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Dedicated to Prof. Emeritus Mihail Megan on the occasion of his 75^{th} *anniversary*

A linear quadratic tracking problem for stochastic systems controlled by impulses. The finite horizon time case

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ABSTRACT. We investigate a problem to solve the linear quadratic tracking problem for stochastic systems controlled by impulses. Two optimal control problems are investigated where the different objective functions are minimized. Explicit formulae for optimal controls are developed. The optimal controllers are computed based on the solution of the backward jump matrix Lyapunov type linear differential equations.

1. INTRODUCTION

The study of impulsive systems have important theoretical and practical value. A real life system may encounter at certain time moments some abrupt changes and from this reason cannot be considered continuously. There are various examples of such systems, see [12]. This impulsive phenomenon, represents the framework of impulsive differential equations, see [2], [3]. In control area, many impulsive control methods have been developed into the framework of optimal control, the so called impulsive control. In fact, optimal control assure the change of states at certain instants instantaneously. Reference [13] present an overview for recent progress on impulsive control systems.

A linear quadratic tracking problem is an important control task. The tracking control constructs a model of controller with state as close as possible to the given reference signal. It can be applied in a wide range of areas such as process control [14], in control of vibrations, diffusion and many other mechanical problems [11]. The primary objective of linear quadratic tracking control is to find an optimal controller to minimize the deviation of the controlled output $z(\cdot)$ from the reference signal $r(\cdot)$. Tracking problems were studied for stochastic and nonlinear systems intensively in the past several years. We can mention here few references of interest, i.e. [4, 5, 6, 15].

The problem of minimization of the mean square value of the deviation of a random signal $z(t_f)$ from a given target is analysed in [7]. In this paper, we investigate a problem to solve the linear quadratic tracking problem for stochastic systems controlled by impulses. We consider a system of affine stochastic differential equations of Itô type controlled by impulses and introduce a set of the admissible controls. Two optimal control problems are investigated where the different objective functions are minimized. Explicit formulae for optimal controls are developed. The optimal controllers are computed based on the solution of the backward jump matrix Lyapunov type linear differential equations.

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2. THE PROBLEM

Let us consider the controlled system described by:

$$\text{(2.1a)}\quad dx(t) = (A_0(t)x(t) + f_0(t))dt + (A_1(t)x(t) + f_1(t))dw(t), \quad kh < t \leq (k+1)h$$

$$x(kh^+) = A_{d0}(k)x(kh) + B_{d0}(k)u(k) + g_0(k) +$$

$$\text{(2.1b)}\quad + w_d(k)(A_{d1}(k)x(kh) + B_{d1}(k)u(k) + g_1(k)), \quad k = 0, 1, \dots, N,$$

$$\text{(2.1c)}\quad z(t) = C(t)x(t), \quad t \in [0, T]$$

where $h > 0$ is fixed and $T > 0$ is such that $Nh < T \leq (N+1)h$. In (2.1), $x(t) \in \mathbb{R}^n$ are the state parameters, $z(t) \in \mathbb{R}^{n_z}$ is the controlled output and $u(k) \in \mathbb{R}^m$ are the control parameters.

In (2.1a), $\{w(t)\}_{t\geq 0}$ is 1-dimensional Wiener process defined on the probability space $(\Omega, \mathfrak{F}, \mathcal{P})$ while in (2.1b) $\{w_d(k)\}_{k\geq 0}$ is a sequence of random variables defined on the same probability space $(\Omega, \mathfrak{F}, \mathcal{P})$. The system (2.1a)-(2.1b) is a system of affine stochastic differential equations of Itó type, controlled by impulses or, equivalently, an impulsive controlled stochastic system (ICSS).

In order to be able to provide a clear description of the class of admissible controls, let us introduce an assumption regarding the properties of the random noises involved in (2.1).

(A1)

(a) $\{w(t)\}_{t\geq 0}$ *is a 1-dimensional standard Wiener process, that is, it is a process of a Brownian motion with the properties:* $w(0) = 0$, $\mathbb{E}[w(t)] = 0$, $\mathbb{E}[(w(t) - w(s))^2] = t - s$, *for all* $t \geq s \geq 0$.

(b) $\{w_d(k)\}_{k\geq 0}$ *are random variables with the properties* $\mathbb{E}[w_d(k)] = 0$ *and*

$$\mathbb{E}[w_d(k)w_d(j)] = \begin{cases} 0, & \textit{if } k \neq j \\ 1, & \textit{if } k = j. \end{cases}$$

(c) $\{w(t)\}_{t\geq 0}$, $\{w_d(k)\}_{k\geq 0}$ *are independent stochastic processes.*

For more details regarding definitions and properties of standard Wiener processes we refer to [9] and [10]. Throughout the paper $\mathbb{E}[\cdot]$ stands for the mathematical expectation. For each $t \geq 0$, $\mathcal{F}_t \subset \mathfrak{F}$ stands for the sigma algebra generated by the random variables $w(s)$, $w_d(k)$ where s and k are such that $0 \leq s \leq t$, $k \in \{0, 1, \dots, N\}$ and $kh < t$. For each t, $\mathcal{F}_t$ is augmented with the set of the events $\mathfrak{F}^0 = \{\theta \in \mathfrak{F} \mid \mathcal{P}(\theta) = 0\}$.

