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Dedicated to Prof. Emeritus Mihail Megan on the occasion of his 75^{th} anniversary

Generalized exponential behavior on the half-line via evolution semigroups

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ABSTRACT. We modify the classical concept of an evolution semigroup associated to an evolution family on the half-line to fit to the general case when linear flows may not agree with the restricted hypothesis of uniform exponential growth. We study the connections between spectral properties of the corresponding generator and a wide class of behavior of the evolution family. As a consequence, we prove that the generalized exponential dichotomy of possible non-invertible evolution families persists under sufficiently small linear perturbations.

1. INTRODUCTION

The non-autonomous linear differential equation

$$\dot{x} = A(t)x,\ t \geq 0,$$

in a Banach space X, is *well-posed* if the existence, uniqueness and continuous dependence on initial data of its solutions are assumed [23]. Under such hypotheses, one can define an evolution family $\{U(t,s)\}_{t\geq s\geq 0}$ on X such that

$$x(t) = U(t,s)x(s),\ \text{for all } t \geq s \geq 0.$$

Thus, the study of non-autonomous linear differential equations extended to the study of evolution families. Recall that an *evolution family* on X is a collection $\mathcal{U} = \{U(t,s)\}_{t\geq s\geq 0}$ of bounded linear operators acting on X such that the following properties hold:

- $U(t,t) = \mathrm{Id}$, $t \geq 0$;
- $U(t,\tau)U(\tau,t_0) = U(t,t_0)$, $t \geq \tau \geq t_0 \geq 0$;
- for each $x \in X$, the mapping $(t,s) \mapsto U(t,s)x$ is continuous on

$$\Delta = \left\{(t,s) \in \mathbb{R}^2 :\ t \geq s \geq 0\right\}.$$

In the asymptotic theory of non-autonomous linear differential equations, a central problem is to find conditions for detecting stability, instability, dichotomy or trichotomy of corresponding evolution families. A significant approach is represented by the so-called input-output technique or admissibility method. More precisely, the study of asymptotic behavior of an evolution family $\mathcal{U}$ reduces to the analysis of the solvability of the integral equation

$$u(t) = U(t,s)u(s) + \int_s^t U(t,\xi)f(\xi)\,d\xi,\ \text{for } t \geq s \geq 0, \tag{1.1}$$

in a wide class of pairs of function spaces, called admissible pairs for $\mathcal{U}$. The contribution of Megan and his collaborators (A.L. Sasu and B. Sasu) to this topic is remarkable and

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provided new classes of methods in approaching the asymptotic behavior of evolution families by means of various admissibility conditions, both in uniform [20, 21, 28] and nonuniform setting [19, 27].

A notable step in this framework was made by Van Minh, Räbiger, and Schnaubelt in [29]. Their approach is based on a connection between the integral equation (1.1) and the generator of the so-called evolution semigroup associated to $\mathcal{U}$. It is known that if $\mathcal{U} = \{U(t,s)\}_{t\geq s\geq 0}$ is an *exponentially bounded evolution family* on X, then one can define a C_0-semigroup $\mathcal{S} = \{S_t\}_{t\geq 0}$ on certain super-space $E(X)$ of functions $u : \mathbb{R}_+ \to X$, by setting

$$S_t u(s) = \begin{cases} U(s, s-t)u(s-t), & s \geq t, \\ U(s,0)u(0), & 0 \leq s < t, \end{cases}$$

called the *evolution semigroup* associated to $\mathcal{U}$. For evolution families on the real line, Räbiger and Schnaubelt extended this concept to a large class of X-valued function spaces, including in particular $C_0(\mathbb{R}, X)$ and $L^p(\mathbb{R}, X)$ [25].

Let us remark that even if the underlying Banach space X is finite-dimensional, the space $E(X)$ where the C_0-semigroup $\mathcal{S}$ acts has infinite dimension. It follows that the arguments and techniques needed for such endeavour are specific to functional analysis. On the other hand, one may notice that this type of analysis reduces in fact the study of a non-autonomous differential equation to the study of an autonomous one, such as $dy/dt = Gy$, where G is the (infinitesimal) generator of $\mathcal{S}$. More precisely, some spectral properties of the generator G has been proved to characterize the exponential behavior of the evolution family $\mathcal{U}$ (see [6, Section 3.3] or [7, 26, 29]). Similar techniques were developed by Latushkin, Randolph, and Schnaubelt for evolution families on the real line [14]. More recently, the concept of an evolution semigroup was extended to nonuniform behavior [1, 15, 16].

In our paper we investigate a more general type of exponential behavior, replacing the standard growth or decay rates $t \mapsto \nu t$ by some well-chosen mappings $t \mapsto \Omega(t)$, $t \geq 0$. These types of asymptotic behavior can occur naturally when all Lyapunov exponents are infinite or they are all zero (see [2]). In particular, if $\Omega(t) = \ln(t+1)$, then one steps over the so-called polynomial behavior, introduced independently by Barreira and Valls [3] and Bento and Silva [5] in the context of nonuniform behavior.

The tool we make use of can be considered as an intersection between the theory of evolution semigroups and input-output methods. Let us point out that we replaced the restricted hypothesis of uniform exponential growth imposed to evolution families with the general condition (2.2) below, which is satisfied by any evolution family generated by a differential equation.

For our purpose, we introduce and analyze a new concept of C_0-semigroup, called the *generalized evolution semigroup*, that generalizes the usual evolution semigroup. We prove that this type of evolution semigroup is similar to a classical one and thus all the known basic properties, such as the spectral mapping theorem, still hold. In Section 3 we study a general exponential behavior on the half-line, using the generalized evolution semigroup previously constructed. The last section is dedicated to the analysis of the persistence of the generalized exponential dichotomy with respect to small linear perturbations. The main result of this section is given by Theorem 4.2. We point out that its proof only uses generalized evolution semigroups (and not a direct method as in [4] or [24] for example).

We believe that the present constructions and techniques may apply as well in the case of a generalized nonuniform behavior. We refer the reader to papers [1, 16] for a better understanding. Hopefully our ideas might inspire potential researches in investigating other more general concepts of exponential behavior.

