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Dedicated to Prof. Emeritus Mihail Megan on the occasion of his 75^{th} anniversary

A dynamic Cournot mixed oligopoly model with time delay for competitors

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ABSTRACT. This paper deals with the analysis of a discrete-time Cournot game with time delay, where the interactions of one public firm and n private firms on the market are considered. The production of the public firm is adjusted based on the past production levels of the private firms. At the same time, the productions of the private firms are updated with respect to the past production of the public firm. Two equilibrium points are determined for the discrete-time nonlinear mathematical model. The analysis of the stability reveals that the boundary equilibrium point is a saddle point, while the positive one, under some conditions, is asymptotically stable for any time delays. Numerical simulations illustrate the complex dynamic behaviour of the system.

1. INTRODUCTION

Oligopoly theory is an area of interest in mathematical economics. Diverse oligopoly models have been considered in the literature, such as single or multi-product models with or without product differentiation. In order to reflect real-world conditions, when there are delays in the decision-making processes, lead time, information implementation or execution time, time delay models are used [12, 16, 17].

Dynamic duopoly games have been studied in the context of quantity-setting firms, in a discrete or continuous time setup. For homogeneous or heterogeneous players the options for their strategies are: naive, adaptive or bounded rational [1, 5, 6, 7, 8, 10, 11, 14, 17, 18].

Regarding dynamic Cournot oligopoly games, in [3], there are three concurrent firms with bounded rationality, all based on the utility CES function. For the associated nonlinear discrete-time mathematical model, the analysis of the equilibria is conducted numerically, where complex behaviour is detected. In [20] the dynamics of a mixed triopoly game is analyzed, where a public firm competes against two private firms. Both quantity and price competition are considered and the equilibrium points are determined, while their local stability is investigated. In [2] n firms compete in a framework of isoelastic demand with non-unitary elasticity and the stability of the Nash equilibrium is analysed. In addition, in [13] it is highlighted that state-owned public firms are significant for their market competitors and private firms might face financial difficulties that could lead for the latter ones to be nationalized.

Motivated by the mixed competition, as well as the delay in the decision making process, the aim of the present paper is to extend the previous studies by analyzing the influence on the stability of the equilibrium of the number of private firms in the market, if information delays are taken into account.

More precisely, we consider one public firm and n private firms producing differentiated products in the framework of a dynamic oligopoly game. The players, considered as

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the firms, are heterogeneous: while the public firm has bounded rationality, the private firms are naive. The public firm makes the output's decision based on the expected marginal payoff, that is, the social surplus, taking into account the past production levels of the private firms [15, 20]. The outputs of the private firms are established using the reaction functions and the past production of the public firm. The main finding is related to characterising the stability of the Nash equilibrium, with respect to the number of the private firms, the degree of product differentiation, the adjustment parameter and the time delays.

The structure of this paper is presented in what follows. The mathematical model is described in Section 2 and two equilibrium points are found. Section 3 deals with the local stability analysis for both the boundary and the positive equilibrium points. Numerical simulations are presented in Section 4, where the theoretical findings are illustrated, followed by conclusions and the discussion of future directions for research.

2. MATHEMATICAL MODEL

Let q_0 be the quantity produced by the public firm with the retail price p_0 and q_i, $i = \overline{1,n}$ the quantity corresponding to the private firm i with the retail price p_i.

The representative consumer maximizes the function [19]:

$$U(q_0, q_1, ..., q_n) - \sum_{i=0}^{n} p_i q_i \ ,$$

where the utility function is supposed to be quadratic and strictly concave, of the form [4, 19, 20]:

$$U(q_0, q_1, ..., q_n) = a \sum_{i=0}^{n} q_i - \frac{b}{2} \left(\sum_{i=0}^{n} q_i^2 + \delta \sum_{i=0}^{n} \sum_{i \neq j} q_i q_j \right),$$

with a, b positive parameters and $\delta \in (0,1)$ the degree of product differentiation.