The set $\mathcal{U}_{ad}$ of the admissible controls, consists of all sequences $\mathbf{u} = \{u(k)\}_{k\in\{0,1,\dots,N\}}$ with the properties that for each k, $u(k) : \Omega \to \mathbb{R}^m$ is the random vector $f_{kh}-$measurable and $\mathbb{E}[|u(k)|^2] < \infty$.

Let $r(\cdot) : [0, T] \to \mathbb{R}^{n_z}$ be a continuous vector valued function which will be called **reference signal**, or simple **reference**.

Our aim is that for a given initial state $x_0 \in \mathbb{R}^n$ to design a control $\tilde{\mathbf{u}} \in \mathcal{U}_{ad}$ which minimizes the deviation of the controlled output $z(\cdot)$ from the reference signal $r(\cdot)$. For a rigorous setting of the tracking problem briefly described before we introduce two performance criteria:

$$\text{(2.2)}\quad J_1(x_0; \mathbf{u}) = \mathbb{E}\left[|z(T; x_0, \mathbf{u}) - \zeta|^2 + \int_0^T |z(t; x_0, \mathbf{u}) - r(t)|^2 dt\right] + \sum_{k=0}^{N} \mathbb{E}[u^\top(k)R(k)u(k)]$$

and

$$J_2(x_0;\mathbf{u}) = \mathbb{E}\left[|z(T;x_0,\mathbf{u})-\zeta|^2 + \int_0^T |z(t;x_0,\mathbf{u})-r(t)|^2 dt\right]$$
$$+\sum_{k=0}^{N}\mathbb{E}[(u(k)-v(k))^\top R(k)(u(k)-v(k))] \qquad (2.3)$$

where $z(t;x_0,\mathbf{u}) = C(t)x(t;x_0,\mathbf{u})$, $t \in [0,T]$, $x(\cdot;x_0,\mathbf{u})$ being the solution of the ICSS (2.1a)-(2.1b) determined by the control $\mathbf{u} \in \mathcal{U}_{ad}$ and satisfying $x(0;x_0,\mathbf{u}) = x_0$.

In (2.2) and (2.3), $\zeta \in \mathbb{R}^{n_z}$ have to be viewed as a target for the terminal value of the controlled output of the ICSS (2.1). In (2.2), the term $\sum_{k=0}^{N}\mathbb{E}[u^\top(k)R(k)u(k)]$ represents a measure of the control effort, while in (2.3), the term $\sum_{k=0}^{N}\mathbb{E}[(u(k)-v(k))^\top R(k)(u(k)-v(k))]$ can be regarded as a penalization of the deviation of the controls $u(k)$ from a given reference $\mathbf{v} = \{v(k)\}_{0\le k\le N}$.

In this work, the finding of the control $\tilde{\mathbf{u}}_l \in \mathcal{U}_{ad}$ which achieves the best tracking of the reference $r(\cdot)$ by the controlled output $z(\cdot;x_0,\mathbf{u})$ of the ICSS (2.1), is done solving one of the two optimal problems asking for the designing of an admissible control which minimizes the cost $J_l(x_0;\cdot)$, $l = 1,2$ along the trajectories of the ICSS (2.1) determined by the admissible controls $\mathbf{u} \in \mathcal{U}_{ad}$. With other words, for each $l \in \{1,2\}$ find the control $\tilde{\mathbf{u}}_l \in \mathcal{U}_{ad}$ which solves the following problem:

$$\textbf{Problem l.}\quad J_l(x_0;\tilde{\mathbf{u}}_l) = \min_{\mathbf{u}\in\mathcal{U}_{ad}} J_l(x_0;\mathbf{u}) \qquad (2.4)$$

Explicit formulae of the controls $\tilde{\mathbf{u}}_l$ which solve **Problem l**, $l \in \{1,2\}$, will be derived in Section 4.

First, in Section 3, we analyse the problem of the global existence on the whole interval $[0,T]$ of the solutions with given terminal values of some backward differential equations with impulses.

For the developments in the next sections, we assume that the coefficients of the system (2.1) and the weights matrices from the performance criteria (2.2) and (2.3) are satisfying the assumption:

(A2)

(a) $A_j(\cdot) : [0,T] \to \mathbb{R}^{n\times n}$, $f_j(\cdot) : [0,T] \to \mathbb{R}^n$, $j = 0,1$, $C(\cdot) : [0,T] \to \mathbb{R}^{n_z\times n}$ *are given continuous functions.*

(b) For each $0 \le k \le N$, $A_{dj}(k) \in \mathbb{R}^{n\times n}$, $B_{dj}(k) \in \mathbb{R}^{n\times m}$, $g_j(k) \in \mathbb{R}^n$, $j = 0,1$, *are known.*

(c) $R(k) \in \mathcal{S}_m$, *is such that* $R(k) > 0$, $0 \le k \le N$.