2. Generalized evolution semigroups

Let $X=(X,\|\cdot\|)$ be a Banach space and let $\mathcal{B}(X)$ be the Banach algebra of all bounded linear operators acting on X. We also consider the following Banach function spaces:

$$C_0(\mathbb{R}_+,X)=\left\{u:\mathbb{R}_+\to X:\ u \text{ is continuous and } \lim_{t\to\infty}u(t)=0\right\},$$
$$C_{00}(\mathbb{R}_+,X)=\{u\in C_0(\mathbb{R}_+,X):\ u(0)=0\},$$

endowed with the sup-norm $\|u\|_\infty=\sup_{t\geq 0}\|u(t)\|$, where $\mathbb{R}_+=[0,\infty)$.

Throughout this paper, *we always assume that* $\omega:\mathbb{R}_+\to\mathbb{R}_+$ *is a continuous function with* $\omega(t)>0$ *for every* $t>0$ *such that* $\int_0^t\omega(\tau)d\tau\to\infty$ *as* $t\to\infty$.

Definition 2.1. We say that an evolution family $\mathcal{U}=\{U(t,s)\}_{t\geq s\geq 0}$ is *ω-exponentially bounded* if there exist constants $\alpha\in\mathbb{R}$ and $K\geq 1$ such that

$$\|U(t,s)\|\leq Ke^{\alpha\int_s^t\omega(\tau)d\tau},\ \text{for all } t\geq s\geq 0. \tag{2.2}$$

The above condition is in fact the hypothesis of ω-bounded growth assumed in [22]. In particular, if ω is bounded, we step over the classical concept of an *exponentially bounded evolution family*.

Notice that any evolution family generated by a differential equation $\dot{x}=A(t)x$, where $\mathbb{R}_+\ni t\mapsto A(t)\in\mathcal{B}(X)$ is continuous in uniform operator topology, is ω-exponentially bounded with $\omega(t)=\|A(t)\|$, provided that $\int_0^t\|A(\tau)\|d\tau\to\infty$ as $t\to\infty$ (see, for instance, [8, p. 101]).

On the other hand, if an evolution family $\mathcal{U}=\{U(t,s)\}_{t\geq s\geq 0}$ (not necessarily generated by a differential equation) is exponentially bounded with constants $\alpha>0$ and $K\geq 1$, then it is also ω-exponentially bounded *for any non-decreasing function* ω satisfying our above assumption, precisely

$$\|U(t,s)\|\leq Ke^{\alpha t_0}e^{\frac{\alpha}{\omega(t_0)}\int_s^t\omega(\tau)d\tau},\ \text{for any fixed } t_0>0 \text{ and for all } t\geq s\geq 0.$$

This is a simple consequence of the following inequality:

$$\int_s^t\omega(\tau)d\tau\geq\omega(t_0)(t-s)-\omega(t_0)t_0,\ \text{for any fixed } t_0>0 \text{ and for all } t\geq s\geq 0. \tag{2.3}$$

Indeed, pick $t_0>0$ and let $t\geq s\geq 0$. If $t_0\leq s$, then $\omega(\tau)\geq\omega(t_0)$ for every $\tau\geq s$, thus

$$\int_s^t\omega(\tau)d\tau-\omega(t_0)(t-s)=\int_s^t\omega(\tau)-\omega(t_0)d\tau\geq 0\geq-\omega(t_0)t_0.$$

For $t_0\in(s,t)$ we get

$$\int_s^t\omega(\tau)d\tau-\omega(t_0)(t-s)\geq\int_s^{t_0}\omega(\tau)-\omega(t_0)d\tau\geq-\omega(t_0)(t_0-s)\geq-\omega(t_0)t_0.$$

Finally, if $t_0\geq t$, then $\int_s^t\omega(\tau)d\tau-\omega(t_0)(t-s)\geq-\omega(t_0)(t-s)\geq-\omega(t_0)t_0$. Hence, (2.3) holds.

In the next result, we define a C_0-semigroup associated to an ω-exponentially bounded evolution family, which may not necessarily be exponentially bounded. For instance, the evolution family $U(t,s)=e^{t^2-s^2}\mathrm{Id}$, $t\geq s\geq 0$, is ω-exponentially bounded with $\omega(t)=2t$ (or, more generally, $\omega(t)=at$ for any fixed $a\in\mathbb{R}$, $a\neq 0$), but it is not exponentially bounded.

Set

$$\Omega(t)=\int_0^t\omega(\tau)d\tau,\ \text{for } t\geq 0.$$

One may easily observe that Ω is a continuously differentiable, strictly increasing and invertible function with $\Omega(0) = 0$ and $\lim_{t\to\infty} \Omega(t) = \infty$. Furthermore, (2.2) is equivalent to

$$\|U(t,s)\| \le Ke^{\alpha(\Omega(t)-\Omega(s))}, \text{ for all } t \ge s \ge 0. \tag{2.4}$$

Theorem 2.1. *Assume that $\mathcal{U} = \{U(t,s)\}_{t\ge s\ge 0}$ is an evolution family on X such that* (2.2) *holds for some $\alpha \in \mathbb{R}$ and $K \ge 1$, that is $\mathcal{U}$ is ω-exponentially bounded. The family $\{T_t\}_{t\ge 0}$, given by*

$$T_t u(s) = \begin{cases} U(s, \Omega^{-1}(\Omega(s)-t))u(\Omega^{-1}(\Omega(s)-t)), & s \ge \Omega^{-1}(t), \\ U(s,0)u(0), & 0 \le s < \Omega^{-1}(t), \end{cases} \tag{2.5}$$

is a C_0-semigroup on $E(X)$, where $E(X)$ is one of the spaces $C_0(\mathbb{R}_+, X)$ or $C_{00}(\mathbb{R}_+, X)$. In fact, $\{T_t\}_{t\ge 0}$ is similar to a classical evolution semigroup on $E(X)$. Furthermore, we have

$$\|T_t\| \le Ke^{\alpha t}, \textit{ for every } t \ge 0. \tag{2.6}$$

Proof. Set

$$V(t,s) = U(\Omega^{-1}(t), \Omega^{-1}(s)), \text{ for } t \ge s \ge 0. \tag{2.7}$$