Thus, the maximization problem:

$$\max \left(U(q_0, q_1, ..., q_n) - \sum_{i=0}^{n} p_i q_i \right)$$

leads to

$$p_i = a - bq_i - b\delta \sum_{j=0, j\neq i}^{n} q_j, \quad i = \overline{0,n}. \tag{2.1}$$

The public firm focuses on the maximization of its payoff, that is, the social surplus [20]:

$$SW(q_0, q_1, .., q_n) = a \sum_{i=0}^{n} q_i - \frac{b}{2} \left(\sum_{i=0}^{n} q_i^2 + \delta \sum_{i=0}^{n} \sum_{i \neq j} q_i q_j \right) - \sum_{i=0}^{n} p_i q_i + \sum_{i=0}^{n} P_i, \tag{2.2}$$

where P_i, $i = \overline{0,n}$ is the profit function of firm i, given by:

$$P_i = (p_i - c_i) q_i, \quad i = \overline{0,n} \tag{2.3}$$

with c_i the marginal cost of firm i. Thus,

$$\frac{\partial SW}{\partial q_0}(q_0, q_1, ..., q_n) = 0,$$

namely

$$a - c_0 - bq_0 - b\delta \sum_{i=1}^{n} q_i = 0. \tag{2.4}$$

The private firm i, $i = \overline{1,n}$ also wants to maximize its payoff, that is, its profit P_i. We assume that all the private firms have the same marginal costs $c_1 = c_2 = ... = c_n = c$ with $a > c_0 \geq c$ and we obtain:

$$p_i - c - bq_i = 0, \quad i = \overline{1,n}. \tag{2.5}$$

Therefore, from (2.1), (2.4) and (2.5) we determine the reaction functions:

$$\begin{cases} q_0 = \dfrac{a_0}{b} - \delta \sum\limits_{i=1}^{n} q_i, \\ q_i = \dfrac{a_1}{2b} - \dfrac{\delta}{2} \sum\limits_{j=0, j\neq i}^{n} q_j, \quad i = \overline{1,n}, \end{cases} \tag{2.6}$$

where $a_0 = a - c_0 > 0$ and $a_1 = a - c > 0$.

The public firm has bounded rationality and it makes the output's decision based on the expected marginal payoff $\dfrac{\partial SW}{\partial q_0}$. Thus, the dynamical equation of the quantity q_0 is:

$$q_0(t+1) = q_0(t) + \alpha q_0(t) \left[a_0 - bq_0(t) - b\delta \sum_{i=1}^{n} q_i(t) \right],$$

where α is the positive adjustment parameter.

The private firm i, $i = \overline{1,n}$ is naive and its output is established using the reaction function given by (2.6). The dynamical equation of the quantity q_i is:

$$q_j(t+1) = \frac{a_1}{2b} - \frac{\delta}{2} \sum_{i=0, i\neq j}^{n} q_i(t), \; j = \overline{1,n}.$$

We consider competitors' time delays in the terms of the bounded rationality, reflected on the ideas presented in [9]. The production of the public firm is adjusted based on the past production levels of the private firms (at time $t - \tau_1$, $\tau_1 > 0$). However, the aim of this study is to focus on the case when the productions of private firms are updated with respect to the past production (at time $t - \tau_0$, $\tau_0 > 0$) of the public firm. Therefore, in this paper, we investigate the following nonlinear discrete-time mathematical model with time delays:

$$\begin{cases} q_0(t+1) = q_0(t) + \alpha q_0(t) \left[a_0 - bq_0(t) - b\delta \sum\limits_{i=1}^{n} q_i(t - \tau_1) \right] \\ q_j(t+1) = \dfrac{a_1}{2b} - \dfrac{\delta}{2} q_0(t - \tau_0) - \dfrac{\delta}{2} \sum\limits_{i=1, i\neq j}^{n} q_i(t) \quad , \quad j = \overline{1,n}. \end{cases} \tag{2.7}$$

The discrete dynamical system (2.7) has two equilibrium points (nonnegative fixed points):

$$E_0 = (0, q^\star, q^\star, ..., q^\star) \qquad \text{and} \qquad E_+ = (q_0^\star, q_1^\star, q_1^\star ..., q_1^\star),$$

where

$$q^\star = \frac{a_1}{b[2+(n-1)\delta]},$$
$$q_0^\star = \frac{[2+(n-1)\delta]a_0 - n\delta a_1}{b[2+(n-1)\delta - n\delta^2]},$$
$$q_1^\star = \frac{a_1 - \delta a_0}{b[2+(n-1)\delta - n\delta^2]}.$$