Throughout this work $\mathcal{S}_q \subset \mathbb{R}^{q\times q}$ is the linear space of symmetric matrices of size $q \times q$.

3. TWO KINDS OF BACKWARD LINEAR DIFFERENTIAL EQUATIONS WITH JUMPS

In this section we consider two types of backward jump linear differential equations whose solutions will be used in the description of the optimal controls $\tilde{\mathbf{u}}_l$ for the **Problem l**, $l = 1,2$, given in (2.4).

3.1. **A backward jump matrix linear differential equation with Riccati type jumping operators.** Based on the matrix coefficients of the ICSS (2.1) we define the following backward jump matrix linear differential equation (BJMLDE) on the space $\mathcal{S}_n$:

$$\begin{aligned} &(3.5a) \quad -\dot{Y}(t) = A_0^\top(t)Y(t) + Y(t)A_0(t) + A_1^\top(t)Y(t)A_1(t) + C^\top(t)C(t), \\ &\qquad kh \le t < (k+1)h \end{aligned}$$

$$\begin{aligned} &(3.5b) \quad Y(kh^-) = \sum_{j=0}^{1} A_{dj}^\top(k)Y(kh)A_{dj}(k) - \left(\sum_{j=0}^{1} A_{dj}^\top(k)Y(kh)B_{dj}(k)\right) \\ &\cdot \left(R(k) + \sum_{j=0}^{1} B_{dj}^\top(k)Y(kh)B_{dj}(k)\right)^{-1} \left(\sum_{j=0}^{1} B_{dj}^\top(k)Y(kh)A_{dj}(k)\right), \quad 0 \le k \le N. \end{aligned}$$

Since the right hand side from (3.5b) has the form of a Riccati type operator arising in connection with a discrete-time linear quadratic optimal control problem, we shall call the BJMLDE (3.5) as a BJMLDE with **Riccati type jumping operator**. Before to study the extensibility of the solutions with given terminal values of a BJMLDE of type (3.5) on the whole interval $[0, T]$, we recall several auxiliary issues already known. For more details we refer the reader to Chapter 2 from [6].

First, let us remark that $\mathcal{S}_n$ has a Hilbert space structure induced by the inner product

$$\langle X, Y\rangle = \ Tr\ [XY], \tag{3.6}$$

for all $X, Y \in \mathcal{S}_n$, where $Tr\ [\cdot]$ is the trace operator. Moreover, $\mathcal{S}_n$ is a real ordered Hilbert space. The order relation on $\mathcal{S}_n$ is induced by the convex cone

$$\mathcal{S}_n^+ = \{X \in \mathcal{S}_n \mid X \ge 0\}.$$

Here $X \ge 0$ means that X is a positive semidefinite matrix.

For each $t \in [0, T]$ we consider the linear operator $X \to \mathcal{L}(t)[X] : \mathcal{S}_n \to \mathcal{S}_n$ defined by

$$\mathcal{L}(t)[X] = A_0(t)X + XA_0^\top(t) + A_1(t)XA_1^\top(t), \quad \text{for all} \quad X \in \mathcal{S}_n. \tag{3.7}$$

The adjoint $\mathcal{L}^*(t)[\cdot]$ of the operator $\mathcal{L}(t)[\cdot]$ with respect to the inner product (3.6) is described by

$$\mathcal{L}^*(t)[Y] = A_0^\top(t)Y + YA_0(t) + A_1^\top(t)YA_1(t). \tag{3.8}$$

With this notation, the differential equation (3.5a) may be rewritten as:

$$-\dot{Y}(t) = \mathcal{L}^*(t)[Y(t)] + C^\top(t)C(t), \quad kh \le t < (k+1)h. \tag{3.9}$$

Let $\mathbf{T}(t, \tau) : \mathcal{S}_n \to \mathcal{S}_n$ be the linear evolution operator on $\mathcal{S}_n$ defined by the linear differential equation

$$\dot{X}(t) = \mathcal{L}(t)[X(t)]. \tag{3.10}$$

This means that

$$\mathbf{T}(t, t_0)[X_0] = X(t; t_0, X_0),$$

for all $t, t_0 \in [0, T]$, $X_0 \in \mathcal{S}_n$, where $X(\cdot; t_0, X_0)$ is the solution of the linear differential equation on $\mathcal{S}_n$ (3.10) which satisfy the initial condition $X(t_0; t_0, X_0) = X_0$.

From Theorem 2.6.1 from [6] applied to the special case of the differential equation (3.10), we obtain:

Corollary 3.1. *Under the assumption **(A2)** (a), both $\mathbf{T}(t, t_0)[\cdot]$ and its adjoint $\mathbf{T}^*(t, t_0)[\cdot]$ are positive operators on the ordered Hilbert space $(\mathcal{S}_n, \mathcal{S}_n^+)$. This means that*

$$\mathbf{T}(t, t_0)[X_0] \ge 0$$

and

$$\mathbf{T}^*(t,t_0)[X_0] \geq 0$$

for all $0 \leq t_0 \leq t \leq T$*, whenever* $X_0 \in \mathcal{S}_n^+$.