One can easily check that $\mathcal{V} = \{V(t,s)\}_{t\ge s\ge 0}$ is an evolution family on X satisfying

$$\|V(t,s)\| \le Ke^{\alpha(t-s)}, \text{ for all } t \ge s \ge 0,$$

and hence $\mathcal{V}$ is exponentially bounded. Let us define the evolution semigroup $\{S_t\}_{t\ge 0}$ on $E(X)$ associated to the evolution family $\mathcal{V}$, that is

$$S_t v(s) = \begin{cases} V(s, s-t)v(s-t), & s \ge t, \\ V(s,0)v(0), & 0 \le s < t, \end{cases}$$

for $t \ge 0$ and $v \in E(X)$. Then, we have

$$S_t v(s) = \begin{cases} U(\Omega^{-1}(s), \Omega^{-1}(s-t))v(s-t), & s \ge t, \\ U(\Omega^{-1}(s), 0)v(0), & 0 \le s < t, \end{cases}$$

and thus

$$S_t v(\Omega(s)) = \begin{cases} U(s, \Omega^{-1}(\Omega(s)-t))v(\Omega(s)-t), & \Omega(s) \ge t, \\ U(s,0)v(0), & \Omega(s) \in [0,t). \end{cases}$$

Letting $v = u \circ \Omega^{-1} \in E(X)$ for $u \in E(X)$, we get $(S_t(u\circ\Omega^{-1}))(\Omega(s)) = T_t u(s)$. This yields

$$(S_t(u \circ \Omega^{-1})) \circ \Omega = T_t u, \text{ for all } t \ge 0 \text{ and } u \in E(X). \tag{2.8}$$

Notice that the mapping

$$\mathcal{F} : E(X) \to E(X), \ \mathcal{F}u = u \circ \Omega^{-1}$$

is a linear isomorphism of the Banach space $E(X)$, with inverse

$$\mathcal{F}^{-1}v = v \circ \Omega.$$

From (2.8) we have

$$T_t u = (S_t(\mathcal{F}u)) \circ \Omega = \mathcal{F}^{-1}(S_t(\mathcal{F}u)), \text{ for all } t \ge 0 \text{ and } u \in E(X),$$

and thus

$$T_t = \mathcal{F}^{-1} S_t \mathcal{F}, \text{ for every } t \ge 0. \tag{2.9}$$

The above relation shows that $\{S_t\}_{t\ge 0}$ and $\{T_t\}_{t\ge 0}$ are similar semigroups (see [9, p. 59]) and in particular $\{T_t\}_{t\ge 0}$ is a C_0-semigroup on $E(X)$.

It remains to prove inequality (2.6). Let $u \in E(X)$ and $t \geq 0$. If $s \geq \Omega^{-1}(t)$, then (2.4) implies

$$\|T_t u(s)\| = \|U(s, \Omega^{-1}(\Omega(s)-t))u(\Omega^{-1}(\Omega(s)-t))\| \leq Ke^{\alpha t}\|u(\Omega^{-1}(\Omega(s)-t))\| \leq Ke^{\alpha t}\|u\|_\infty.$$

On the other hand, for $s \in [0, \Omega^{-1}(t))$ we have

$$\|T_t u(s)\| = \|U(s,0)u(0)\| \leq Ke^{\alpha\Omega(s)}\|u(0)\| \leq Ke^{\alpha t}\|u\|_\infty,$$

and thus (2.6) holds. □

Throughout this paper, we call the C_0-semigroup in (2.5) as the *generalized evolution semigroup* on $E(X)$ associated to the ω-exponentially bounded evolution family $\mathcal{U}$.

For each fixed $t_0 \geq 0$ we set

$$C_0(t_0) = \left\{u : [t_0, \infty) \to X : \ u \text{ is continuous and } \lim_{t\to\infty} u(t) = 0\right\}.$$

Lemma 2.1. *Let $\mathcal{U} = \{U(t,s)\}_{t\geq s\geq 0}$ be an evolution family on X and let $\mathcal{V} = \{V(t,s)\}_{t\geq s\geq 0}$ be the evolution family defined by (2.7). The following statements are equivalent:*

(i) *for every $t_0 \geq 0$ and for every $f \in C_0(t_0)$ there exists $u \in C_0(t_0)$ such that*

$$u(t) = U(t,s)u(s) + \int_s^t \omega(\xi)U(t,\xi)f(\xi)\,d\xi, \text{ for } t \geq s \geq t_0;$$

(ii) *for every $t_0 \geq 0$ and for every $f \in C_0(t_0)$ there exists $v \in C_0(t_0)$ such that*

$$v(t) = V(t,s)v(s) + \int_s^t V(t,\xi)f(\xi)\,d\xi, \text{ for } t \geq s \geq t_0.$$

Proof. We only prove $(i) \Rightarrow (ii)$ as the converse implication can be proved similarly. Let $t_0 \geq 0$ and $f \in C_0(t_0)$. Then $f \circ \Omega \in C_0(\Omega^{-1}(t_0))$ and by (i) there exists $u \in C_0(\Omega^{-1}(t_0))$ such that

$$u(t) = U(t,s)u(s) + \int_s^t \omega(\xi)U(t,\xi)f(\Omega(\xi))\,d\xi, \text{ for all } t \geq s \geq \Omega^{-1}(t_0).$$

For $t \geq s \geq t_0$ one has

$$\begin{aligned} u(\Omega^{-1}(t)) &= U(\Omega^{-1}(t), \Omega^{-1}(s))u(\Omega^{-1}(s)) + \int_{\Omega^{-1}(s)}^{\Omega^{-1}(t)} \Omega'(\xi)U(\Omega^{-1}(t),\xi)f(\Omega(\xi))\,d\xi \\ &= V(t,s)u(\Omega^{-1}(s)) + \int_s^t V(t,\tau)f(\tau)\,d\tau, \end{aligned}$$

therefore (ii) holds for $v = u \circ \Omega^{-1} \in C_0(t_0)$. □

Remark 2.1. Relation (2.9) is of a significant importance because it gives the connection between the generator $(G, D(G))$ of the generalized evolution semigroup on $E(X)$ and the generator $(A, D(A))$ of the corresponding classical evolution semigroup associated to the evolution family $\mathcal{V}$ defined by (2.7). Indeed, we have

$$G = \mathcal{F}^{-1}A\,\mathcal{F}, \text{ with domain } D(G) = \left\{u \in E(X) : \ u \circ \Omega^{-1} \in D(A)\right\}, \tag{2.10}$$

consequently $\sigma(G) = \sigma(A)$.

Let us remark that the above similarity property permits to directly deduce the spectral mapping theorem for generalized evolution semigroups on $C_{00}(\mathbb{R}_+, X)$ from [7, Theorem 2.1] or [29, Corollary 2.4] (see also [26] for some important spectral symmetry properties).

