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Dedicated to Prof. Emeritus Mihail Megan on the occasion of his 75^{th} anniversary

Common fixed point theory for a pair of multivalued operators

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ABSTRACT. The aim of this paper is to present a common fixed point theory for a pair of multivalued operators in complete metric spaces. Existence and stability results for the common fixed point problem will be discussed. The approach is based on some fixed point results for multivalued graphical contractions and for multivalued operators of Feng-Liu type. Our results extend to the multivalued case some recent results in the literature.

1. INTRODUCTION

The multivalued analogous of the famous Banach-Cacciopoli contraction principle was proved by Nadler [7] in 1969. One year later, Covitz and Nadler [4] proved that Nadler's contraction principle also holds without the hypothesis of boundedness for the values of the multivalued operator. There are many extensions, in various directions, of this principle. Of these extensions, we mention here the graph contraction principle [12] and the case of Feng-Liu operators [6]. For other extensions and generalizations see e.g. [2], [3], [5], [9], [15], [17], [18], [19], [23], [24].

The aim of this paper is to present a common fixed point theory for a pair of multivalued operators. Existence and stability results for the common fixed point problem will be discussed. The approach is based on some fixed point results for multivalued graphical contractions and for multivalued operators of Feng-Liu type. Our results extend to the multivalued case some recent results given in [14].

2. PRELIMINARIES

Let (M, d) be a metric space and $P(M)$ be the family of all nonempty subsets of M. The following family of sets is important in our considerations:

$$P_{cl}(M) := \{A \in P(M) |\ A \text{ is closed}\}.$$

In the same framework, we also consider the following concepts:

- the distance from a point $a \in M$ to a set $B \in P(M)$

$$D : M \times P(M) \to \mathbb{R}_+,\ D(a, B) := \inf\{d(a, b) \mid b \in B\}.$$

- the excess functional of A over B generated by d

$$e : P(M) \times P(M) \to \mathbb{R}_+ \cup \{+\infty\},\ e(A, B) := \sup\{D(a, B), a \in A\}.$$

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- the Hausdorff-Pompeiu functional generated by d

$$H : P(M) \times P(M) \to \mathbb{R}_+ \cup \{+\infty\},\ H(A,B) = \max\{e(A,B), e(B,A)\}.$$

In the same setting, if $F : M \to P(M)$ is a multivalued operator, then $x \in M$ is called a fixed point for F if $x \in F(x)$. The set

$$Fix(F) := \{x \in M |\ x \in F(x)\}$$

is the fixed point set of F. We also denote by

$$Graph(F) := \{(x,y) \in M \times M |\ y \in F(x)\}$$

the graph of the multivalued operator F.
If $F, G : M \to P(M)$ are two multivalued operators, we denote by

$$ComFix(F,G) := \{x \in M |\ x \in F(x) \cap G(x)\}$$

the set of all common fixed points for F and G.
If W, Z are nonempty sets and $T : W \to P(Z)$ is a multivalued operator, then a selection of T is a single-valued operator $t : W \to Z$ such that

$$t(x) \in T(x), \text{ for every } x \in W.$$

Definition 2.1. ([19]) Let (M,d) be a metric space. Then $F : M \to P(M)$ is called a multivalued weakly Picard operator if for each $x \in M$ and each $y \in F(x)$ there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X such that

(i) $x_0 = x,\ x_1 = y$;
(ii) $x_{n+1} \in F(x_n)$, for all $n \in \mathbb{N}$;
(iii) the sequence $\{x_n\}_{n\in\mathbb{N}}$ is convergent and its limit is a fixed point of F.

Notice that a sequence satisfying (i) and (ii) is called an iterative sequence of Picard type for F.

Let (M,d) be a metric space and $F : M \to P(M)$ be a multivalued weakly Picard operator. Then, we define the multivalued operator $F^\infty : Graph(F) \to P(Fix(F))$ by the formula $F^\infty(x,y) = \{x^* \in Fix(F) \mid$ there exists a sequence of successive approximations of F starting from (x,y) converging to $x^*\}$.

If $F : M \to P(M)$ is a multivalued weakly Picard operator, then F is a called a multivalued weakly c-Picard operator if $c > 0$ and there exists a selection f^∞ of F^∞ such that

$$d(x, f^\infty(x,y)) \le cd(x,y), \text{ for all } (x,y) \in Graph(F).$$

The following retraction-displacement condition will be important for our main results.