As $\delta \in (0,1)$, it follows that E_+ is a positive equilibrium point if and only if the following assumptions are satisfied:

$$\text{(A.1)} \quad [2+(n-1)\delta]a_0 > n\delta a_1 , \qquad \text{(A.2)} \quad a_1 > \delta a_0 .$$

3. Local stability analysis

The liniarized system at one of the equilibrium points $E = (q_0^e, q_1^e, q_1^e, ..., q_1^e) \in \{E_0, E_+\}$ is of the form:

$$y(t+1) = A^e y(t) - By(t-\tau_0) - C^e y(t-\tau_1) \tag{3.8}$$

where

$$y(t) = \begin{bmatrix} q_0(t) - q_0^e & q_1(t) - q_1^e & \dots & q_n(t) - q_1^e \end{bmatrix}^T$$

and the matrices A, B, C are given below:

$$A^e = \begin{bmatrix} 1+\alpha(a_0 - 2bq_0^e - nb\delta q_1^e) & 0 & 0 & \dots & 0 \\ 0 & 0 & -\delta/2 & \dots & -\delta/2 \\ 0 & -\delta/2 & 0 & \dots & -\delta/2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -\delta/2 & -\delta/2 & \dots & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 & \dots & 0 \\ \delta/2 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \delta/2 & 0 & \dots & 0 \end{bmatrix} \quad \text{and} \quad C^e = \begin{bmatrix} 0 & b\alpha\delta q_0^e & \dots & b\alpha\delta q_0^e \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}$$

The characteristic equation of system (3.8) can be obtained using the $\mathcal{Z}$-transform method, and is given as follows:

$$\det\left(A^e - B\lambda^{-\tau_0} - C^e\lambda^{-\tau_1} - \lambda I\right) = 0,$$

or equivalently:

$$\left(\lambda - \frac{\delta}{2}\right)^{n-1}\left[nb\alpha q_0^e \frac{\delta^2}{2}\lambda^{-\tau_0-\tau_1} - (\lambda - 1 - \alpha(a_0 - 2bq_0^e - nb\delta q_1^e))\left(\lambda + (n-1)\frac{\delta}{2}\right)\right] = 0. \tag{3.9}$$

In what follows, we analyze each of the equilibrium points E_0 and E_+.

3.1. The boundary equilibrium E_0.

Theorem 3.1. *If assumption* (A.1) *holds, the boundary equilibrium point E_0 is a saddle point.*

Proof. At the boundary equilibrium point E_0, as $q_0^e = 0$ and $q_1^e = q^\star$, the characteristic equation (3.9) reduces to the polynomial equation:

$$\left(\lambda - \frac{\delta}{2}\right)^{n-1}\left(\lambda - 1 - \alpha\frac{(2+(n-1)\delta)a_0 - n\delta a_1}{2+(n-1)\delta}\right)\left(\lambda + (n-1)\frac{\delta}{2}\right) = 0. \tag{3.10}$$

Hence, the roots of the characteristic polynomial are: $\lambda_0 = \dfrac{\delta}{2}$ of multiplicity $(n-1)$, $\lambda_1 = 1 + \dfrac{(2+(n-1)\delta)a_0 - n\delta a_1}{2+(n-1)\delta}$ and $\lambda_2 = -(n-1)\dfrac{\delta}{2}$.

Taking into account assumption $(A.1)$, it can be easily seen that $\lambda_1 > 1$. On the other hand, $\lambda_0 \in (0,1)$, and hence, the equilibrium E_0 is a saddle point of system (2.7). □

3.2. **The positive equilibrium E_+.**

For the positive equilibrium, as $q_0^e = q_0^\star$ and $q_1^e = q_1^\star$, the characteristic equation (3.9) becomes

$$\left(\lambda - \frac{\delta}{2}\right)^{n-1}\left[n\beta\frac{\delta^2}{2}\lambda^{-\tau_0-\tau_1} - (\lambda - 1 + \beta)\left(\lambda + (n-1)\frac{\delta}{2}\right)\right] = 0, \tag{3.11}$$

where

$$\beta = \alpha\frac{[2+(n-1)\delta]a_0 - n\delta a_1}{2+(n-1)\delta - n\delta^2} > 0. \tag{3.12}$$