Now we are in position to prove the main result of this section.

Proposition 3.1. *Let* $Y(\cdot;T,Y_T)$ *be the solution of the BJMLDE (3.5) with the terminal value* $Y(T;T,Y_T) = Y_T$*. Under the assumption* ***(A2)*** *the solution* $Y(\cdot;T,Y_T)$ *is defined on the whole interval* $[0,T]$ *if* $Y_T \geq 0$*. Moreover, this solution has the properties:*

(i) $Y(t;T,Y_T) \geq 0$*, for all* $t \in [0,T]$;

(ii) $Y(\cdot;T,Y_T)$ *is differentiable on each interval* $(kh,(k+1)h)$, $0 \leq k \leq N-1$ *and on* (Nh,T) *and is right continuous on each* kh *with possible discontinuities in* kh, $0 \leq k \leq N$.

Proof. Using the version (3.9) of (3.5a) we obtain the following representation formula of the solution $Y(\cdot;T,Y_T)$ on the interval $[kh,(k+1)h)$:

$$Y(t;T,Y_T) = \mathbf{T}^*((k+1)h,t)[Y((k+1)h^-;T,Y_T)] + \int_t^{(k+1)h} \mathbf{T}^*(s,t)[C^\top(s)C(s)]ds. \tag{3.11}$$

When $k = N$ in (3.11), $(k+1)h$ will be T. From (3.5b) and (3.11) one sees that the extensibility of the solution $Y(\cdot;T,Y_T)$ on each of the intervals $[kh,(k+1)h)$ depends upon the invertability of the matrices

$$R(k+1) + \sum_{j=0}^{1} B_{dj}^\top(k+1)Y((k+1)h;T,Y_T)B_{dj}(k+1), \quad 0 \leq k \leq N-1.$$

The right hand side of (3.5b) is the Schur complement of the 22-block in the matrix

$$\Gamma(k) = \begin{pmatrix} \sum\limits_{j=0}^{1} A_{dj}^\top(k)Y(kh)A_{dj}(k) & \sum\limits_{j=0}^{1} A_{dj}^\top(k)Y(kh)B_{dj}(k) \\ \sum\limits_{j=0}^{1} B_{dj}^\top(k)Y(kh)A_{dj}(k) & R(k) + \sum\limits_{j=0}^{1} B_{dj}^\top(k)Y(kh)B_{dj}(k) \end{pmatrix}.$$

We rewrite this matrix in the form

$$\Gamma(k) = \sum_{j=0}^{1} (A_{dj}(k) \;\; B_{dj}(k))^\top Y(kh)(A_{dj}(k) \;\; B_{dj}(k)) + \begin{pmatrix} 0 & 0 \\ 0 & R(k) \end{pmatrix}. \tag{3.12}$$

From (3.11) for $k = N$ we obtain via Corollary 3.1 that

$$Y(t;T,Y_T) \geq 0, \quad \text{for all } \; Nh \leq t \leq T.$$

Hence,

$$R(N) + \sum_{j=0}^{1} B_{dj}^\top(N)Y(Nh)B_{dj}(N) > 0$$

and from (3.12) written for $k = N$ we have $\Gamma(N) \geq 0$. Applying, for example Lemma 2.3 from [8] (or Theorem 1 from [1]) in the case of the matrix $\Gamma(N)$ we deduce that

$$\sum_{j=0}^{1} A_{dj}^\top(N)Y(Nh;T,Y_T)A_{dj}(N) - \left(\sum_{j=0}^{1} A_{dj}^\top(N)Y(Nh;T,Y_T)B_{dj}(N)\right)$$
$$\cdot \left(R(N) + \sum_{j=0}^{1} B_{dj}^\top(N)Y(Nh;T,Y_T)B_{dj}(N)\right)^{-1} \left(\sum_{j=0}^{1} B_{dj}^\top(N)Y(Nh;T,Y_T)A_{dj}(N)\right) \geq 0.$$

Thus (3.5b) written for $k = N$ yields

$$Y(Nh^-; T, Y_T) \geq 0.$$

Further, (3.11) written for $k = N - 1$ gives

$$Y(t; T, Y_T) \geq 0, \quad \text{for all} \quad (N-1)h \leq t \leq Nh.$$

Let us assume by induction that for a $k \in \{N, N-1, \dots, 1, 0\}$ we have that

$$Y(t; T, Y_T) \geq 0, \quad \text{for all} \quad t \in [kh, (k+1)h).$$

Based on (c) from assumption **(A2)** we deduce that

$$R(k) + \sum_{j=0}^{1} B_{dj}^{\top}(k) Y(kh; T, Y_T) B_{dj}(k) > 0 \tag{3.13}$$

and from (3.12) we may infer that $\Gamma(k) \geq 0$. Invoking again Lemma 2.3 from [8] (or Theorem 1 from [1]) in the case of the matrix $\Gamma(k)$ we obtain via (3.13) that the right hand side of (3.5b) is a positive semidefinite matrix. Hence,

$$Y(kh^-; T, Y_T) \geq 0.$$

This allows us to conclude via (3.11), written for k replaced by $k - 1$, that

$$Y(t; T, Y_T) \geq 0, \quad \text{for all} \quad t \in [(k-1)h, kh).$$

So, the inductive process allows us to conclude that $Y(\cdot; T, Y_T)$ can be extended to the whole interval $[0, T]$ and it is positive semidefinite. Thus the proof is complete because (ii) follows directly from (3.11). □

