \dot{\psi}(s, t) &= 0, && s \in I_2(t), \qquad (2.23) \\ \dot{\xi}(s, t) &= F_x(\varphi(s, t), \psi(s, t), \xi(s, t)), && s \in I_2(t), \\ \varphi(0, t) &= x(t),\ \psi(0, t) = y(t),\ \xi(0, t) = z(t), && t \in I_1. \qquad (2.24) \end{aligned}$$

Above, we assume that $F_x(x^0, y^0, z^0) \neq 0$, I_1, I_2 denote local existence intervals and we use standard notations for derivatives with respect to t, s.

It is known that the double torus has planes of symmetry and we refer to the "horizontal" one and the "vertical" one, dividing the double torus in two equal "torus like" surfaces. We notice two properties:

i) the orthogonal projection of the double torus on the "horizontal" symmetry plane is a plane domain with two holes and its exterior boundary is included in the intersection of the double torus with this plane.

ii) the intersection of each plane parallel to the "vertical" symmetry plane with the double torus surface has one or two components. All of them have at least one point in common with the curve giving the exterior boundary mentioned at i).
We consider that these two properties ensure in fact the global character of the parametrization obtained via (2.21)-(2.24), see Fig.1 below. Notice that i), ii) are valid for the n-torus too (adapting correspondingly the definition of the "vertical" plane, which is no more a symmetry plane). In the case of the torus, point ii) is satisfied by any "vertical" symmetry plane. These two orthogonal planes give as well the coordinates used in (2.21)-(2.24). The solutions of the Hamiltonian systems (2.21)-(2.22), (2.23)-(2.24) are periodic, as argued before. The initial condition for (2.21)-(2.22) is $(2, 0.31623, 0)$ and it belongs to the exterior boundary defined at i). The points marked by $\times$ belong to this horizontal curve that is parametrized by the solution of the first Hamiltonian system (2.21)-(2.22). Through each such point as initial condition, the iterated Hamiltonian system (2.23)-(2.24) is solved and we get, in this way, the global parametrization of the implicitly defined double torus (2.20). We have used a very coarse discretization, in order to ensure, in Fig. 1, the visibility of all the details. In particular, around the central zone of the double torus this mesh should be denser for a more accurate description, but this would have obscured other parts.

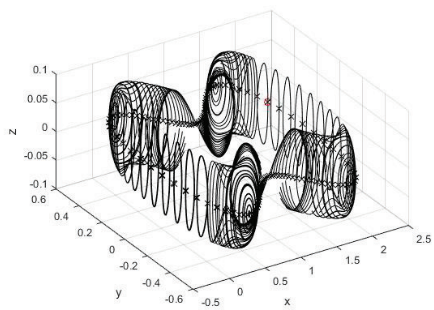

FIGURE 1. The double torus.

The extension of the properties i), ii) to a more general geometric setting, to curvilinear coordinates, to higher dimension and/or codimension of the implicitly defined manifolds and the construction of their global parametrizations, is an open question.

3. ARBITRARY DIMENSION

In this section, we extend the analysis of the periodicity for the systems (2.12), (2.13) to Hamiltonian systems in arbitrary dimension. We consider the general setting from [19]. Let d be a natural number, $D \subset R^d$ be a bounded domain and $F_1, F_2, \ldots, F_{d-1} : \overline{D} \to R$ belong to $C^1(\overline{D})$. We assume that

$$\frac{D(F_1, F_2, \ldots, F_{d-1})}{D(x_1, x_2, \ldots, x_{d-1})} \neq 0 \text{ in } x^0 = (x_1^0, x_2^0, \ldots, x_d^0) \in D, \tag{3.25}$$

$$F_1(x^0) = F_2(x^0) = \ldots = F_{d-1}(x^0) = 0. \tag{3.26}$$

The condition (3.25) is valid on a neighbourhood $V \subset D$, $x^0 \in V$, due to $F-1, F_2, \ldots, F_{d-1} \in C^1(\overline{D})$.

We introduce the linear algebraic system with unknown $v(x) \in R^d$, any $x \in V$, given by:

$$v(x) \cdot \nabla F_j(x) = 0,\ j = \overline{1, d-1}. \tag{3.27}$$

Due to (3.25), the vectors $\nabla F_j(x)$, $j = \overline{1, d-1}$ are independent and the system (3.27) has a unique (up to multiplication by scalars) nontrivial solution $v(x) \in R^d$ and $v(\cdot) \in C(V)$ by $F_j \in C^1(\overline{D})$, $j = \overline{1, d-1}$ and the Cramer rule. We denote by $A(x)$ the nonsingular Jacobian matrix defined in (3.25) and we fix the last component of $v(x)$ to be $\det A(x)$. Then, the first $d-1$ components of $v(x)$ are uniquely determined by (3.27).