We recall that $s(B) = \sup\{Re\lambda : \ \lambda \in \sigma(B)\}$ is the *spectral bound* of a linear operator $B : D(B) \subseteq X \to X$ and $r(B) = \sup\{|\lambda| : \ \lambda \in \sigma(B)\}$ is the *spectral radius* of B.

Corollary 2.1. *Let $\{T_t\}_{t\geq 0}$ be the generalized evolution semigroup on $C_{00}(\mathbb{R}_+, X)$ associated to an ω-exponentially bounded evolution family $\mathcal{U}$ and let G_{00} denote its generator. The spectra of the operators T_t and G_{00} have some important symmetry properties: $\sigma(T_t)$ is rotationally invariant for each $t > 0$, that is*

$$\lambda\sigma(T_t) = \sigma(T_t), \text{ for every } \lambda \in \mathbb{C} \text{ with } |\lambda| = 1,$$

and $\sigma(G_{00})$ is invariant under translations along the imaginary axis, i.e. $\sigma(G_{00}) = \sigma(G_{00}) + i\mathbb{R}$. Moreover, $\sigma(T_t) = \{\lambda \in \mathbb{C} : |\lambda| \leq r(T_t)\}$ is a disc centered at the origin for each $t > 0$, $\sigma(G_{00}) = \{\lambda \in \mathbb{C} : Re\lambda \leq s(G_{00})\}$ is a left half-plane, and the spectral mapping theorem holds for the generalized evolution semigroup, that is

$$\sigma(T_t) \setminus \{0\} = e^{t\sigma(G_{00})}, \text{ for every } t > 0. \tag{2.11}$$

Remark 2.2. An important consequence of identity (2.11) is that the growth bound

$$\omega(\mathcal{T}) = \inf\{\alpha \in \mathbb{R} : \text{ there exists } K \geq 1 \text{ such that (2.6) holds}\}$$

of the generalized evolution semigroup $\mathcal{T} = \{T_t\}_{t\geq 0}$ on $C_{00}(\mathbb{R}_+, X)$ coincides with the spectral bound $s(G_{00})$ of the corresponding generator, i.e. $\omega(\mathcal{T}) = s(G_{00})$.

Let G_0 and G_{00} be the generators of the generalized evolution semigroups $\{T_t\}_{t\geq 0}$ on $C_0(\mathbb{R}_+, X)$ and $C_{00}(\mathbb{R}_+, X)$, respectively. Notice that $D(G_{00}) = D(G_0) \cap C_{00}(\mathbb{R}_+, X)$ and $G_{00}u = G_0u$ for $u \in D(G_{00})$. Furthermore, if $u \in D(G_0)$, then $G_0u(0) = 0$, therefore $\text{Range}(G_0) \subseteq C_{00}(\mathbb{R}_+, X)$.

Lemma 2.2. (a) *Let $u \in C_0(\mathbb{R}_+, X)$ and $f \in C_{00}(\mathbb{R}_+, X)$. Then $u \in D(G_0)$ and $G_0u = -f$ if and only if*

$$u(t) = U(t, s)u(s) + \int_s^t \omega(\xi)U(t, \xi)f(\xi)\, d\xi, \text{ for } t \geq s \geq 0. \tag{2.12}$$

(b) *Let $u, f \in C_{00}(\mathbb{R}_+, X)$. Then $u \in D(G_{00})$ and $G_{00}u = -f$ if and only if*

$$u(t) = \int_0^t \omega(\xi)U(t, \xi)f(\xi)\, d\xi, \text{ for } t \geq 0.$$

Proof. We only prove (a). Let $u \in D(G_0)$ and $f \in C_{00}(\mathbb{R}_+, X)$ such that $G_0u = -f$. Using (2.10) and the expression of the operator $\mathcal{F}$ in the proof of Theorem 2.1, we have

$$v = u \circ \Omega^{-1} \in D(A_0) \text{ and } A_0v = -f \circ \Omega^{-1},$$

where $(A_0, D(A_0))$ stands for the generator $(A, D(A))$ whenever $E(X) = C_0(\mathbb{R}_+, X)$. Then, Lemma 1.1 in [29] implies

$$v(t) = V(t, s)v(s) + \int_s^t V(t, \tau)f(\Omega^{-1}(\tau))d\tau, \ t \geq s \geq 0,$$

which is equivalent to

$$u(\Omega^{-1}(t)) = U(\Omega^{-1}(t), \Omega^{-1}(s))u(\Omega^{-1}(s)) + \int_s^t U(\Omega^{-1}(t), \Omega^{-1}(\tau))f(\Omega^{-1}(\tau))d\tau, \ t \geq s \geq 0.$$

Let $t \geq s \geq 0$. Replacing in the above relation t and s by $\Omega(t)$ and $\Omega(s)$, respectively, we get

$$\begin{aligned} u(t) &= U(t, s)u(s) + \int_{\Omega(s)}^{\Omega(t)} U(t, \Omega^{-1}(\tau))f(\Omega^{-1}(\tau))d\tau \\ &= U(t, s)u(s) + \int_s^t \Omega'(\xi)U(t, \xi)f(\xi)d\xi, \end{aligned}$$

and thus formula (2.12) holds. The converse can be proved by reversing all the above arguments. $\square$

3. Generalized exponential behavior on the half-line

For non-autonomous differential equations, the classical theory of exponential stability may look as too restrictive, therefore it is important to search for more general behavior.

Definition 3.2. An evolution family $\mathcal{U} = \{U(t,s)\}_{t\geq s\geq 0}$ is called:

- *ω-exponentially stable* if there exist constants $\nu > 0$ and $N \geq 1$ such that

$$\|U(t,s)\| \leq Ne^{-\nu \int_s^t \omega(\tau)d\tau},\ t \geq s \geq 0; \tag{3.13}$$

- *ω-exponentially expansive* if the operators $U(t,s)$ are invertible for $t \geq s \geq 0$ and there exist constants $\nu > 0$ and $N \geq 1$ such that

$$\|U(s,t)\| \leq Ne^{-\nu \int_s^t \omega(\tau)d\tau},\ t \geq s \geq 0,$$

  where $U(s,t) = U(t,s)^{-1}$.