3.2. **A backward jump linear differential equation on the space R^n.** Beside the BJMLDE with Riccati type jumping operator of type (3.5), in the derivation of the optimal controls which solve the optimal control problems (2.4), an important role will be played by a backward jump linear differential equation of the form:

$$-\dot{\varphi}(t) = A_0^{\top}(t)\varphi(t) + h(t), \quad kh \leq t < (k+1)h \tag{3.14a}$$

$$\varphi(kh^-) = (A_{d0}(k) + B_{d0}(k) F^Y(k))^{\top} \varphi(kh) + \xi(k) \tag{3.14b}$$

where $F^Y(k)$ is associated to a solution $Y(t) = Y(t; T, Y_T)$ of the BJMLDE (3.5) by:

$$F^Y(k) \triangleq -\left(R(k) + \sum_{j=0}^{1} B_{dj}^{\top}(k) Y(kh) B_{dj}(k) \right)^{-1} \left(\sum_{j=0}^{1} B_{dj}^{\top}(k) Y(kh) A_{dj}(k) \right). \tag{3.15}$$

From Proposition 3.1 it follows that if the terminal value Y_T of the solution $Y(\cdot)$ lies in $\mathcal{S}_n^+$ then (3.13) holds and in this case, $F^Y(k)$ can be computed for any $0 \leq k \leq N$ via (3.15). In (3.14a), $h(\cdot) : [0, T] \to \mathbb{R}^n$ is a continuous vector valued function and in (3.14b), $\xi(k) \in \mathbb{R}^n$ are given vectors. In the construction of the optimal controls $h(\cdot)$ will have an explicit formula depending upon the reference signal $r(\cdot)$ as well as the functions $f_j(\cdot)$ arising in (2.1a) and $\xi(k)$ will have an explicit formula involving the vectors $g_j(k)$ arising in (2.1b). Since (3.14) is a linear equation all solutions with given terminal values at T are defined on the whole interval $[0, T]$ provided that the matrices $F^Y(k)$ are defined for any $0 \leq k \leq N$.

4. The solution of the tracking problems

Let $\tilde{Y}(t) \triangleq Y(t;T,C^\top(T)C(T))$ be the solution of the BJMLDE (3.5) satisfying $\tilde{Y}(T) = C^\top(T)C(T)$. Applying Proposition 3.1 in the case $Y_T = C^\top(T)C(T) \geq 0$ we deduce that $\tilde{Y}(\cdot)$ is defined on the whole interval $[0,T]$ and satisfies

$$\tilde{Y}(t) \geq 0, \quad \text{for all} \quad t \in [0,T].$$

Let $\tilde{F}(\cdot) \in \mathbb{R}^{m\times n}$ be computed as in (3.15) for $Y(\cdot)$ replaced by $\tilde{Y}(\cdot)$. Let $\tilde{\varphi}(\cdot) : [0,T] \to \mathbb{R}^n$ be the solution of the following problem with given terminal value (TVP)

$$\begin{aligned}
&(4.16a) \qquad -\dot{\varphi}(t) = A_0^\top(t)\varphi(t) + C^\top(t)r(t) - \tilde{Y}(t)f_0(t) - A_1^\top(t)\tilde{Y}(t)f_1(t),\\
&\qquad\qquad\qquad kh \leq t < (k+1)h\\
&(4.16b) \qquad \varphi(kh^-) = (A_{d0}(k) + B_{d0}(k)\tilde{F}(k))^\top\varphi(kh)\\
&\qquad\qquad\qquad - \sum_{j=0}^{1}(A_{dj}(k) + B_{dj}(k)\tilde{F}(k))^\top\tilde{Y}(kh)g_j(k)\\
&(4.16c) \qquad \varphi(T) = C^\top(T)\zeta.
\end{aligned}$$

Further, employing the functions $\tilde{Y}(\cdot)$ and $\tilde{\varphi}(\cdot)$ we introduce a new function $\tilde{\mu}(\cdot) : [0,T] \to \mathbb{R}$ defined by

$$\begin{aligned}
&(4.17) \qquad \dot{\tilde{\mu}}(t) + |r(t)|^2 - 2f_0^\top(t)\tilde{\varphi}(t) + f_1^\top(t)\tilde{Y}(t)f_1(t) = 0, \quad t \in [0,T)\\
&\qquad\qquad \tilde{\mu}(T) = |\zeta|^2.
\end{aligned}$$