Definition 2.2. ([12]) Let (M,d) be a metric space and let $F : M \to P(M)$ be a multivalued operator such that $Fix(F) \neq \emptyset$. We say that F satisfies the strong retraction-displacement condition if there exist $c > 0$ and a set retraction $r : M \to Fix(F)$ such that

$$d(x, r(x)) \le cD(x, F(x)), \text{ for all } x \in M. \tag{2.1}$$

The concept is generated by the situations in which for every $x \in M$ there exists a sequence of successive approximations starting from x which converges to a fixed point of F. Then $r : M \to Fix(F)$ is defined by $r(x) := \{x^* \in Fix(F) \mid$ there exists a sequence of successive approximations of F starting from x converging to $x^*\}$.

Some typical conditions in fixed point theory for a multivalued operator are recalled now.

Definition 2.3. *Let (M,d) be a metric space and $F : M \to P(M)$. Then, the multivalued operator F is called:*

(a) *α-contraction if $\alpha \in]0,1[$ and $H(F(x_1),F(x_2)) \leq \alpha d(x_1,x_2)$, for all $(x_1,x_2) \in M \times M$;*

(b) *graphical α-contraction if $\alpha \in]0,1[$ and*

$$H(F(x_1),F(x_2)) \leq \alpha d(x_1,x_2), \text{ for all } (x_1,x_2) \in Graph(F);$$

(c) *α-contraction of Feng-Liu type if, for $b \in]0,1[$, there exists $\alpha \in]0,b[$ such that, for each $x \in X$ there is $y \in I_b^x := \{y \in F(x) : bd(x,y) \leq D_d(x,F(x))\}$ for which the following condition is satisfied*

$$D(y,F(y)) \leq \alpha d(x,y).$$

Remark 2.1. ([12]) It can be checked that any multivalued α-contraction is a multivalued graphical α-contraction and any multivalued graphical α-contraction is a multivalued α-contraction of Feng-Liu type, but not reversely, see [6], [12], [13].

The following results are the main fixed point theorems for multivalued graphical contractions, respectively for multivalued Feng-Liu type operators.

Theorem 2.1. *Let (M,d) be a complete metric space and $F : M \to P(M)$ be a multivalued graphical α-contraction with closed graph. Then, the following conclusions hold:*

(a) $Fix(F) \neq \emptyset$;

(b) for each $(x,y) \in Graph(F)$ there exists an iterative sequence $(x_n)_{n\in\mathbb{N}}$ of Picard type for F starting from (x,y) (i.e., $x_0 = x, x_1 = y, x_{n+1} \in F(x_n), n \in \mathbb{N}$) which converges to a fixed point $x^*(x,y)$ of F;

(c) there exists a selection $f^\infty : Graph(F) \to Fix(F)$ of F^∞ such that

$$d(x, f^\infty(x,y)) \leq \frac{1}{1-\alpha} d(x,y), \text{ for all } (x,y) \in Graph(F).$$

In other words, any multivalued graphical α-contraction on a complete metric space is a multivalued weakly $\frac{1}{1-\alpha}$-Picard operator, provided the set $Graph(F)$ is closed.

Theorem 2.2. ([6], [13]) *Let (M,d) be a complete metric space and $F : M \to P_{cl}(M)$ be a multivalued α-contraction of Feng-Liu type with constant $\alpha < b$. Suppose that either the mapping $g : M \to \mathbb{R}_+$ defined by $g(x) = D(x,F(x))$ is lower semi-continuous or F has closed graph. Then, the following conclusions hold:*

(i) $Fix(F) \neq \emptyset$;

(ii) for every $x_0 \in X$ there exists an iterative sequence $\{x_n\}_{n\in\mathbb{N}}$ of Picard type for F starting from x_0 (i.e., $x_{n+1} \in F(x_n), n \in \mathbb{N}$) which converges to a fixed point $x^*(x_0)$ of F and the following relation holds

$$d(x_0, x^*(x_0)) \leq \frac{1}{1-\frac{\alpha}{b}} d(x_0,x_1).$$

In other words, any multivalued α-contraction of Feng-Liu type on a complete metric space satisfies the strong retraction-displacement condition, provided the set $Graph(F)$ is closed or F is H-upper semicontinuous (which assures that the functional $g(x) = D(x,F(x))$ is lower semicontinuous).

3. Main results

Let (M,d) be a metric space and $F,G : M \to P(M)$ be two multivalued operators. We are interested in some sufficient conditions on M, F and G assuring the existence of a common fixed point for F and G, i.e., an element $x^* \in M$ with the property that

$$x^* \in F(x^*) \cap G(x^*). \tag{3.2}$$

Then, some stability properties for the common fixed point problem (3.2) will be studied. For related results in common fixed point theory see [1], [20]-[22].

