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Dedicated to Prof. Emeritus Mihail Megan on the occasion of his 75^{th} anniversary

General analytical solution of fractional Klein–Gordon equation in a spherical domain

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ABSTRACT. Time-fractional Klein–Gordon equation in a sphere is considered for the case of central symmetry under the time-variable Dirichlet condition. The time-fractional derivative with the power-law kernel is used. The Laplace transform and convenient transformations of the independent variable and unknown function are used to determine the general analytical solution of the problem in the Laplace domain. In order to obtain the solution in the real domain, the inverse Laplace transforms of two functions of exponential type whose expressions are new in the literature have been determined. The similar solution for ordinary Klein–Gordon equation is a limiting case of general solution but a simpler form for this solution is provided. The convergence of general solution to the ordinary solution and the effects of fractional parameter on this solution are graphically underlined.

1. INTRODUCTION

The Klein–Gordon equation has been used since 1925 to describe de Broglie waves of the electron in the hydrogen atom [9]. It is worth pointing out that this equation is also suitable in the description of some vibrating systems in the classical mechanics, such as vibrating flexible strings [10]. This equation, whose general form is given by the relation

$$\frac{\partial^2 u(\mathbf{x},t)}{\partial t^2} = a\Delta u(\mathbf{x},t) - bu(\mathbf{x},t),\ (\mathbf{x},t) \in D \times [0,\infty),\ D \subset R^3;\ a,b>0, \tag{1.1}$$

is also used in the nonlinear optics, the quantum field theory, solid-state physics, and classical mechanics as well. If the domain D is a sphere of radius R, considering a spherical coordinate system and the case of the central symmetry, Eq. (1.1) takes the simple form

$$\frac{\partial^2 u(r,t)}{\partial t^2} = a\left[\frac{\partial^2 u(r,t)}{\partial r^2} + \frac{2}{r}\,\frac{\partial u(r,t)}{\partial r}\right] - bu(r,t);\ r \in (0,R),\ t \in (0,\infty). \tag{1.2}$$

Often a mathematical model put forward to describe a given phenomenon is replaced by another mathematical model which is able to better describe the complex properties of different physical processes. It should be noted that the fractional differential equations obtained as generalizations of some classical differential equations have numerous applications in physics, plasma physics, chemistry, rheology, geophysics, biology, bioengineering, engineering, finance, and medicine [6, 15, 2, 4, 11].

There are many non-homogeneous media where the transport phenomena exhibit anomalous properties. The experiments and accurate measurements have shown that the mathematical models based on fractional differential equations are more suitable to describe complex processes at different scales.

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In this paper we consider the time-fractional differential equation

(1.3) $^{C}D_t^{\alpha}u(\mathbf{x},t) = a\Delta u(\mathbf{x},t) - bu(\mathbf{x},t),\ (\mathbf{x},t) \in D \times [0,\infty),\ D \subset R^3;\ a,b > 0,\ \alpha \in (1,2],$

that is an extension of the hyperbolic equation (1.1). In this equation the operator $^{C}D_t^{\alpha}u(\mathbf{x},t)$, $\alpha \in (1,2]$ denotes time-fractional Caputo derivative defined as [15, 5]

$$^{C}D_t^{\alpha}u(\mathbf{x},t) = \begin{cases} \dfrac{1}{\Gamma(2-\alpha)}\displaystyle\int_0^t (t-\tau)^{1-\alpha}\frac{\partial^2 u(\mathbf{x},\tau)}{\partial \tau^2}\,d\tau,\ \alpha \in (1,2), \\ \dfrac{\partial^2 u(\mathbf{x},t)}{\partial t^2},\ \alpha = 2, \end{cases} \tag{1.4}$$

where $\Gamma(\zeta) = \int_0^{\infty} t^{\zeta-1}e^{-t}dt,\ \mathrm{Re}(\zeta) > 0$ is the Euler integral of second kind.