It follows that $\lambda_0 = \dfrac{\delta}{2} \in (0,1)$ is a root of the characteristic equation (3.11) and hence, the stability of the equilibrium point E_+ is determined by the roots of the following equation:

$$n\beta\frac{\delta^2}{2}\lambda^{-\tau_0-\tau_1} - (\lambda - 1 + \beta)\left(\lambda + (n-1)\frac{\delta}{2}\right) = 0. \tag{3.13}$$

Theorem 3.2. *If assumptions $(A.1)$ and $(A.2)$ hold, when $\tau_0 = \tau_1 = 0$, the equilibrium point E_+ is asymptotically stable if and only if the following inequalities are satisfied:*

$$\delta < \frac{2}{n-1} \qquad \textit{and} \qquad \beta < \frac{4-2(n-1)\delta}{2-(n-1)\delta + n\delta^2}. \tag{3.14}$$

Proof. If $\tau_0 = \tau_1 = 0$ the equation (3.13) becomes

$$n\beta\frac{\delta^2}{2} - (\lambda - 1 + \beta)\left(\lambda + (n-1)\frac{\delta}{2}\right) = 0$$

which is equivalently written as:

$$\lambda^2 + \left((n-1)\frac{\delta}{2} + \beta - 1\right)\lambda - \frac{\delta}{2}\big(n\beta\delta + (n-1)(1-\beta)\big) = 0.$$

Hence, from the Schur-Cohn stability criterion, the roots of the above quadratic polynomial are in the open unit circle if and only if

$$\left|(n-1)\frac{\delta}{2} + \beta - 1\right| < 1 - \frac{\delta}{2}\big(n\beta\delta + (n-1)(1-\beta)\big) < 2,$$

which can be equivalently expressed as (3.14). □

Remark 3.1. For the case of two private firms ($n = 2$), which has been previously investigated in [20], the inequalities (3.14) reduce to:

$$\beta < \frac{4-2\delta}{2-\delta+2\delta^2} \qquad \text{where} \qquad \beta = \alpha\frac{(2+\delta)a_0 - 2\delta a_1}{2+\delta-2\delta^2}. \tag{3.15}$$

We point out that there is an error in the formulation of Theorem 2 (inequality (21)) of [20]. However, a careful analysis shows that the Schur-Cohn inequalities (20) from the same paper reduce to (3.15).

Theorem 3.3. *If assumptions $(A.1)$ and $(A.2)$ hold and the inequalities* (3.14) *are satisfied, the equilibrium point E_+ is asymptotically stable for any time delays τ_0 and τ_1.*

Proof. From Theorem 3.2, we know that if assumptions $(A.1)$, $(A.2)$ and inequalities (3.14) hold, the equilibrium E_+ is asymptotically stable for $\tau_0 = \tau_1 = 0$. Assuming that asymptotic stability is lost for certain values of the time delays, it follows that there exist critical values (τ_0^*, τ_1^*) of the time delays, such that the equation (3.13) has roots λ on the unit circle: $|\lambda| = 1$. Hence, we have:

$$n\beta\frac{\delta^2}{2}\lambda^{-\tau_0-\tau_1} = (\lambda - 1 + \beta)\left(\lambda + (n-1)\frac{\delta}{2}\right),$$

and taking the absolute value of both sides of this equation, we obtain:

$$|\lambda - 1 + \beta|\left|\lambda + (n-1)\frac{\delta}{2}\right| = n\beta\frac{\delta^2}{2}.$$

As $|\lambda| = 1$, denoting $\mu = \Re(\lambda) \in [-1, 1]$, this leads to:

$$\left[2(\beta-1)\mu + (\beta-1)^2 + 1\right]\left[(n-1)\delta\mu + (n-1)^2\frac{\delta^2}{4} + 1\right] = n^2\beta^2\frac{\delta^4}{4}. \tag{3.16}$$

We denote by $P(\mu)$ the quadratic polynomial given by the left- hand side of the above equation. The roots of this polynomial are:

$$\mu_1 = -\frac{(\beta-1)^2+1}{2(\beta-1)} \qquad \text{and} \qquad \mu_2 = -\frac{(n-1)^2\delta^2+4}{4(n-1)\delta}$$

It can be easily seen that $\mu_2 < -1$, while $\mu_1 < -1$ for $\beta > 1$ and $\mu_1 > 1$ for $\beta < 1$.