We set

$$(4.18) \qquad \Pi_d(k;\tilde{Y}(kh)) \triangleq R(k) + \sum_{j=0}^{1} B_{dj}^\top(k)\tilde{Y}(kh)B_{dj}(k), \quad 0 \leq k \leq N.$$

With this notation, (3.13) written for $Y(t)$ replaced by $\tilde{Y}(t)$ becomes:

$$(4.19) \qquad \Pi_d(k;\tilde{Y}(kh)) > 0.$$

4.1. **The solution of the first tracking problem.** We consider $\tilde{\mathbf{u}} = \{\tilde{u}(k)\}_{0\leq k\leq N}$ defined as

$$(4.20) \qquad \tilde{u}(k) \triangleq \tilde{F}(k)\tilde{x}(kh) + \tilde{\Psi}(k)$$

where $\tilde{F}(k)$ is computed as in (3.15) for $Y(\cdot)$ replaced by $\tilde{Y}(\cdot)$ and

$$(4.21) \qquad \tilde{\Psi}(k) \triangleq \Pi_d^{-1}(k;\tilde{Y}(kh)) \left(B_{d0}^\top(k)\tilde{\varphi}(kh) - \sum_{j=0}^{1} B_{dj}^\top(k)\tilde{Y}(kh)g_j(k) \right),$$

$k \in \{0,1,\ldots,N\}$. In (4.20) $\tilde{x}(kh)$ are the values at impulsive instants times $t_k = kh$ of the solution of the following initial value problem (IVP)

$$\begin{aligned}
&(4.22a) \quad dx(t) = (A_0(t)x(t) + f_0(t))dt + (A_1(t)x(t) + f_1(t))dw(t), \quad kh < t \leq (k+1)h\\
&(4.22b) \quad x(kh^+) = [A_{d0}(k) + B_{d0}(k)\tilde{F}(k) + w_d(k)(A_{d1}(k) + B_{d1}(k)\tilde{F}(k))]x(kh)\\
&\qquad\qquad + (B_{d0}(k) + w_d(k)B_{d1}(k))\Psi(k) + g_0(k) + w_d(k)g_1(k), \quad 0 \leq k \leq N,\\
&(4.22c) \quad x(0) = x_0.
\end{aligned}$$

Now we are in position to state and prove a result which provides the solution of the first tracking problem asking for the minimization of the objective function (2.2).

Theorem 4.1. *Under the assumptions **(A1)** and **(A2)**, the optimal control problem which is requiring the minimization of the objective function $J_1(x_0;\mathbf{u})$ along of the trajectories of the ICSS (2.1) determined by the controls $\tilde{\mathbf{u}} = \{\widetilde{u(k)}\}_{0\le k\le N}$. This control is described by (4.20)-(4.22). The minimal value of the performance criterion (2.2) is:*

$$
\begin{aligned}
J_1(x_0;\tilde{\mathbf{u}}) &= x_0^\top \tilde{Y}(0^-)x_0 - 2x_0^\top \tilde{\varphi}(0^-) + \tilde{\mu}(0) + \sum_{k=0}^{N}\Bigg[\sum_{j=0}^{1} g_j^\top(k)\tilde{Y}(kh)g_j(k) - 2g_0^\top(k)\tilde{\varphi}(kh) - \\
&- (B_{d0}^\top(k)\tilde{\varphi}(kh) - \sum_{j=0}^{1} B_{dj}^\top(k)\tilde{Y}(kh)g_j(k))^\top \Pi_d^{-1}(k;\tilde{Y}(kh))(B_{d0}^\top(k)\tilde{\varphi}(kh) \\
&- \sum_{j=0}^{1} B_{dj}^\top(k)\tilde{Y}(kh)g_j(k))\Bigg]
\end{aligned}
\tag{4.23}
$$

Proof. We consider the function $V(\cdot,\cdot):[0,T]\times\mathbb{R}^n\to\mathbb{R}$ defined by

$$V(t,x) = x^\top \tilde{Y}(t)x - 2x^\top \tilde{\varphi}(t) + \tilde{\mu}(t).$$

Applying the Ito formula (see Theorem 5.2.1 from [10]) on intervals of the form $[\tau_1,\tau_2]\subset[kh,(k+1)h]$ and letting $\tau_1\to kh$ and $\tau_2\to(k+1)h$ we obtain via (3.5a), (4.16a) and (4.17) that

$$
\begin{aligned}
&\int_{kh}^{(k+1)h} \mathbb{E}[|z(t;x_0,\mathbf{u}) - r(t)|^2]dt + \mathbb{E}[V((k+1)h^-, x((k+1)h;x_0,\mathbf{u}))] \\
&= \mathbb{E}[V(kh, x(kh^+;x_0,\mathbf{u}))]
\end{aligned}
\tag{4.24}
$$

for all $0\le k\le N$ with the convention that $(k+1)h = T$ when $k=N$. Substituting $x(kh^+;x_0,\mathbf{u})$ in (4.24), using (2.1b), (3.5b) and (4.16b) after some several algebraic calculations we obtain that