The kernel $^{C}h(t,\alpha)$ of the Caputo derivative (1.4) is

$$^{C}h(t,\alpha) = \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \tag{1.5}$$

and the operator (1.4) can be written as a convolution product, namely

$$^{C}D_t^{\alpha}u(\mathbf{x},t) = {}^{C}h(t,\alpha) * \frac{\partial^2 u(\mathbf{x},t)}{\partial t^2} = \int_0^t {}^{C}h(t-\tau,\alpha)\frac{\partial^2 u(\mathbf{x},\tau)}{\partial \tau^2}\,d\tau,\ \alpha \in (1,2). \tag{1.6}$$

Using Eq. (1.6) and some properties of the Laplace transform [7], one obtains

$$\begin{aligned} L\{^{C}D_t^{\alpha}u(\mathbf{x},t)\} &= L\{^{C}h(t,\alpha)\}L\left\{\frac{\partial^2(\mathbf{x},t)}{\partial t^2}\right\} \\ &= s^{\alpha}\bar{u}(\mathbf{x},s) - s^{\alpha-1}u(\mathbf{x},0) - s^{\alpha-2}\left.\frac{\partial u(\mathbf{x},t)}{\partial t}\right|_{t=0};\ \alpha \in (1,2], \end{aligned} \tag{1.7}$$

where $\bar{u}(\mathbf{x},s) = L\{u(\mathbf{x},t)\} = \int_0^t u(\mathbf{x},t)\exp(-st)dt$ denotes the Laplace transform of the unknown function $u(\mathbf{x},t)$.

The time-fractional differential equation (1.3) has been studied in several publications. Some nonlinear fractional Klein–Gordon equations in Caputo sense have been analytically and numerically studied in [1]. The authors used the fractional reduced differential transform method that provides the solutions as convergent series. Kheiri et al. [12] studied a non-homogeneous fractional Klein–Gordon equation with Dirichlet, Neumann, and Robin boundary conditions using the separating variables method. Other studies regarding the Klein–Gordon equation can be found in the references [3, 13, 8, 14].

In this paper, an initial-boundary value problem for the equation (1.3) with the Dirichlet boundary condition variable in time is studied in a spherical domain with central symmetry. Using Laplace transform and suitable transformations of the unknown function and independent variable the corresponding equation is transformed into a modified Bessel equation with known solution. To change the Laplace transforms in the real domain, two adequate inverses Laplace transforms are determined. Making $\alpha = 2$ in the analytical solution corresponding to the fractional case, the solution of classical Klein–Gordon equation in the same domain can be obtained. However, for completion and convenience, an equivalent but simpler form of this solution is also established. The convergence of general solution to this simple form of the classical Klein–Gordon equation is graphically proved.

2. SOLUTION OF FRACTIONAL KLEIN–GORDON EQUATION IN A SPHERICAL DOMAIN

In this section we determine the analytical solution of the fractional differential equation (1.3) in a sphere of radius R. Considering the case of central symmetry, this equation can be written in the next form (see also Eq. (1.2))

$$ {}^{C}D_t^{\alpha}u(r,t) = a\left[\frac{\partial^2 u(r,t)}{\partial r^2} + \frac{2}{r}\,\frac{\partial u(r,t)}{\partial r}\right] - bu(r,t);\ r \in (0,R),\ t \in (0,\infty),\ \alpha \in (1,2]. \tag{2.8}$$

Along with this fractional differential equation, the following initial-boundary conditions are considered

$$ u(r,0) = 0,\ \left.\frac{\partial u(r,t)}{\partial t}\right|_{t=0} = 0;\ \ r \in [0,R], \tag{2.9}$$

$$ u(R,t) = f(t);\ \ t > 0. \tag{2.10}$$

Here, $f(\cdot)$ is a piecewise continuous function of exponential order as $t \to \infty$.

Applying the Laplace transform to Eq. (2.8) and using the identity (1.7) as well as the initial conditions (2.9), one obtains the transformed equation

$$ \frac{\partial^2 \bar{u}(r,s)}{\partial r^2} + \frac{2}{r}\,\frac{\partial \bar{u}(r,s)}{\partial r} - \gamma(s)\bar{u}(r,s) = 0;\ r \in (0,R), \tag{2.11}$$

where $\gamma(s) = (s^{\alpha} + b)/a$. The function $\bar{u}(r,s)$ has to satisfy the boundary condition

$$ \bar{u}(R,s) = \bar{f}(s), \tag{2.12}$$

where $\bar{f}(s) = L\{f(t)\}$ is the Laplace transform of $f(t)$.