Taking into account inequalities (3.14), we deduce that $\beta < 2$ and:

$$\begin{aligned}
P(-1) &= (2-\beta)^2\left(1-(n-1)\frac{\delta}{2}\right)^2 > \left(2 - \frac{4-2(n-1)\delta}{2-(n-1)\delta+n\delta^2}\right)^2\left(1-(n-1)\frac{\delta}{2}\right)^2 \\
&> n^2\delta^4\left(\frac{2-(n-1)\delta}{2-(n-1)\delta+n\delta^2}\right)^2 > n^2\beta^2\frac{\delta^4}{4}\ , \\
P(1) &= \beta^2\left(1+(n-1)\frac{\delta}{2}\right)^2 > n^2\beta^2\frac{\delta^4}{4}\ .
\end{aligned}$$

Case $\beta > 1$. As both roots of $P(\mu)$ are smaller than -1, the polynomial $P(\mu)$ is increasing on $[-1, 1]$. Hence, for any $\mu \in [-1, 1]$, we deduce:

$$P(\mu) \geq P(-1) > n^2\beta^2\frac{\delta^4}{4}.$$

Case $\beta < 1$. In this case, we have a concave parabola $P(\mu)$ with the roots on either side of the interval $[-1, 1]$. Hence, for any $\mu \in [-1, 1]$, we have:

$$P(\mu) \geq \min\{P(-1), P(1)\} > n^2\beta^2\frac{\delta^4}{4}.$$

Consequently, equality (3.16) cannot take place, and hence, the equilibrium point E_+ is asymptotically stable for any time delays τ_0 and τ_1. □

Remark 3.2. Based on Theorem 3.3, we deduce that the time delays may have a stabilizing effect on the positive equilibrium point E_+. Inequalities (3.14) provide the delay-independent stability regions of E_+, which have been exemplified in Figure 1, for $a_0 = 2$ and $a_1 = 2.5$. We notice that as the number n of private firms increases, while smaller values of δ are required for the stability of the positive equilibrium, slightly larger values of α are admissible.

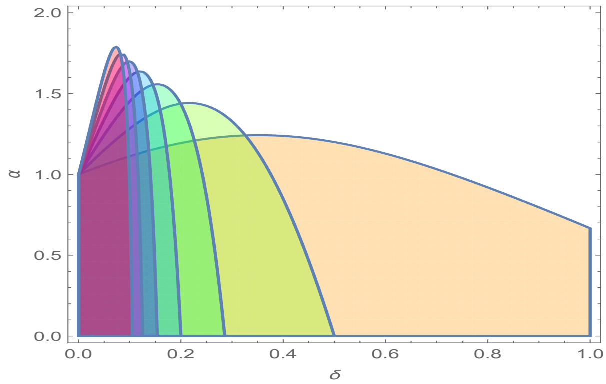

FIGURE 1. Stability regions (independent of time delays (τ_0, τ_1)) of the positive equilibrium point E_+ of system (2.7) in the (δ, α) parameter plane, for different value of $n \in \{2, 5, 8, 11, 14, 17, 20\}$ (colored orange to red). Here, $a_0 = 2$ and $a_1 = 2.5$.

Remark 3.3. If $\tau_0 + \tau_1$ is **even**, a flip bifurcation takes place in a neighborhood of the equilibrium point E_+ exactly at the boundary of the stability region given by inequalities (3.14), i.e. when

$$\beta = \frac{4 - 2(n-1)\delta}{2 - (n-1)\delta + n\delta^2} .$$

This easily follows from the fact that the reduced characteristic equation (3.13) has a root $\lambda = -1$ if and only if the above equality holds.