Theorem 3.3. *Let (M,d) be a metric space and $F,G : M \to P_{cl}(M)$ be two multivalued operators. We suppose that there exists $\alpha \in]0, \frac{1}{2}[$ such that*

$$H(F(x),G(y)) \leq \alpha\left(D(x,F(x)) + D(y,G(y))\right), \text{ for every } x,y \in M. \tag{3.3}$$

Then:

(a) $ComFix(F,G) = Fix(F) = Fix(G)$.

If, additionally, the space (M,d) is complete and F and G have closed graph, then:

(b) $ComFix(F,G) \neq \emptyset$;

(c) for each $x_0 \in M$, there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ with

$$x_{2n+1} \in F(x_{2n}), \quad x_{2n+2} \in G(x_{2n+1}), \quad \text{for all } n \in \mathbb{N},$$

converging to a common fixed point for F and G;

(d) for each $y_0 \in X$, there exists a sequence $(y_n)_{n\in\mathbb{N}}$ with

$$y_{2n+1} \in G(y_{2n}), \quad y_{2n+2} \in F(y_{2n+1}), \quad \text{for all } n \in \mathbb{N},$$

converging to a common fixed point for F and G;

(e) if $\alpha < \frac{1}{3}$, then F and G are multivalued graphical contractions;

(f) F and G are multivalued quasi-contractions;

(g) if $\alpha < \frac{1}{3}$, then F and G are multivalued weakly c-Picard operator with $c := \frac{(1-\alpha)^2}{1-3\alpha}$;

Proof. (a) Let $u \in Fix(F)$. we will show that $u \in Fix(G)$. Indeed, we have

$$D(u,G(u)) \leq H(F(u),G(u)) \leq \alpha\left(D(u,F(u)) + D(u,G(u))\right) = \alpha D(u,G(u)).$$

Thus, $D(u,G(u)) = 0$ and hence $u \in G(u)$. In a similar way, we can get that $Fix(G) \subset Fix(F)$.

(b)-(c) Let $x_0 \in M$ and $x_1 \in F(x_0)$ be arbitrarily chosen. Then, for $1 < q < \frac{1}{2\alpha}$, there exists $x_2 \in G(x_1)$ such that

$$d(x_1,x_2) \leq qH(F(x_0),G(x_1)) \leq \alpha q\left(D(x_0,F(x_0)) + D(x_1,G(x_1))\right) \leq$$
$$\alpha q\left(d(x_0,x_1) + d(x_1,x_2)\right).$$

Hence

$$d(x_1,x_2) \leq \frac{\alpha q}{1-\alpha q} d(x_0,x_1).$$

Denote $\beta := \frac{\alpha q}{1-\alpha q} < 1$. Then, inductively, we can generate a sequence $\{x_n\}_{n\in\mathbb{N}}$ defined by

$$x_{2n+1} \in F(x_{2n}), \quad x_{2n+2} \in G(x_{2n+1}), \quad \text{for all } n \in \mathbb{N},$$

with the property that

$$d(x_n,x_{n+1}) \leq \beta^n d(x_0,x_1), n \in \mathbb{N}.$$

By an usual approach, it follows that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in the complete metric space (M,d). Let $x^* \in M$ be its limit. We have

$$D(x^*,F(x^*)) \leq d(x^*,x_{2n+2}) + D(x_{2n+2},F(x^*)) \leq d(x^*,x_{2n+2}) + H(F(x^*),G(x_{2n+1})) \leq$$
$$d(x^*,x_{2n+2}) + \alpha\left(D(x^*,F(x^*)) + D(x_{2n+1},G(x_{2n+1})\right) \leq$$
$$d(x^*,x_{2n+2}) + \alpha\left(D(x^*,F(x^*)) + d(x_{2n+1},x_{2n+2})\right).$$

Letting $n \to \infty$, we get that $x^* \in F(x^*)$. In a similar way, we have

$$D(x^*,G(x^*)) \leq d(x^*,x_{2n+1}) + D(x_{2n+1},G(x^*)) \leq d(x^*,x_{2n+1}) + H(F(x_{2n}),G(x^*)) \leq$$
$$d(x^*,x_{2n+1}) + \alpha\left(D(x_{2n},F(x_{2n}) + D(x^*,G(x^*))\right) \leq$$
$$d(x^*,x_{2n+1}) + \alpha\left(d(x_{2n},x_{2n+1}) + D(x^*,G(x^*))\right).$$

Letting $n \to \infty$, we get that $x^* \in G(x^*)$. Thus $x^* \in ComFix(F,G)$.