$$
\begin{aligned}
&\int_{kh}^{(k+1)h} \mathbb{E}[|z(t;x_0,\mathbf{u} - r(t)|^2]dt + \mathbb{E}[u^\top(k)R(k)u(k)] \\
&= \mathbb{E}[V(kh^-, x(kh;x_0,\mathbf{u}))] - \mathbb{E}[V((k+1)h^-, x((k+1)h;x_0,\mathbf{u}))] \\
&+ \mathbb{E}[(u(k)-\hat{u}(k))^\top \Pi_d(k,\tilde{Y}(kh))(u(k)-\hat{u}(k))] + \gamma(k)
\end{aligned}
\tag{4.25}
$$

for all $\mathbf{u}\in\mathcal{U}_{ad}$, where we denoted

$$\hat{u}(k) = \tilde{F}(k)x(kh;k_0,\mathbf{u}) + \tilde{\Psi}(k) \tag{4.26}$$

and

$$\gamma(k) := \sum_{j=0}^{1} g_j^\top(k)\tilde{Y}(kh)g_j(k) - 2g_0^\top\tilde{\varphi}(kh) - \tilde{\Psi}^\top(k)\Pi_d(k,\tilde{Y}(kh))\tilde{\Psi}(k). \tag{4.27}$$

Summing up (4.25) from $k=0$ to $k=N$ we obtain according to (2.2), that

$$
\begin{aligned}
J_1(x_0;\mathbf{u}) =& V(0^-,x_0) + \sum_{k=0}^{N}\mathbb{E}[(u(k)-\hat{u}(k))^\top \Pi_d(k,\tilde{Y}(kh))(u(k)-\hat{u}(k))] \\
&+ \sum_{k=0}^{N}\gamma(k)
\end{aligned}
\tag{4.28}
$$

for all $\mathbf{u} \in \mathcal{U}_{ad}$. Invoking (4.19) we deduce that

$$J_1(x_0, \mathbf{u}) \geq V(0^-, x_0) + \sum_{k=0}^{N} \gamma(k), \quad \text{for all } \ \mathbf{u} \in \mathcal{U}_{ad}. \tag{4.29}$$

On the other hand, from the uniqueness of the solution of the ICSS (2.1) we deduce that $x(\cdot; x_0, \tilde{\mathbf{u}})$ coincides to the solution $\tilde{x}(\cdot)$ of the IVP (4.22). Comparing (4.26) and (4.20) we may infer that $\hat{u}(k) = \tilde{u}(k)$, when $u(k) = \tilde{u}(k)$. Thus (4.28) yields:

$$J_1(x_0; \mathbf{u}) \geq J_1(x_0; \tilde{\mathbf{u}}), \quad \text{for all } \ \mathbf{u} \in \mathcal{U}_{ad}. \tag{4.30}$$

Hence, $\tilde{\mathbf{u}}$ described in (4.20)-(4.22) achieves the minimal value of (2.2). Moreover, comparing (4.21) and (4.27), we may conclude that the right hand side of (4.30) coincides to the right hand side of (4.23). The proof is complete. □

4.2. **The solution of the second tracking problem.** Setting $\nu(k) := u(k) - v(k)$, $0 \leq k \leq N$, we can see that the minimization of $J_2(x_0; \mathbf{u})$ along of the trajectories of the ICSS (2.1) is equivalent to the minimization of

$$J_3(x_0; \boldsymbol{\nu}) \triangleq \mathbb{E}[|z(T; x_0, \boldsymbol{\nu}) - \zeta|^2 + \int_0^T |z(t; x_0, \nu) - r(t)|^2 dt] + \sum_{k=0}^{N} \mathbb{E}[\nu^\top(k) R(k) \nu(k)] \tag{4.31}$$

along the trajectories of the following ICSS:

$$dx(t) = (A_0(t)x(t) + f_0(t))dt + (A_1(t)x(t) + f_1(t))dw(t), \quad kh < t \leq (k+1)h \tag{4.32a}$$

$$\begin{aligned} x(kh^+) = {} & A_{d0}(k)x(kh) + B_{d0}(k)\nu(k) + \tilde{g}_0(k) + w_d(k)(A_{d1}(k)x(kh) \\ & + B_{d1}(k)\nu(k) + \tilde{g}_1(k)), \quad k \in \{0, 1, \ldots, N\} \end{aligned} \tag{4.32b}$$

$$x(0) = x_0, \tag{4.32c}$$

where we have denoted

$$\tilde{g}_j(k) \triangleq g_j(k) + B_{dj}(k)v(k), \quad j = 0, 1. \tag{4.33}$$

So, to obtain the solution of the second tracking problem we can apply the result derived in Theorem 4.1 in the case of the problem asking for the minimization of the objective function (4.31) along the trajectories of the ICSS (4.32) determined by the set of admissible controls $\mathcal{U}_{ad}$.