Making the following changes of independent variable and unknown function

$$ r = \frac{1}{\sqrt{\gamma(s)}}\,z,\ \bar{u}(r,s) = \frac{\sqrt[4]{\gamma(s)}}{\sqrt{z}}\,\overline{U}(z,s), \tag{2.13}$$

in Eq. (2.11), one obtain the next modified Bessel equation

$$ z^2\,\frac{\partial^2 \overline{U}(z,s)}{\partial z^2} + z\,\frac{\partial \overline{U}(z,s)}{\partial z} - \left(z^2 + \frac{1}{4}\right)\overline{U}(z,s) = 0. \tag{2.14}$$

The general solution of Eq. (2.14) is given by the relation [16]

$$ \overline{U}(z,s) = C_1(s)I_{1/2}(z) + C_2(s)K_{1/2}(z), \tag{2.15}$$

where $I_{1/2}(\cdot)$ and $K_{1/2}(\cdot)$ are modified Bessel functions of first and second kind of order $1/2$ and the functions $C_1(\cdot)$ and $C_2(\cdot)$ will be determined from the boundary condition. On the basis of the relations (2.13) we find that

$$ \bar{u}(r,s) = \frac{1}{\sqrt{r}}\,\left[C_1(s)I_{1/2}\left(r\sqrt{\gamma(s)}\right) + C_2(s)K_{1/2}\left(r\sqrt{\gamma(s)}\right)\right]. \tag{2.16}$$

Now, using the identities (A1) from Appendix, it results that

$$ \bar{u}(r,s) = \frac{1}{r\,\sqrt[4]{\gamma(s)}}\left[\sqrt{\frac{2}{\pi}}\,C_1(s)\sinh\left(r\sqrt{\gamma(s)}\right) + \sqrt{\frac{\pi}{2}}\,C_2(s)e^{-r\sqrt{\gamma(s)}}\right]. \tag{2.17}$$

In order to have a solution $\bar{u}(r,s)$ with finite values inside the sphere the function $C_2(s)$ has to be identically equal to zero. Consequently, the solution of the equation (2.11) is

$$ \bar{u}(r,s) = \frac{C_1(s)}{r}\sqrt{\frac{2}{\pi\sqrt{\gamma(s)}}}\,\sinh\left[r\sqrt{\gamma(s)}\right]. \tag{2.18}$$

Imposing the boundary condition (2.10), it results that

$$ \bar{u}(r,s) = \frac{R}{r}\,\frac{\sinh\left[r\sqrt{\gamma(s)}\right]}{\sinh\left[R\sqrt{\gamma(s)}\right]}\,\bar{f}(s);\ r \in (0,R), \tag{2.19}$$

or equivalently

$$\bar{u}(r,s)=\frac{R}{r}\,\bar{f}(s)\sum_{k=0}^{\infty}\left[e^{[r-(2k+1)R]\sqrt{\gamma(s)}}-e^{-[r+(2k+1)R]\sqrt{\gamma(s)}}\right]. \tag{2.20}$$

Applying the inverse Laplace transform to Eq. (2.20), it results that

$$u(r,t)=\frac{R}{r}\,f(t)*\sum_{k=0}^{\infty}\left[L^{-1}\left\{e^{[r-(2k+1)R]\sqrt{\gamma(s)}}\right\}-L^{-1}\left\{e^{-[r+(2k+1)R]\sqrt{\gamma(s)}}\right\}\right], \tag{2.21}$$

where "$*$" denotes the convolution product of the two functions and $L^{-1}\{\cdot\}$ means the inverse Laplace transform.