(d) can be proved in a similar way to (c).

(e) For $(x, y) \in Graph(F)$, we will estimate

$$H(F(x), F(y)) \leq H(F(x), G(y)) + H(F(y), G(y)).$$

On one hand, we have

$$H(F(x), G(y)) \leq \alpha\left(D(x, F(x)) + D(y, G(y))\right) \leq \alpha d(x, y) + \alpha H(F(x), G(y)).$$

Thus

$$H(F(x), G(y)) \leq \frac{\alpha}{1-\alpha} d(x, y).$$

On the other hand, we have

$$\begin{aligned} H(F(y), G(y)) &\leq \alpha\left(D(y, F(y)) + D(y, G(y))\right) \leq \alpha\left(H(F(x), F(y)) + H(F(x), G(y))\right) \\ &\leq \alpha H(F(x), F(y)) + \frac{\alpha^2}{1-\alpha} d(x, y). \end{aligned}$$

By the above relations, we get

$$H(F(x), F(y)) \leq \frac{\alpha}{1-\alpha} d(x, y) + \alpha H(F(x), F(y)) + \frac{\alpha^2}{1-\alpha} d(x, y).$$

Thus, we conclude that

$$H(F(x), F(y)) \leq \frac{\alpha^2 + \alpha}{(1-\alpha)^2} d(x, y). \tag{3.4}$$

Since $\gamma := \frac{\alpha^2+\alpha}{(1-\alpha)^2} < 1$ we get that F is a multivalued graphical γ-contraction. A similar approach gives us that G is a multivalued graphical γ-contraction.

(f) Let $x \in M$ and denote by x^* a common fixed point for F and G. Then, we have

$$\begin{aligned} D(F(x), x^*) &\leq H(F(x), F(x^*)) \leq H(F(x), G(x^*)) + H(F(x^*), G(x^*)) \\ &\leq \alpha\left(D(x, F(x)) + D(x^*, G(x^*)) + D(x^*, F(x^*)) + D(x^*, G(x^*))\right) \\ &= \alpha D(x, F(x)) \leq \alpha\left(d(x, x^*) + D(x^*, F(x))\right). \end{aligned}$$

Thus

$$D(F(x), x^*) \leq \frac{\alpha}{1-\alpha} d(x, x^*),$$

showing that F is a quasi-contraction with constant $\frac{\alpha}{1-\alpha} < 1$.

(g) Let us prove that F and G are a multivalued weakly c-Picard operator with $c := \frac{(1-\alpha)^2}{1-3\alpha}$. By (b) and (c) we get that F and G are multivalued weakly Picard operators. By (e) we obtain that F and G are multivalued graphical γ-contractions. Using Theorem 2.1 (c), we get that F and G are multivalued weakly $\frac{1}{1-\gamma}$-Picard operators. □

Example 3.1. *Let $F, G : [0,1] \to P([0,1])$ be given by*

$$F(x) := \begin{cases} [\frac{1}{2}, 1], & x = 0 \\ \{\frac{1}{2}\}, & x \in]0,1[\\ \{0, \frac{1}{2}\}, & x = 1. \end{cases}$$

$$G(y) := \begin{cases} [\frac{1}{2}, 1], & y = 0 \\ \{1-y\}, & y \in]0,1[\\ \{0, \frac{1}{2}\}, & y = 1. \end{cases}$$

Then, F and G satisfy all the assumption of the above theorem and we have $Fix(F) = Fix(G) = Com(F, G) = \{\frac{1}{2}\}$.

We recall now some stability concepts. We consider first the concept of Ulam-Hyers stability for the multivalued common fixed point inclusion (see also [8]).

Definition 3.4. Let (M, d) be a metric space and $F, G : M \to P(M)$ be two multivalued operators. The multivalued common fixed point inclusion $x \in F(x) \cap G(x)$ is called Ulam-Hyers stable if there exists $C > 0$, such that for every $\varepsilon > 0$ and for each ε-common fixed point $u \in M$ of F and G (i.e., $D(u, F(u) \cap G(u)) \leq \varepsilon$), there exists $x^* \in ComFix(F, G)$ such that

$$d(u, x^*) \leq C\varepsilon.$$

The well-posedness of the common fixed point inclusion $x \in F(x) \cap G(x)$ is defined as follows (see also [10]).