Furthermore, we say that $\mathcal{U}$ has an *ω-exponential dichotomy* if:

(a) there exist projections $P(t) : X \to X$, and write $Q(t) = \mathrm{Id} - P(t)$, for $t \geq 0$, such that

$$P(t)U(t,s) = U(t,s)P(s)$$

and the restriction $U_Q(t,s) : Range(Q(s)) \to Range(Q(t))$ of $U(t,s)$ is invertible for all $t \geq s \geq 0$;

(b) there exist constants $\nu > 0$ and $N \geq 1$ such that

$$\|U(t,s)P(s)\| \leq Ne^{-\nu \int_s^t \omega(\xi)\,d\xi} \text{ and } \|U_Q(s,t)Q(t)\| \leq Ne^{-\nu \int_s^t \omega(\xi)\,d\xi},$$

where $U_Q(s,t) = U_Q(t,s)^{-1}$, for $t \geq s \geq 0$.

To the best of our knowledge this type of behavior has been introduced by Martin in [17] (see also [12, 13, 22]).

The above concepts generalize the usual exponential behavior considering for instance $\omega(t) = 1$ for $t \geq 0$. On the other hand, setting

$$\omega(t) = \frac{1}{t+1},\ t \geq 0,$$

one may step over the polynomial behavior (see [10, 11, 18]). Furthermore, (2.3) shows that if in particular the function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ is non-decreasing, then any ω-exponentially stable, ω-exponentially expansive or ω-exponentially dichotomic evolution family, with constants $\nu > 0$ and $N \geq 1$, admits the corresponding uniform exponential behavior with new constants $Ne^{\nu\omega(t_0)t_0}$ and $\nu\omega(t_0)$ for any fixed $t_0 > 0$.

Remark 3.3. One may easily check that if $\mathcal{U}$ is an evolution family on X and $\mathcal{V}$ is the evolution family given by (2.7), then the following statements hold:

(i) $\mathcal{U}$ is ω-exponentially stable if and only if $\mathcal{V}$ is exponentially stable;
(ii) $\mathcal{U}$ is ω-exponentially expansive if and only if $\mathcal{V}$ is exponentially expansive;
(iii) $\mathcal{U}$ has an ω-exponential dichotomy with respect to projections $P(t)$ if and only if $\mathcal{V}$ has an exponential dichotomy with projections $\widetilde{P}(t) = P(\Omega^{-1}(t))$, $t \geq 0$.

Next claim offers a simple characterization of ω-exponential stability.

Proposition 3.1. *An evolution family $\mathcal{U} = \{U(t,s)\}_{t\geq s\geq 0}$ is ω-exponentially stable if and only if it is ω-exponentially bounded and the growth bound $\omega(\mathcal{T})$ of the induced generalized evolution semigroup $\mathcal{T} = \{T_t\}_{t\geq 0}$ on $C_{00}(\mathbb{R}_+, X)$ is negative, that is $\mathcal{T}$ is exponentially stable.*

Proof. Assume that $\mathcal{U}$ is ω-exponentially stable, that is there exist $\nu > 0$ and $N \geq 1$ such that (3.13) holds, which in particular means that $\mathcal{U}$ is ω-exponentially bounded. From (2.6) we have

$$\|T_t\| \leq Ne^{-\nu t}, \text{ for every } t \geq 0,$$

and this yields that $\omega(\mathcal{T}) < 0$. Conversely, assume that $\mathcal{U}$ is ω-exponentially bounded and (2.6) holds for some $\alpha < 0$ and $K \geq 1$. Let $x \in X$ with $\|x\| = 1$. For $t \geq s \geq 0$ choose $u \in C_{00}(\mathbb{R}_+, X)$ such that $\|u\|_\infty = 1$ and $u(s) = x$. Then

$$\|U(t,s)x\| = \|U(t,s)u(s)\| = \|T_{\Omega(t)-\Omega(s)}u(t)\| \leq \|T_{\Omega(t)-\Omega(s)}u\|_\infty \leq Ke^{\alpha(\Omega(t)-\Omega(s))},$$

which proves that $\mathcal{U}$ is ω-exponentially stable. □

An immediate consequence of the above result and Remark 2.2 is that *an ω-exponentially bounded evolution family is ω-exponentially stable if and only if the spectral bound $s(G_{00})$ is negative.*

In the following we assume $\mathcal{U} = \{U(t,s)\}_{t\geq s\geq 0}$ *to be an ω-exponentially bounded evolution family on X.* Let G_0 and G_{00} be the generators of the generalized evolution semigroups $\mathcal{T} = \{T_t\}_{t\geq 0}$ on $C_0(\mathbb{R}_+, X)$ and $C_{00}(\mathbb{R}_+, X)$, respectively. Using the results due to Van Minh, Räbiger, and Schnaubelt in [29], we completely characterize ω-exponential stability, ω-exponential expansiveness, and ω-exponential dichotomy of $\mathcal{U}$ in terms of some spectral or admissibility conditions.

Using Remark 3.3, [29, Theorem 2.2], Remark 2.1, and Lemma 2.2 (b), a characterization of ω-exponential stability is present in the next result.

Corollary 3.2. *The following statements are pairwise equivalent:*

(i) *$\mathcal{U}$ is ω-exponentially stable;*

(ii) *G_{00} is invertible;*

(iii) *For every $f \in C_{00}(\mathbb{R}_+, X)$ the function $t \mapsto (\mathcal{G}f)(t) = \int_0^t \omega(\xi)U(t,\xi)f(\xi)d\xi$ belongs to $C_{00}(\mathbb{R}_+, X)$.*

Notice that if the evolution family $\mathcal{U} = \{U(t,s)\}_{t\geq s\geq 0}$ is ω-exponentially stable, then $\mathcal{G}$ is a linear operator acting on $C_{00}(\mathbb{R}_+, X)$. Explicitly

$$(\mathcal{G}f)(t) = \int_0^{\Omega(t)} U(t, \Omega^{-1}(\Omega(t) - \tau))f(\Omega^{-1}(\Omega(t) - \tau))d\tau = \int_0^\infty T_\tau f(t)d\tau, \ t \geq 0,$$

and, using standard arguments from the theory of C_0-semigroups, we get that $\mathcal{G} = -G_{00}^{-1}$ (see, for instance, [9, Theorem II.1.10]).

On the other hand, using the uniform boundedness principle one may easily prove that condition (iii) of Corollary 3.2 is equivalent to the boundedness of $\mathcal{G}$ on $C_{00}(\mathbb{R}_+, X)$.

From Remark 3.3, Lemma 2.1, and [29, Theorem 2.5] we get a complete characterization of ω-exponential expansiveness.