Consequently, in order to determine the problem solution $u(r,t)$, we need the inverse Laplace transform of the exponential function $\exp[-c\sqrt{\gamma(s)}]$. To find it, the auxiliary function $\bar{h}(c,s)=\exp[-c\sqrt{s}]$ (whose inverse Laplace transform $h(c,t)$ is given by Eq. (A2) from Appendix) and the property of the inverse Laplace transform of the composite functions will be used. More exactly, the inverse Laplace transform of $\exp[-c\sqrt{\gamma(s)}]$ is

$$L^{-1}\left\{\exp\left[-c\sqrt{\gamma(s)}\right]\right\}=L^{-1}\left\{\bar{h}(c,\gamma(s)\right\}=\int_0^{\infty}h(c,\xi)\varphi(\xi,t)d\xi, \tag{2.22}$$

where the function $\varphi(\xi,t)$ is given by the relation

$$\begin{aligned}\varphi(\xi,t)&=L^{-1}\left\{\exp\left[-\xi\gamma(s)\right]\right\}=L^{-1}\left\{\exp\left(-\frac{\xi(s^{\alpha}+b)}{a}\right)\right\}\\&=\exp\left(-\frac{b\xi}{a}\right)L^{-1}\left\{\exp\left(-\tfrac{\xi s^{\alpha}}{a}\right)\right\}.\end{aligned} \tag{2.23}$$

Lemma 2.1. *The inverse Laplace transform of the function $\overline{\psi}(\xi,s)=\exp(-\xi s^{a}/a)$ is*

$$\psi(\xi,t)=L^{-1}\left\{\exp\left(-\frac{\xi s^{\alpha}}{a}\right)\right\}=-\sum_{k=1}^{\infty}\frac{1}{k!\Gamma(\alpha k)}\left(-\frac{\xi}{a}\right)^{k}\int_0^{\infty}\frac{z^{\alpha k-1/2}}{\sqrt{t}}\,J_1(2\sqrt{tz})dz \tag{2.24}$$

and the function $\varphi(\xi,t)$ is given by the relation

$$\varphi(\xi,t)=\psi(\xi,t)\exp[-b\xi/a]. \tag{2.25}$$

Proof. Let us consider the function

$$\bar{g}(s)=\exp\left(-\frac{\xi}{as^{\alpha}}\right)=1+\sum_{k=1}^{\infty}\left(-\frac{\xi}{a}\right)^{k}\frac{1}{k!s^{\alpha k}}, \tag{2.26}$$

whose inverse Laplace transform is given by the expression

$$g(t)=L^{-1}\{\bar{g}(s)\}=\delta(t)+\sum_{k=1}^{\infty}\left(-\frac{\xi}{a}\right)^{k}\frac{t^{\alpha k-1}}{k!\Gamma(\alpha k)}, \tag{2.27}$$

where $\delta(\cdot)$ is the Dirac' distribution and $\Gamma(\cdot)$ is the well known Gamma function.

Now, using the (A3) property and the identity $(A4)_1$ from Appendix and bearing in mind Eq. (2.27), it results that

$$\begin{aligned}L^{-1}\left\{\frac{1}{s}\exp\left(-\frac{\xi s^{\alpha}}{a}\right)\right\}&=L^{-1}\left\{\frac{1}{s}\,\bar{g}\left(\frac{1}{s}\right)\right\}=\int_0^{\infty}J_0\left(2\sqrt{tz}\right)g(z)dz\\&=1+\sum_{k=1}^{\infty}\frac{1}{k!\Gamma(\alpha k)}\left(-\frac{\xi}{a}\right)^{k}\int_0^{\infty}z^{\alpha k-1}J_0\left(2\sqrt{tz}\right)dz.\end{aligned} \tag{2.28}$$

On the other hand,

$$L^{-1}\left\{\frac{1}{s}\exp\left(-\frac{\xi s^{\alpha}}{a}\right)\right\}=L^{-1}\left\{\frac{1}{s}\,\overline{\psi}(\xi,s)\right\}=\int_0^t L^{-1}\{\overline{\psi}(\xi,s)\}(\tau)d\tau=\int_0^t \psi(\xi,\tau)d\tau. \tag{2.29}$$

From the last two relations it clearly results

$$\int_0^t \psi(\xi,\tau)d\tau = 1+\sum_{k=1}^{\infty}\frac{1}{k!\Gamma(\alpha k)}\left(-\frac{\xi}{a}\right)^k \int_0^{\infty} z^{\alpha k-1}J_0\left(2\sqrt{tz}\right)dz. \tag{2.30}$$

By deriving the last relation with respect to t and using the equality $(A4)_2$ from Appendix, one obtains the equality (2.24).