Remark 3.4. On the other hand, if $\tau_0 + \tau_1$ is **odd**, a flip bifurcation takes place in a neighborhood of the equilibrium point E_+ if and only if

$$\beta = \frac{4 - 2(n-1)\delta}{2 - (n-1)\delta - n\delta^2} .$$

In this case, we can notice that the stability region for the delays (τ_0, τ_1) corresponding to the equilibrium point E_+ is larger than the delay-independent stability region provided by inequalities (3.14).

4. NUMERICAL EXAMPLES

To illustrate our theoretical findings, we consider the case of $n = 4$ private firms and one public firm, fixing the following parameters: $a_0 = 2$, $a_1 = 2.5$, $\beta = 1$ and $\delta = 0.4$. For these values of the parameters, the positive equilibrium point is:

$$E_+ = (0.9375, 0.664, 0.664, 0.664, 0.664).$$

Based on inequalities (3.14), we deduce that the positive equilibrium E_+ is asymptotically stable, for any choice of the time delays τ_0 and τ_1 if $\alpha < \alpha^\star = 1.18519$. From Remark 3.3, it can be seen that when $\tau_0 + \tau_1$ is even, a flip bifurcation takes place in a neighborhood of the positive equilibrium at the critical value $\alpha^\star$ of the parameter α. This is in line with the bifurcation diagram with respect to the parameter α for the special case $\tau_0 = \tau_1 = 1$, represented in Figure 2. This flip bifurcation is followed by a period doubling bifurcation at $\alpha \simeq 1.39$ and a Neimark-Sacker bifurcation of the period-4 point at $\alpha \simeq 1.52$.

On the other hand, from Remark 3.4, it follows that when $\tau_0 + \tau_1$ is odd, the stability of the positive equilibrium E_+ is not lost at $\alpha = \alpha^\star$. Indeed, the bifurcation diagram for the

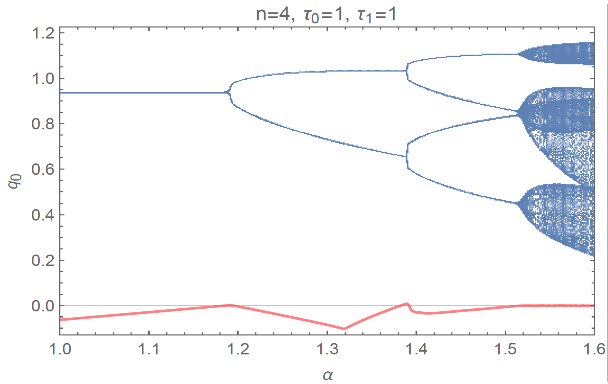

FIGURE 2. Bifurcation diagram and largest Lyapunov exponent (shown in red) for system (2.7) with $n = 4$ private firms and one public firm, with respect to α. Fixed parameter values: $a_0 = 2$, $a_1 = 2.5$, $\beta = 1$ and $\delta = 0.4$. Time delays: $\tau_0 = \tau_1 = 1$.

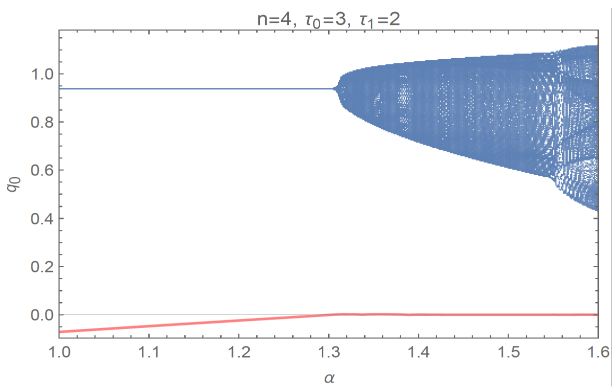

FIGURE 3. Bifurcation diagram and largest Lyapunov exponent (shown in red) for system (2.7) with $n = 4$ private firms and one public firm, with respect to α. Fixed parameter values: $a_0 = 2$, $a_1 = 2.5$, $\beta = 1$ and $\delta = 0.4$. Time delays: $\tau_0 = 3$, $\tau_1 = 2$.

case $\tau_0 = 3$, $\tau_1 = 2$ shown in Figure 3. Instead, a Neimark-Sacker bifurcation takes place at $\alpha \simeq 1.31$, and a stable limit cycle is formed (the largest Lyapunov exponent is null).