We introduce now the autonomous Cauchy problem in D:

$$y'(t) = v(y(t)),\ t \in I, \tag{3.28}$$

$$y(0) = x^0. \tag{3.29}$$

The local existence for (3.28), (3.29) is ensured by the Peano theorem and we also get

Proposition 3.2. *For every* $j = \overline{1, d-1}$, *we have*

$$F_j(y(t)) = 0,\ t \in I. \tag{3.30}$$

Relation (3.30) and (3.26) show that the Hamiltonian mappings F_j, $j = \overline{1, d-1}$ are constant along the trajectory $y(\cdot)$ of the Hamiltonian system (3.28), (3.29), on the maximal existence interval I. Due to (3.30) and an argument based on the implicit functions theorem around x^0, we also get the uniqueness of the solution of (3.28), (3.29), although the right-hand side is just continuous. More general Hamiltonian systems are studied in [19], where Prop. 3.2 is proved.

The geometric interpretation of the above equations is that each relation $F_j(x) = 0$, $j = \overline{1, d-1}$ defines a hipersurface around x^0 and $\nabla F_j(x)$ is the normal vector to it, in $x \in V$. If C is the d dimensional curve in D, passing through x^0 and determined by the intersection of all these hipersurfaces, then $v(x)$ obtained by (3.27) is its tangent vector and $y(t)$, $t \in I$, is a parametrization of C around x^0.

We impose the supplementary hypothesis

$$\sum_{j=1}^{d-1} |F_j(x)| > 0,\ x \in \partial D. \tag{3.31}$$

Then, by (3.31) and Prop 3.2, we know that $C \cap \partial D = \emptyset$. Since D is bounded, again by standard results in the theory of ODE's (Barbu [2], Thm. 9) we get that (3.28), (3.29) has the global existence property and $I = (-\infty, +\infty)$. We underline that even for $I = R$, the solution of (3.28), (3.29) may give just a local parametrization of C, around x^0 (for instance, if C contains an equilibrium point $\overline{x}$, for (3.28), then $y(t)$ approaches $\overline{x}$ for $t \to \infty$, it is possible, without reaching it in fact).

We strengthen hypothesis (3.25):

$$\frac{D(F_1, F_2, \ldots, F_{d-1})}{D(x_1, x_2, \ldots, x_{d-1})} \neq 0 \text{ in any } x \in \overline{C}. \tag{3.32}$$

By (3.32), there are nonsingular minors of order $d-2$ in the above Jacobian matrix. We assume that

$$\frac{D(F_1, F_2, \ldots, F_{d-2})}{D(x_1, x_2, \ldots, x_{d-2})} \neq 0 \text{ in any } x \in \overline{C}. \tag{3.33}$$

In (3.32), (3.33) we have assumed that the nonsingular minor is of this form. Notice that (3.32), (3.33) are standard assumptions in global inversion theorems, [9], [16].

Assuming again the global implicit functions theorem to be valid under condition (3.33), we can express the first $d-2$ arguments $x_1, \ldots, x_{d-2}$ as functions of the last two arguments x_{d-1}, x_d and the system (3.28), (3.29) can be reduced to the last two equations:

$$y'_{d-1}(t) = v_{d-1}(f_1(y_{d-1}(t), y_d(t)), \ldots, f_{d-2}(y_{d-1}(t), y_d(t)), y_{d-1}(t), y_d(t)), \tag{3.34}$$

$$y'_d(t) = v_d(f_1(y_{d-1}(t), y_d(t)), \ldots, f_{d-2}(y_{d-1}(t), y_d(t)), y_{d-1}(t), y_d(t)), \tag{3.35}$$

$$y_{d-1}(0) = x^0_{d-1}, \; y_d(0) = x^0_d. \tag{3.36}$$

We also know that the solution of (3.27) satisfies $v_d(x) = \det A(x)$ and it is not null on $\overline{C}$ by condition (3.32). That is, the system in R^2 given by (3.34)-(3.36) has no equilibrium points and the Poincaré-Bendixson theorem can be applied, under the condition $F_1, F_2, \ldots, F_{d-1} \in C^2(\overline{D})$, which ensures $v \in C^1(V)$. We show that the existence of limit cycles is not possible, for (3.34)-(3.36).

We also know, due to Prop. 3.2, that:

$$F_{d-1}(f_1(y_{d-1}(t), y_d(t)), \ldots, f_{d-2}(y_{d-1}(t), y_d(t)), y_{d-1}(t), y_d(t)) = 0. \tag{3.37}$$

Then, the tangential derivative of the (3.37) function

$$F_{d-1}(f_1(y_{d-1}, y_d), \ldots, f_{d-2}(y_{d-1}, y_d), y_{d-1}, y_d) : \Pr \overline{C} \to R \tag{3.38}$$

along the trajectory $(y_{d-1}(t), y_d(t))$, $t \in R$, is null. Here

$$\Pr \overline{C} = \{(y_{d-1}, y_d) \in R^2; \; \exists (y_1, \ldots, y_{d-2}) : \; (y_1, y_2, \ldots, y_{d-2}, y_{d-1}, y_d) \in \overline{C}\}.$$