Now, from Eqs. (2.21), (2.22), (2.25) and (A2) from Appendix it clearly results that

$$\begin{aligned} u_f(r,t) \;&= \frac{R}{2r\sqrt{\pi}}\,f(t)*\sum_{k=0}^{\infty}\int_0^{\infty}\left\{[(2k+1)R-r]\exp\left[-\frac{[(2k+1)R-r]^2}{4\xi}\right]\right.\\ &\left.-[(2k+1)R+r]\exp\left[-\frac{[(2k+1)R+r]^2}{4\xi}\right]\right\}\frac{\psi(\xi,t)\exp(-b\xi/a)}{\xi\sqrt{\xi}}\,d\xi, \end{aligned} \tag{2.31}$$

where, for difference, $u_f(r,t)$ denotes the solution of fractional Klein–Gordon equation.

3. Solution of classical Klein–Gordon equation

Although the general solution given by Eq. (2.31) is also valid for the classical Klein–Gordon equation corresponding to $\alpha = 2$, we shall establish an equivalent but simpler form for the solution of this equation. In order to do this, let us observe that for $\alpha = 2$ Eq. (2.20) can be written in the form

$$\bar{u}(r,s) = \frac{R}{r}\,\bar{f}(s)\sum_{k=0}^{\infty}\{\bar{w}[d_1(r),s]-\bar{w}[d_2(r),s]\}, \tag{3.32}$$

where $d_1(r) = [(2k+1)R-r]/\sqrt{a}$ and $d_2(r) = [(2k+1)R+r]/\sqrt{a}$,

$$\bar{w}(d,s) = \exp(-d\sqrt{s^2+b}). \tag{3.33}$$

□

Lemma 3.2. *The inverse Laplace transform of the function $\bar{w}(d,s)$ is*

$$w(d,t) = L^{-1}\{\bar{w}(d,s)\} = \delta(t-d) - \frac{d\sqrt{b}}{\sqrt{t^2-d^2}}\,J_1\left[\sqrt{b(t^2-d)^2)}\right]. \tag{3.34}$$

Proof. Rewriting the function $\bar{w}(d,s)$ in a convenient form, namely

$$\bar{w}(d,s) = \exp(-ds)+\left\{\exp\left[d\left(s-\sqrt{s^2-\left(i\sqrt{b}\right)^2}\right)\right]-1\right\}\exp(-ds) \tag{3.35}$$

and using the relations (A5) and (A6) from Appendix, it results that its inverse Laplace transform $w(d,t)$ is given by the relation

$$\begin{aligned} w(d,t) \;&= \delta(t-d)+\delta(t-d)*\frac{id\sqrt{b}}{\sqrt{t^2+2dt}}\,I_1\left[i\sqrt{bt(t+2d)}\right]\\ &= \delta(t-d)+\frac{id\sqrt{b}}{\sqrt{t^2-d^2}}\,I_1\left[i\sqrt{b(t^2-d^2)}\right]. \end{aligned} \tag{3.36}$$

Now, using the identities (A7) from Appendix, one obtains for $w(d,t)$ the simpler form from the equality (3.34) and this lemma is proved.

Finally, applying the inverse Laplace transform to Eq. (3.32) and bearing in mind Eqs. (3.34) and (A6) from Appendix, we obtain for the solution of the classical Klein–Gordon equation (1.2) the following expression

$$
\begin{aligned}
u_c(r,t) &= \frac{R}{r}\sum_{k=0}^{\infty} f(t) * \left\{ w_1\left(\frac{(2k+1)R+r}{\sqrt{a}}, t\right) - w_1\left(\frac{(2k+1)R-r}{\sqrt{a}}, t\right)\right\} \\
&+ \frac{R}{r}\sum_{k=0}^{\infty}\left\{ f\left[t - \frac{(2k+1)R-r}{\sqrt{a}}\right] - f\left[t - \frac{(2k+1)R+r}{\sqrt{a}}\right]\right\},
\end{aligned}
\tag{3.37}
$$

where $u_c(r,t)$ denotes the solution of classical Klein–Gordon equation and

$$
w_1(d,t) = \frac{d\sqrt{b}}{\sqrt{t^2-d^2}}\, J_1\left[\sqrt{b(t^2-d^2)}\right]. \tag{3.38}
$$