Definition 3.5. Let (M, d) be a metric space and let $F, G : M \to P(M)$ be two multivalued operators such that $ComFix(F, G) \neq \emptyset$ and there exists a set retraction $r : M \to ComFix(F, G)$. Then, the common fixed point inclusion $x \in F(x) \cap G(x)$ is called well-posed in the sense of Reich and Zaslavski ([16]) if for each $x^* \in ComFix(F)$ and for any sequence $\{u_n\}_{n\in\mathbb{N}} \subset r^{-1}(x^*)$ such that

$$D(u_n, (F \cap G)(u_n)) \to 0 \text{ as } n \to \infty,$$

we have that

$$u_n \to x^* \text{ as } n \to \infty.$$

Finally, we recall the notion of Ostrowski stability property for a common fixed point inclusion (see also [11]).

Definition 3.6. Let (M, d) be a metric space and let $F, G : M \to P(M)$ be two multivalued operators such that $ComFix(F, G) \neq \emptyset$ and there exists a set retraction $r : M \to ComFix(F, G)$. Then, the common fixed point inclusion $x \in F(x) \cap G(x)$ is said to have the Ostrowski stability property if for each $x^* \in ComFix(F, G)$ and for any sequence $\{w_n\}_{n\in\mathbb{N}} \subset r^{-1}(x^*)$ such that

$$D(w_{n+1}, (F \cap G)(w_n)) \to 0 \text{ as } n \to \infty,$$

we have that

$$w_n \to x^* \text{ as } n \to \infty.$$

The following result from [13] gives an important property of multivalued contractions of Feng-Liu type.

Theorem 3.4. [13] *Let (M, d) be a complete metric space and $F : M \to P_{cl}(M)$ be a multivalued α-contraction of Feng-Liu type with constant $\alpha < b$. Suppose that either the mapping $g : M \to \mathbb{R}_+$ defined by $g(x) = D(x, F(x))$ is lower semi-continuous or F has closed graph. Then, $Fix(F) \neq \emptyset$ and there exists a set retraction $r : M \to Fix(F)$ such that F satisfies the following strong retraction-displacement condition*

$$D(x, r(x)) \leq \frac{1}{b - \alpha} D(x, F(x)), \text{ for every } x \in M.$$

From Theorem 3.3 (e) we know that F and G are multivalued graphical γ-contractions (with $\gamma := \frac{\alpha^2+\alpha}{(1-\alpha)^2}$), provided $\alpha < \frac{1}{3}$. Thus, F and G are multivalued γ-contraction of Feng-Liu type with constant $\gamma < b$. By Theorem 3.4, we obtain that F and G satisfies the strong retraction-displacement condition (2.1). Thus, the following result can be obtained.

Theorem 3.5. *Let (M, d) be a complete metric space and $F, G : M \to P_{cl}(M)$ be two multivalued operators. We suppose that there exists $\alpha \in]0, \frac{1}{3}[$ such that*

$$H(F(x), G(y)) \leq \alpha \left(D(x, F(x)) + D(y, G(y))\right), \text{ for every } x, y \in M. \tag{3.5}$$

Then, the following conclusions hold:

(i) the common fixed point inclusion $x \in F(x) \cap G(x)$ is Ulam-Hyers stable;

(ii) the common fixed point inclusion $x \in F(x) \cap G(x)$ is well-posed in the sense of Reich and Zaslavski.

Proof. (i) Let $\gamma := \frac{\alpha^2+\alpha}{(1-\alpha)^2} < 1$ and $b > \gamma$. Let $\varepsilon > 0$ and $u \in M$ such that

$$D(u, F(u) \cap G(u)) \leq \varepsilon.$$

Let $x^* := r(u)$. Then, by the strong retraction-displacement condition (2.1) for F we have

$$d(u, x^*) \leq \frac{1}{b-\gamma} D(u, F(u)) \leq \frac{1}{b-\gamma} D(u, F(u) \cap G(u)) \leq \frac{1}{b-\gamma} \varepsilon.$$

A similar approach holds for G.

(ii) Let $x^* \in ComFix(F,G)$ and $\{u_n\}_{n\in\mathbb{N}} \subset r^{-1}(x^*)$ such that

$$D(u_n, (F \cap G)(u_n)) \to 0 \text{ as } n \to \infty.$$

Then

$$d(u_n, x^*) \leq \frac{1}{b-\gamma} D(u_n, F(u_n)) \leq \frac{1}{b-\gamma} D(u_n, F(u_n) \cap G(u_n)) \to 0, \text{ as } n \to \infty.$$

□

Remark 3.2. It is an open question to obtain the Ostrowski stability property for the common fixed point inclusion (3.2). See also [11].

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