First, let us update the IVPs (4.16) and (4.17) according to the coefficients of the ICSS (4.32)-(4.33). Thus TVP (4.16) is replaced by

$$\begin{aligned} -\dot{\varphi}(t) = {} & A_0^\top(t)\varphi(t) + C^\top(t)r(t) - \tilde{Y}(t)f_0(t) - A_1^\top(t)\tilde{Y}(t)f_1(t), \\ & kh \leq t < (k+1)h \end{aligned} \tag{4.34a}$$

$$\begin{aligned} \varphi(kh^-) = {} & (A_{d0}(k) + B_{d0}(k)\tilde{F}(k))^\top \varphi(kh) \\ & - \sum_{j=0}^{1} (A_{dj}(k) + B_{dj}(k)\tilde{F}(k))^\top \tilde{Y}(kh)(g_j(k) + B_{dj}(k)v(k)), \quad 0 \leq k \leq N, \end{aligned} \tag{4.34b}$$

$$\varphi(T) = C^\top(T)\zeta. \tag{4.34c}$$

If $\tilde{\tilde{\varphi}}(\cdot)$ is a solution of the TVP (4.34) we can define the function $\tilde{\tilde{\mu}}(\cdot) : [0, T] \to \mathbb{R}$ by

$$\begin{aligned} & \frac{d}{dt}\tilde{\tilde{\mu}}(t) + |r(t)|^2 - 2f_0^\top(t)\tilde{\tilde{\varphi}}(t) + f_1^\top(t)\tilde{Y}(t)f_1(t) = 0, \\ & \tilde{\tilde{\mu}}(T) = |\zeta|^2 \end{aligned} \tag{4.35}$$

which is the new version of (4.17). Let $\tilde{\tilde{\mathbf{u}}} = \{\tilde{\tilde{u}}(k)\}_{0\leq k\leq N}$ be defined by

$$\tilde{\tilde{u}}(k) \triangleq v(k) + \tilde{F}(k)\tilde{\tilde{x}}(kh) + \tilde{\tilde{\Psi}}(k) \tag{4.36}$$

where

$$\tilde{\tilde{\Psi}}(k) \triangleq \Pi_d^{-1}(k;\tilde{Y}(kh))\left(B_{d0}^\top(k)\tilde{\tilde{\varphi}}(kh) - \sum_{j=0}^{1} B_{dj}^\top(k)\tilde{Y}(kh)(g_j(k) + B_{dj}(k)v(k))\right), \tag{4.37}$$
$$k \in \{0,1,\ldots,N\}.$$

In (4.36), $\tilde{\tilde{x}}(kh)$ are the values at impulsive instant times $t_k = kh$ of the solution of the following ICSS:

$$dx(t) = (A_0(t)x(t) + f_0(t))dt + (A_1(t)x(t) + f_1(t))dw(t), \quad kh < t \leq (k+1)h \tag{4.38a}$$
$$\begin{aligned} x(kh^+) = {} & [A_{d0}(k) + B_{d0}(k)\tilde{F}(k) + w_d(k)(A_{d1}(k) + B_{d1}(k)\tilde{F}(k))]x(kh) \\ & + B_{d0}(k)\tilde{\tilde{\Psi}}(k) + \tilde{g}_0(k) + w_d(k)(B_{d1}(k)\tilde{\tilde{\Psi}}(k) + \tilde{g}_1(k)), \quad k \in \{0,1,\ldots,N\} \end{aligned} \tag{4.38b}$$
$$x(0) = x_0, \tag{4.38c}$$

Now, we are in position to state the result which provides the optimal control for the second tracking problem investigated in this work.

Theorem 4.2. *Under the assumptions **(A1)** and **(A2)**, the tracking problem asking for the minimization of the objective function $J_2(x_0;\mathbf{u})$ along the trajectories of the ICSS (2.1) determined by the controls $\mathbf{u} \in \mathcal{U}_{ad}$ has a unique optimal control. The optimal control $\tilde{\tilde{\mathbf{u}}} = \{\tilde{\tilde{u}}(k)\}_{0\leq k\leq N}$ is described by (4.36)-(4.38). The minimal value of the cost functional is:*

$$\begin{aligned} J_2(x_0;\tilde{\tilde{\mathbf{u}}}) = {} & x_0^\top\tilde{Y}(0^-)x_0 - 2x_0^\top\tilde{\tilde{\varphi}}(0^-) + \tilde{\tilde{\mu}}(0) \\ & + \sum_{k=0}^{N}\Big[(g_j(k) + B_{dj}(k)v(k))^\top\tilde{Y}(kh)(g_j(k) + B_{dj}(k)v(k)) \\ & -2(g_0(k) + B_{d0}(k)v(k))^\top\tilde{\tilde{\varphi}}(kh) - \tilde{\tilde{\Psi}}^\top(k)\Pi_d(k;\tilde{Y}(kh))\tilde{\tilde{\Psi}}(k)\Big]. \end{aligned}$$

Proof. The proof is obtained directly applying Theorem 4.1 to the auxiliary tracking problem described by the objective function (4.31) and the controlled system (4.32). □

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