4. Some numerical results and conclusions

In this work fractional Klein–Gordon equation with time-variable boundary condition has been analytically investigated in a spherical domain with central symmetry. The exact solution given by the equality (2.31) has been obtained using the Laplace transform and convenient changes of the spatial variable and the unknown function. It reduces to the corresponding solution of the ordinary Klein–Gordon equation if the fractional parameter $\alpha = 2$. However, for this equation an equivalent but simpler form of solution has been separately determined and due to the generality of the boundary condition (2.10) the problems in discussion are completely solved. As a check of results that have been here obtained, Figs. 1 and 2 were drawn to show that the solution $u_f(r,t)$ corresponding to the fractional equation converges to the solution $u_c(r,t)$ of the classical Klein–Gordon equation if the fractional parameter $\alpha \to 2$, $R = 2$, $a = 3$, $b = 0.5$, $f(t) = 5\exp(-t)$.

In Fig. 1 are presented the profiles of $u_f(r,t)$ (for three increasing values of the fractional parameter α) and $u_c(r,t)$ versus r at two values of the time t, while Fig. 2 provides the profiles of the same solutions versus t for two distinct values of the radial coordinate r. In all cases, as it was to be expected, the diagrams of $u_f(r,t)$ corresponding to increasing values of $\alpha < 2$ tend to overlap over the curve corresponding to the classical solution when the fractional parameter $\alpha \to 2$. In addition, the curves from Fig. 2 show that both solutions tend to the zero asymptotic value for large values of the time t and fixed values of the radial coordinate r. This property can be analytically proved using the equality (2.19) and the relation (A8) from Appendix regarding the function $f(\cdot)$ and its Laplace transform. Indeed, using Eqs. (2.19) and (A8), one obtains

$$
\begin{aligned}
\lim_{s\to 0} s\bar{u}(r,s) &= \lim_{s\to 0} s\bar{f}(s) \lim_{s\to 0}\frac{R}{r}\,\frac{\sinh\left(r\sqrt{a^{-1}(s^\alpha+b)}\right)}{\sinh\left(R\sqrt{a^{-1}(s^\alpha+b)}\right)} \\
&= \frac{R}{r}\,\frac{\sinh\left(r\sqrt{b/a}\right)}{\sinh\left(R\sqrt{b/a}\right)}\,\lim_{t\to\infty} f(t).
\end{aligned}
\tag{4.39}
$$

In our case, since $\lim_{t\to\infty} f(t) = 0$, the two solutions tend to zero for large values of t.

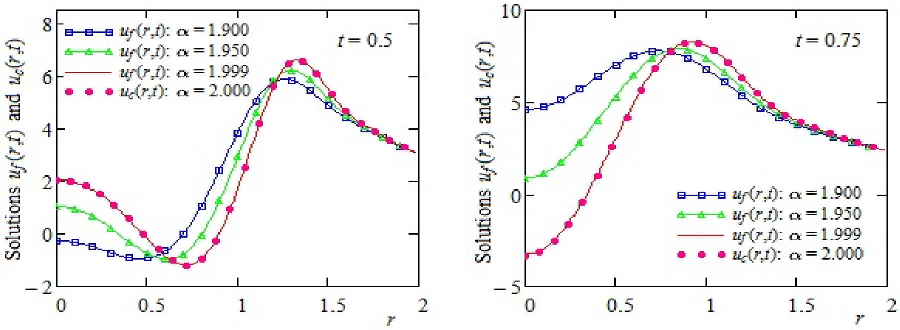

FIGURE 1. Convergence of the solution $u_f(r,t)$ to $u_c(r,t)$ versus r, for two values of the time t when the fractional parameter $\alpha \to 2$.