We compute now the derivative of the composed mapping (3.38), with respect to the before last variable. We redenote $(y_{d-1}, y_d) = (x_{d-1}, x_d)$ to avoid confusion with the solution of (3.34)-(3.36) and we denote by α_1 the derivative of (3.38) with respect to x_{d-1}:

$$\begin{aligned} \alpha_1 = {} & \nabla F_{d-1}(f_1(x_{d-1}, x_d), \ldots, f_{d-2}(x_{d-1}, x_d), x_{d-1}, x_d) \cdot \\ & \cdot \left(\frac{\partial f_1}{\partial x_{d-1}}, \frac{\partial f_2}{\partial x_{d-1}}, \ldots, \frac{\partial f_{d-2}}{\partial x_{d-1}}, 1, 0 \right) = \partial_{d-1} F_{d-1}(f_1(\cdot,\cdot), \ldots, f_{d-2}(\cdot,\cdot), x_{d-1}, x_d) + \\ & + \sum_{j=1}^{d-2} \partial_j F_{d-1}(f_1(\cdot,\cdot), \ldots, f_{d-2}(\cdot,\cdot), x_{d-1}, x_d) \frac{\partial f_j}{\partial x_{d-1}}. \end{aligned} \tag{3.39}$$

In (3.39), we use the well known formulas for the derivation of implicit functions:

(3.40) $$\frac{\partial f_1}{\partial x_{d-1}} = -\frac{\dfrac{D(F_1,\dots,F_{d-2})}{D(x_{d-1},x_2,\dots,x_{d-2})}}{\dfrac{D(F_1,\dots,F_{d-2})}{D(x_1,x_2,\dots,x_{d-2})}}$$

and $\dfrac{\partial f_j}{\partial x_{d-1}}$ is obtained by replacing the position of x_{d-1} in (3.40), in $D(x_{d-1}, x_2, \dots, x_{d-2})$, to be in the place of x_j ($j = \overline{1, d-2}$).

Notice that we also have the development:

(3.41) $$0 \neq \frac{D(F_1,F_2,\dots,F_{d-1})}{D(x_1,x_2,\dots,x_{d-1})}(x) = \sum_{i=1}^{d-1} \frac{\partial F_{d-1}}{\partial x_i}(x)\ \mathrm{cof}(A(x))_{d-1,i},$$

where $A(x)$ is the Jacobian matrix of order $(d-1)\times(d-1)$ and $\mathrm{cof}(A(x))_{d-1,i}$ is the corresponding cofactor [10].

Lemma 3.1. *The gradient of the composed function (3.38) is not null on* $\mathrm{Pr}\,\overline{C}$*.*

Proof. By (3.39)-(3.41) and using hypothesis (3.32), (3.33), we get that $\alpha_1 \neq 0$ on $\mathrm{Pr}\,\overline{C}$ since we have shown

$$\alpha_1 = \frac{D(F_1,F_2,\dots,F_{d-1})}{D(x_1,x_1,\dots,x_{d-1})} \Big/ \frac{D(F_1,\dots,F_{d-2})}{D(x_1,x_2,\dots,x_{d-2})}.$$

Here, we have to notice that, in (3.40) and its comment, moving the column corresponding to x_{d-1} from the position j to its place (the last column) needs $d-2-j$ changes of sign. Combined with the sign in (3.40) we get $d-1-j$ changes of sign and this is the same as $d-1+j$ changes of sign, exactly as in the definition of the cofactor. □

Proposition 3.3. *Under hypotheses* $F_1,\dots,F_{d-1} \in C^2(\overline{D})$ *and (3.26), (3.31), (3.32), (3.33) global, the unique solution of (3.28), (3.29) is periodic.*

Proof. The periodicity is obtained first for the system (3.34)-(3.36). Namely, the limit cycle situation from the Poincaré-Bendixson theorem is not possible here. This follows by a contradiction argument as in Prop. 2.1, due to Lemma 3.1. If we assume that the solution of (3.34)-(3.36) is a limit cycle, by taking the normals to the ω-limit trajectory in two points $(\tilde{x},\tilde{y})$ and $(\hat{x},\hat{y})$, we infer that also the normal derivative of the composed function (3.38) is null at least in one point. Therefore, in that point the gradient of (3.38) is null and Lemma 3.1 is contradicted.

Then, the periodicity of the solution for (3.28), (3.29) follows due to the equivalence with (3.34)-(3.36). □

Remark 3.2. The parametrization of the closed curve C, expressed by (3.28), (3.29), is global, due to Prop. 3.3. This justifies the necessity of the global assumption in the application of the implicit function theorem via (3.33). For manifolds of higher dimension in R^d, the parametrizations via iterated Hamiltonian systems, constructed in [19], are just local.

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