Corollary 3.3. *$\mathcal{U}$ is ω-exponentially expansive if and only if for all $t_0 \geq 0$ and $f \in C_0(t_0)$ there exists a unique $u \in C_0(t_0)$ such that*

$$u(t) = U(t,s)u(s) + \int_s^t \omega(\xi)U(t,\xi)f(\xi)\,d\xi, \text{ for } t \geq s \geq t_0. \tag{3.14}$$

For fixed $t_0 \geq 0$ we define the mapping $I_{t_0} : D(I_{t_0}) \subseteq C_0(t_0) \to C_0(t_0)$ by $I_{t_0}u = f$ if u and f satisfy the integral equation (3.14). Then the above result shows in fact that *$\mathcal{U}$ is ω-exponentially expansive if and only if I_{t_0} is invertible for every $t_0 \geq 0$.*

Remark 2.1 implies $\mathrm{Range}(G_0) = \mathrm{Range}(\mathcal{F}^{-1}A_0)$, where A_0 is the generator of the classical evolution semigroup on $C_0(\mathbb{R}_+, X)$ associated to the evolution family $\mathcal{V}$ defined by (2.7). Using Remark 3.3, Lemma 2.1 for $t_0 = 0$, and [29, Theorem 4.3], we deduce the following characterization of ω-exponential dichotomy.

Corollary 3.4. *The following statements are pairwise equivalent:*

(i) $\mathcal{U}$ *has an ω-exponential dichotomy;*

(ii) $\mathrm{Range}(G_0) = C_{00}(\mathbb{R}_+, X)$ *and the stable subspace of initial data*

$$X_s(0) = \{x \in X : \sup_{t\geq 0} \|U(t,0)x\| < \infty\}$$

is complemented in X;

(iii) I_0 *is surjective and $X_s(0)$ is complemented in X.*

4. ROUGHNESS OF ω-EXPONENTIAL DICHOTOMY

The goal of this section is to establish a roughness result for ω-exponential dichotomies of possible non-invertible evolution families using Corollary 3.4, that is the persistence of ω-exponential dichotomy under sufficiently small linear perturbations.

To the best of our knowledge such technique is new, even for the classical concept of exponential dichotomy. For a completely different proof we refer the reader to [4].

Assume that $\mathcal{U} = \{U(t,s)\}_{t\geq s\geq 0}$ is an ω-exponentially bounded evolution family on X with constants $\alpha \in \mathbb{R}$ and $K \geq 1$, and let $\mathcal{E}$ be the space of all strongly continuous operator-valued function $B : \mathbb{R}_+ \to \mathcal{B}(X)$ such that

$$\|B\|_* = \sup_{t>0} \frac{\|B(t)\|}{\omega(t)} < \infty.$$

Evidently, $\mathcal{E}$ is a Banach space endowed with the norm $\|\cdot\|_*$. On the other hand, for each fixed $B \in \mathcal{E}$ the linear Volterra integral equation

$$U_B(t,s)x = U(t,s)x + \int_s^t U(t,\xi)B(\xi)U_B(\xi,s)xd\xi, \; x \in X, \tag{4.15}$$

has a unique solution, which defines an evolution family $\mathcal{U}_B = \{U_B(t,s)\}_{t\geq s\geq 0}$ on X (see [24] and references therein).

We prove the following preliminary result:

Lemma 4.3. *For each fixed $B \in \mathcal{E}$ the perturbed evolution family $\mathcal{U}_B$ remains ω-exponentially bounded, precisely*

$$\|U_B(t,s)\| \leq K e^{(\alpha + K\|B\|_*)\int_s^t \omega(\xi)d\xi}, \; t \geq s \geq 0. \tag{4.16}$$

Proof. Using (2.2) and (4.15), one gets

$$\|U_B(t,s)x\| \leq K e^{\alpha\int_s^t \omega(\xi)d\xi}\|x\| + K\|B\|_* \int_s^t \omega(\xi)e^{\alpha\int_\xi^t \omega(\tau)d\tau}\|U_B(\xi,s)x\|d\xi.$$

For any fixed $x \neq 0$ and $s \geq 0$, we set $x(t) = \frac{\|U_B(t,s)x\|}{\|x\|}$ for $t \geq s$. Then

$$x(t) \leq K e^{\alpha\int_s^t \omega(\xi)d\xi} + K\|B\|_* \int_s^t \omega(\xi)e^{\alpha\int_\xi^t \omega(\tau)d\tau}x(\xi)d\xi,$$

which when multiplied by $e^{-\alpha\int_s^t \omega(\xi)d\xi}$ becomes

$$x(t)e^{-\alpha\int_s^t \omega(\xi)d\xi} \leq K + K\|B\|_* \int_s^t \omega(\xi)e^{-\alpha\int_s^\xi \omega(\tau)d\tau}x(\xi)d\xi.$$

Putting $y(t) = x(t)e^{-\alpha\int_s^t \omega(\xi)d\xi}$ we obtain

$$y(t) \le K + K\|B\|_* \int_s^t \omega(\xi)y(\xi)d\xi.$$

Grönwall's inequality in [8, Corollary 2.1] implies

$$y(t) \le Ke^{K\|B\|_* \int_s^t \omega(\xi)d\xi},$$

therefore (4.16) is a direct consequence of the above inequality. $\square$

Theorem 4.2. *If the evolution family $\mathcal{U}$ has an ω-exponential dichotomy and the perturbation operator B in $\mathcal{E}$ is sufficiently small, that is $\|B\|_* \le \delta$ for sufficiently small $\delta > 0$, then the perturbed evolution family $\mathcal{U}_B$ still exhibits an ω-exponential dichotomy.*

Proof. Notice that for

$$\varphi(t) = N\int_0^t \omega(\xi)e^{-\nu\int_\xi^t \omega(\tau)d\tau}d\xi + N\int_t^\infty \omega(\xi)e^{-\nu\int_t^\xi \omega(\tau)d\tau}d\xi,\ t \ge 0,$$

one has

$$\varphi(t) \le \frac{2N}{\nu},\ \text{for every } t \ge 0,$$

where $\nu > 0$ and $N \ge 1$ are given by Definition 3.2. For the sake of convenience, we divide the proof into several steps.