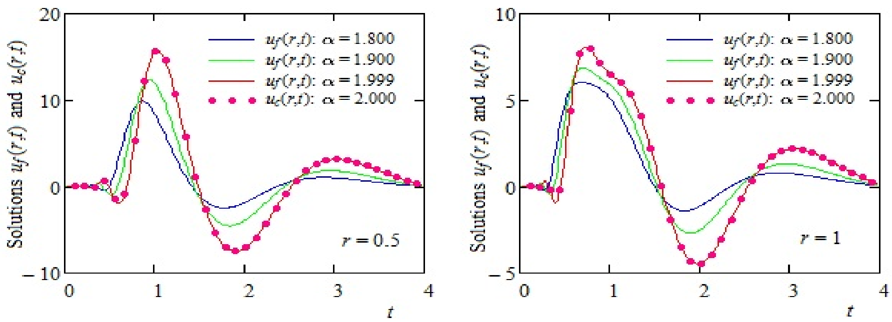

FIGURE 2. Convergence of the solution $u_f(r,t)$ to $u_c(r,t)$ versus t, for two values of the spatial variable r when the fractional parameter $\alpha \to 2$.

The variation of the general solution $u_f(r,t)$ with the fractional parameter α in different spatial positions and for two values of the time t is plotted in Fig 3. From this figure it results that $u_f(r,t)$ is a decreasing function with regard to both variables. It is almost constant for values of α from one up to a critical value α_c and then grows for increasing values of this parameter less or equal to two. All graphical representations have been here prepared for $R = 2$, $a = 3$, $b = 0.5$, $f(t) = 5\exp(-t)$.

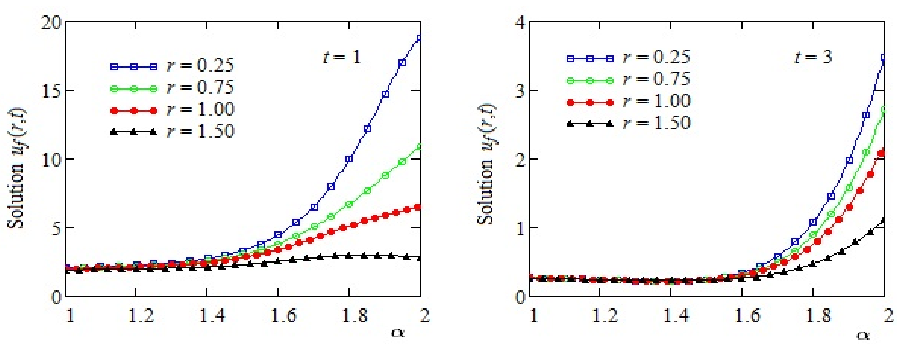

FIGURE 3. Profiles of solution $u_f(r,t)$ versus the fractional parameter $\alpha \in (1,2]$, for two values of the time t and different values of the spatial variable r.

Appendix

$$(A1)\qquad I_{1/2}(z)=\sqrt{\frac{2}{\pi z}}\sinh(z),\ K_{1/2}(z)=\sqrt{\frac{2}{\pi z}}\,e^{-z},$$

$$(A2)\qquad H(c,t)=L^{-1}\left\{\exp\left(-c\sqrt{s}\right)\right\}=\frac{c}{2t\sqrt{\pi t}}\exp\left(-\frac{c^2}{4t}\right),\mathrm{Re}(c^2)>0,$$

$$(A3)\qquad L^{-1}\left\{\frac{1}{s}\,\bar{g}\left(\frac{1}{s}\right)\right\}=\int_0^\infty J_0\left(2\sqrt{tz}\right)g(z)dz \text{ if } \bar{g}(s)=L\{g(t)\},$$

$$(A4)\qquad \int_0^\infty J_0\left(2\sqrt{tz}\right)\delta(z)dz=J_0(0)=1,\ \frac{d}{dz}\left[J_0(u(z))\right]=-J_1(u(z))\,\frac{du(z)}{dz}$$

$$(A5)\qquad \begin{aligned} L^{-1}\{\exp(-ds)\} &= \delta(t-d), L^{-1}\left\{\exp\left[d\left(s-\sqrt{s^2-(ib)^2}\right)\right]-1\right\} \\ &= \frac{ibd}{\sqrt{t^2+2dt}}\,I_1\left(ib\sqrt{t^2+2dt}\right), \end{aligned}$$

$$(A6)\qquad f(t)*\delta(t-d)=f(t-d),$$

$$(A7)\qquad I_1(iz)=\frac{1}{i}\,J_1(-z),\ \ J_1(-z)=-J_1(z).$$

$$(A8)\qquad \lim_{t\to\infty}f(t)=\lim_{s\to0}s\bar{f}(s).$$

References

[1] Abuteen, E.; Freihat, A.; Al-Smadi, M.; Khalil, H.; Khan, R. A. Approximate series solution of nonlinear, fractional Klein-Gordon equations using fractional reduced differential transform method. *J. Math. Stat.* **12** (2016), no. 1, 23–33, DOI: 10.3844/jmssp.2016.23.33.