For larger values of the time delays, chaotic behavior can also be observed. For instance, for $\tau_0 = \tau_1 = 7$, the bifurcation diagram and the largest Lyapunov exponent are plotted in Figure 4. For $\alpha > 1.58$ the largest Lyapunov exponent is positive. Different dynamical regimes are presented in Figure 5, ranging from periodic orbits to chaotic attractors.

5. CONCLUSIONS

The present paper has investigated the dynamics of an oligopoly game with product differentiation, where a state-owned public firm and n private firms coexist. For the corresponding discrete-time mathematical model with time delays, two equilibrium points have been determined and the local stability has been analysed. The boundary equilibrium point E_0 is a saddle point. If there is no delay, under some conditions, the positive

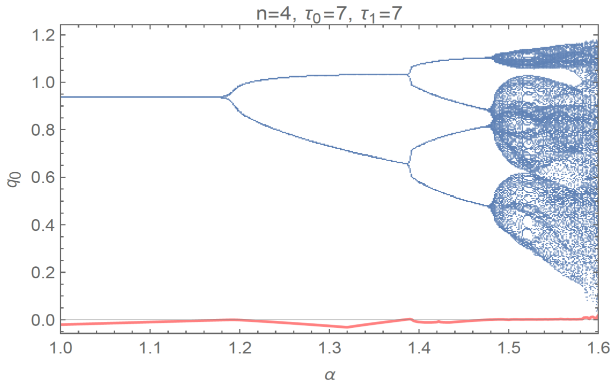

FIGURE 4. Bifurcation diagram and largest Lyapunov exponent (shown in red) for system (2.7) with $n = 4$ private firms and one public firm, with respect to α. Fixed parameter values: $a_0 = 2$, $a_1 = 2.5$, $\beta = 1$ and $\delta = 0.4$. Time delays: $\tau_0 = \tau_1 = 7$.

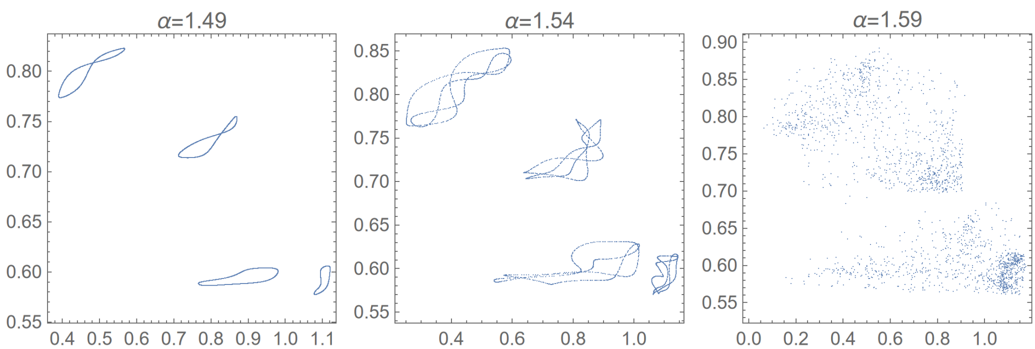

FIGURE 5. Phase portraits for system (2.7) with $n = 4$ private firms and one public firm, for various values of α. Fixed parameter values: $a_0 = 2$, $a_1 = 2.5$, $\beta = 1$ and $\delta = 0.4$. Time delays: $\tau_0 = \tau_1 = 7$.

equilibrium E_+ is asymptotically stable. Moreover, we have found the conditions so that E_+ is asymptotically stable, regardless of the time delays. We have shown that the time delays may have a stabilizing effect on the positive equilibrium point E_+. We have observed that if the number of private firms increases, smaller values of the degree of product differentiation are required for the stability of the positive equilibrium, correlated with slightly larger values of the adjustment parameter. Numerical simulations reveal complex dynamic behavior and the presence of chaotic regimes for sufficiently large values of the time delays.

Our findings generalize some results of the existing literature [20], and can be developed in the following ways: obtaining a complete picture of the Neimark-Sacker bifurcations occurring in a neighborhood of E_+; understanding the possible routes towards chaotic behavior in terms of the number of private firms and the time delays; investigating a similar mathematical model in which the n private firms do not have all-to-all connection.

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