Step 1. If

$$\frac{2N\delta}{\nu} < 1, \tag{4.17}$$

then for any fixed $f \in C_{00}(\mathbb{R}_+, X)$ there exists a unique function $u_f \in C_0(\mathbb{R}_+, X)$ with

$$u_f(t) = \int_0^\infty \Gamma(t,\xi)B(\xi)u_f(\xi)d\xi + \int_0^\infty \omega(\xi)\Gamma(t,\xi)f(\xi)d\xi,\ t \ge 0.$$

Here

$$\Gamma(t,s) = \begin{cases} U(t,s)P(s), & t > s \ge 0, \\ -U_Q(t,s)Q(s), & 0 \le t < s, \end{cases}$$

stands for the Green function corresponding to the ω-exponential dichotomous evolution family $\mathcal{U}$.

Proof. For any fixed $f \in C_{00}(\mathbb{R}_+, X)$, we define the operator $\mathcal{K} : C_0(\mathbb{R}_+, X) \to C_0(\mathbb{R}_+, X)$ by

$$(\mathcal{K}u)(t) = \int_0^\infty \Gamma(t,\xi)B(\xi)u(\xi)d\xi + \int_0^\infty \omega(\xi)\Gamma(t,\xi)f(\xi)d\xi,\ u \in C_0(\mathbb{R}_+, X),\ t \ge 0.$$

Clearly

$$\|(\mathcal{K}u)(t)\| \le \delta\varphi(t)\|u\|_\infty + \frac{2N}{\nu}\|f\|_\infty \le \frac{2N\delta}{\nu}\|u\|_\infty + \frac{2N}{\nu}\|f\|_\infty,$$

thus $\mathcal{K}$ is well-defined. If the estimation (4.17) is satisfied, from

$$\|\mathcal{K}u - \mathcal{K}v\|_\infty \le \frac{2N\delta}{\nu}\|u - v\|_\infty,\ \text{for all } u, v \in C_0(\mathbb{R}_+, X),$$

we conclude that $\mathcal{K}$ is a contraction, with a unique fixed point $u_f \in C_0(\mathbb{R}_+, X)$. $\square$

Step 2. $\mathrm{Range}(G_0^B) = C_{00}(\mathbb{R}_+, X)$.

Proof. Let $f \in C_{00}(\mathbb{R}_+, X)$. We prove that $G_0^B u = -f$ for $u \in D(G_0^B)$ satisfying the integral equation

$$u(t) = \int_0^\infty \Gamma(t,\xi)B(\xi)u(\xi)d\xi + \int_0^\infty \omega(\xi)\Gamma(t,\xi)f(\xi)d\xi$$

or, equivalently,

$$\begin{aligned}u(t) = &\int_0^t U(t,\xi)P(\xi)B(\xi)u(\xi)d\xi - \int_t^\infty U_Q(t,\xi)Q(\xi)B(\xi)u(\xi)d\xi \\ &+ \int_0^t \omega(\xi)U(t,\xi)P(\xi)f(\xi)d\xi - \int_t^\infty \omega(\xi)U_Q(t,\xi)Q(\xi)f(\xi)d\xi,\ t \geq 0.\end{aligned} \tag{4.18}$$

We first remark that under the assumption of the first step, the above integral equation has a unique solution $u \in C_0(\mathbb{R}_+, X)$. Since $\mathcal{U}_B$ is ω-exponentially bounded (see Lemma 4.3), by Lemma 2.2 (a) it suffices to prove that

$$u(t) = U_B(t,s)u(s) + \int_s^t \omega(\xi)U_B(t,\xi)f(\xi)d\xi,\ \text{for } t \geq s \geq 0.$$

Identities (4.15) and (4.18) yield

$$\begin{aligned}&U_B(t,s)u(s) + \int_s^t \omega(\xi)U_B(t,\xi)f(\xi)d\xi \\ &= U(t,s)u(s) + \int_s^t U(t,\xi)B(\xi)U_B(\xi,s)u(s)d\xi + \int_s^t \omega(\xi)U_B(t,\xi)f(\xi)d\xi \\ &= \int_0^s U(t,\xi)P(\xi)B(\xi)u(\xi)d\xi - \int_s^t U(t,\xi)Q(\xi)B(\xi)u(\xi)d\xi - \int_t^\infty U_Q(t,\xi)Q(\xi)B(\xi)u(\xi)d\xi \\ &+ \int_0^s \omega(\xi)U(t,\xi)P(\xi)f(\xi)d\xi - \int_s^t \omega(\xi)U(t,\xi)Q(\xi)f(\xi)d\xi - \int_t^\infty \omega(\xi)U_Q(t,\xi)Q(\xi)f(\xi)d\xi \\ &+ \int_s^t U(t,\xi)B(\xi)U_B(\xi,s)u(s)d\xi + \int_s^t \omega(\xi)U(t,\xi)f(\xi)d\xi \\ &+ \int_s^t \omega(\xi)\left(\int_\xi^t U(t,\eta)B(\eta)U_B(\eta,\xi)f(\xi)d\eta\right)d\xi.\end{aligned}$$

According to Fubini's theorem, one gets

$$\begin{aligned}&U_B(t,s)u(s) + \int_s^t \omega(\xi)U_B(t,\xi)f(\xi)d\xi \\ &\quad = u(t) - \int_s^t U(t,\xi)P(\xi)B(\xi)u(\xi)d\xi - \int_s^t U(t,\xi)Q(\xi)B(\xi)u(\xi)d\xi \\ &\quad - \int_s^t \omega(\xi)U(t,\xi)P(\xi)f(\xi)d\xi - \int_s^t \omega(\xi)U(t,\xi)Q(\xi)f(\xi)d\xi \\ &\quad + \int_s^t U(t,\xi)B(\xi)U_B(\xi,s)u(s)d\xi + \int_s^t \omega(\xi)U(t,\xi)f(\xi)d\xi \\ &\quad + \int_s^t U(t,\xi)B(\xi)\left(\int_s^\xi \omega(\eta)U_B(\xi,\eta)f(\eta)d\eta\right)d\xi \\ &\quad = u(t) + \int_s^t U(t,\xi)B(\xi)\left(U_B(\xi,s)u(s) + \int_s^\xi \omega(\eta)U_B(\xi,\eta)f(\eta)d\eta - u(\xi)\right)d\xi.\end{aligned}$$