[2] Atanackovic, T. M.; Pilipovic, S.; Stankovic, B.; Zorica, D. *Fractional Calculus with Applications in Mechanics: Vibrations and Diffusion Processes.* Wiley-ISTE, London, 2014, ISBN: 978-1-848-21417-0.

[3] Bansu, H.; Kumar, S. Numerical solution of space-time fractional Klein-Gordon equation by radial basis functions and Chebyshev polynomials. *Int. J. Appl. Comput. Math.* **7** (2021), no. 5, 201, DOI: 10.1007/s40819-021-01139-7.

[4] Baleanu, D.; Sajjadi, S. S.; Jajarmi, A.; Defterli, O.; Asad, J. H. The fractional dynamics of a linear triatomic molecule. *Rom. Rep. Phys.* **73** (2021), 105.

[5] Caputo, M. Linear model of dissipation whose Q is almost frequency independent-II. *Geophys. J. Int.* **13** (1967), no. 5, 529–539, DOI:10.1111/j.1365-246x.1967.tb02303.x.

[6] Datsko, B.; Podlubny, I.; Povstenko, Y. Time-fractional diffusion-wave equation with mass absorption in a sphere under harmonic impact. *Mathematics* **7** (2019), no. 5, 433, DOI: 10.3390/math7050433.

[7] Eiderman, V. *An Introduction to Complex Analysis and the Laplace Transform.* Chapman and Hall/CRC, New York, 2021, ISBN-13 978-1000511123, eBook-ISBN 9780367810283.

[8] El-Dib, Y. O.; Elgazery, N. S.; Mady, A. A. Nonlinear dynamical analysis of a time-fractional Klein–Gordon equation. *Pramana-J. Phys.* **95** (2021), 154, DOI: 10.1007/s12043-021-02184-z.

[9] Kragh, H. Erwin Schrödinger and the wave equation: The crucial phase. *Centaurus* **26** (1982), 154–197, DOI: 10.1111/j.1600-0498.1982.tb00659.x.

[10] Gravel, P.; Gauthier, C. Classical applications of the Klein–Gordon equation. *Am. J. Phys.* **79** (2011), no. 5, 447–453, DOI: 10.1119/1.3559500.

[11] Hristov, J. Double integral-balance method to the fractional subdiffusion equation: Approximate solutions, optimization problems to be resolved and numerical simulations. *J. Vib. Control* **23** (2017), no. 17, 2795–2818, DOI: 10.1177/1077546315622773.

[12] Kheiri, H.; Shahi, S.; Mojaver, A. Analytical solutions for the fractional Klein-Gordon equation. *Comput. Methods Differ. Equ.* **2** (2014), no. 2, 99–114.

[13] Khan, N. A.; Rasheed, S. Analytical solutions of linear and nonlinear Klein-Fock-Gordon equation. *Nonlinear Eng.* **4** (2015), no. 1, 43-–48, DOI: 10.1515/nleng-2014-0028.
[14] Shen, J.; Gu, X. M. Two finite difference methods based on an H2N2 interpolation for two-dimensional time fractional mixed diffusion equation and diffusion-wave equations. *Discrete Continuous Dyn. Syst. Ser. B* **27** (2022), no. 2, 1179-1207, DOI: 10.3934/dcdsb.2021086.
[15] Tarasov, V. E. *Fractional Dynamics: Applications of Fractional Calculus to Dynamics of Particles, Fields and Media.* Springer. Heidelberg, Dordrecht, London, New York, 2010, ISBN 97R-1-042-14002-0, e-ISBN 978-3-642-14003-7.
[16] Watson, G. N. *A treatise on the Theory of Bessel Functions.* 2nd ed., Cambridge, University Press, 1966.

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