If we set

$$x(t) = U_B(t,s)u(s) + \int_s^t \omega(\xi)U_B(t,\xi)f(\xi)d\xi - u(t), \text{ for } t \geq s \geq 0,$$

then the above arguments imply

$$x(t) = \int_s^t U(t,\xi)B(\xi)x(\xi)d\xi, \ t \geq s \geq 0.$$

The uniqueness of the solution of the above Volterra-type equation results in $x(t) = 0$ for all $t \geq s \geq 0$, which proves our claim. □

We show that $X_s^B(0)$ is complemented. For each $x \in X$ we consider the operator $\mathcal{J} : C_0(\mathbb{R}_+, X) \to C_0(\mathbb{R}_+, X)$, defined by

$$(\mathcal{J}u)(t) = U(t,0)P(0)x + \int_0^\infty \Gamma(t,\xi)B(\xi)u(\xi)d\xi. \tag{4.19}$$

As

$$\|\mathcal{J}u - \mathcal{J}v\|_\infty = \|\mathcal{K}u - \mathcal{K}v\|_\infty \leq \frac{2N\delta}{\nu}\|u - v\|_\infty, \ u, v \in C_0(\mathbb{R}_+, X),$$

whenever $\frac{2N\delta}{\nu} < 1$, $\mathcal{J}$ is a contraction, with a unique fixed point $u_x \in C_0(\mathbb{R}_+, X)$. Observe that the operator $x \mapsto u_x$ is linear and bounded. Put

$$\widetilde{P}x = u_x(0).$$

Identity (4.19) yields

$$\widetilde{P}x = P(0)x - \int_0^\infty U_Q(0,\xi)Q(\xi)B(\xi)u_x(\xi)d\xi.$$

Then $P(0)\widetilde{P} = P(0)$. The uniqueness of the fixed-point $u_{x-P(0)x}$ also implies $\widetilde{P}P(0) = \widetilde{P}$, thus

$$\widetilde{P} = \widetilde{P}P(0) = \widetilde{P}P(0)\widetilde{P} = \widetilde{P}\widetilde{P},$$

that is $\widetilde{P}$ is a bounded projection on X.

Step 3. $X_s^B(0) = Range(\widetilde{P})$, hence $X_s^B(0)$ is complemented.

Proof. Clearly, for any $x \in X$ one has

$$u_x(t) = U(t,0)u_x(0) + \int_0^t U(t,\xi)B(\xi)u_x(\xi)d\xi, \text{ for every } t \geq 0.$$

The uniqueness of the solution of the integral equation (4.15) at $s = 0$ implies

$$u_x(t) = U_B(t,0)u_x(0) = U_B(t,0)\widetilde{P}x,$$

and consequently $Range(\widetilde{P}) \subseteq X_s^B(0)$.

We prove the converse inclusion. Assume that there exists $x_0 \in X_s^B(0) \setminus Range(\widetilde{P})$. Evidently $x_0 \neq 0$, and let $\widehat{x_0}$ be the one-dimensional subspace generated by x_0. Since $x_0 \in Range(Q(0))$, we can write

$$Range(Q(0)) = \widehat{x_0} \oplus Z \text{ (direct sum)}$$

and set P_0 be the bounded projection with $Range(P_0) = \widehat{x_0}$. We also set

$$\widehat{P} = P_0 + \widetilde{P}.$$

As $P_0\widetilde{P} = \widetilde{P}P_0 = 0$, we get that $\widehat{P}$ is a bounded projection on X. Multiplying at left the identity

$$U_B(\tau,0)\widehat{P}x = U(\tau,0)\widehat{P}x + \int_0^\tau U(\tau,\xi)B(\xi)U_B(\xi,0)\widehat{P}xd\xi,\ \tau \geq 0,\ x \in X,$$

by $U_Q(t,\tau)$, $0 \leq t \leq \tau$, we obtain

$$U(t,0)Q(0)\widehat{P}x = U_Q(t,\tau)Q(\tau)U_B(\tau,0)\widehat{P}x - \int_0^t U(t,\xi)Q(\xi)B(\xi)U_B(\xi,0)\widehat{P}xd\xi$$
$$- \int_t^\tau U_Q(t,\xi)Q(\xi)B(\xi)U_B(\xi,0)\widehat{P}xd\xi.$$

Since $\|U_Q(t,\tau)Q(\tau)\| \leq Ne^{-\nu\int_t^\tau \omega(\xi)d\xi} \to 0$ as $\tau \to \infty$ and the mapping $\tau \mapsto U_B(\tau,0)\widehat{P}x$ is bounded (as $\widehat{P}x \in X_s^B(0)$), for $\tau \to \infty$ we get

$$U(t,0)Q(0)\widehat{P}x = -\int_0^t U(t,\xi)Q(\xi)B(\xi)U_B(\xi,0)\widehat{P}xd\xi$$
$$- \int_t^\infty U_Q(t,\xi)Q(\xi)B(\xi)U_B(\xi,0)\widehat{P}xd\xi.$$

Successively one has

$$\begin{aligned}
U(t,0)P(0)\widehat{P}x &+ \int_0^\infty \Gamma(t,\xi)B(\xi)U_B(\xi,0)\widehat{P}xd\xi \\
&= U(t,0)P(0)\widehat{P}x + \int_0^t U(t,\xi)P(\xi)B(\xi)U_B(\xi,0)\widehat{P}xd\xi \\
&\quad - \int_t^\infty U_Q(t,\xi)Q(\xi)B(\xi)U_B(\xi,0)\widehat{P}xd\xi \\
&= U(t,0)P(0)\widehat{P}x + \int_0^t U(t,\xi)P(\xi)B(\xi)U_B(\xi,0)\widehat{P}xd\xi \\
&\quad + U(t,0)Q(0)\widehat{P}x + \int_0^t U(t,\xi)Q(\xi)B(\xi)U_B(\xi,0)\widehat{P}xd\xi \\
&= U(t,0)\widehat{P}x + \int_0^t U(t,\xi)B(\xi)U_B(\xi,0)\widehat{P}xd\xi = U_B(t,0)\widehat{P}x.
\end{aligned}$$

Since $P(0)\widehat{P}x = P(0)x$, the above considerations imply that $v_x(t) = U_B(t,0)\widehat{P}x$ is a fixed point of the operator $\mathcal{J}$. Thus $\widetilde{P} = \widehat{P}$ and consequently $x_0 = 0$, which is false. This yields that $X_s^B(0) \subseteq Range(\widetilde{P})$. □

Using Lemma 4.3, Step 2 and Step 3, the final conclusion is a direct consequence of Corollary 3.4. □

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