CARPATHIAN J. MATH.
Volume **38** (2022), No. 3,
Pages 737 -762

Online version at https://semnul.com/carpathian/
Print Edition: ISSN 1584 - 2851; Online Edition: ISSN 1843 - 4401
DOI: https://doi.org/10.37193/CJM.2022.03.18

Dedicated to Prof. Emeritus Mihail Megan on the occasion of his 75^{th} anniversary

Admissibility and Polynomial Dichotomy of Discrete Nonautonomous Systems

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ABSTRACT. We give new admissibility criteria for dichotomic behaviours of discrete nonautonomous systems, in infinite dimensional spaces. First, we present admissibility conditions for uniform and exponential dichotomy. Next, our study is focused on polynomial dichotomy, providing new characterizations for this notion by means of some double admissibilities. We obtain two categories of criteria for polynomial dichotomy, based on input-output conditions imposed to some suitable systems such that, for each one, the input sequences belong to certain ℓ^p -spaces and the outputs are bounded. We point out the importance of the assumptions regarding the complementarity of the stable subspaces at the initial time and we also discuss the relevance of the concept of solvability (unique or not) in the admissibility criteria for polynomial dichotomies on the half-line. All the results are obtained in the general case, without any additional hypotheses on the systems coefficients and without assuming any growth type properties for the associated propagators. Furthermore, as an application of the admissibility results we establish a robustness property of the polynomial dichotomy under small perturbations.

1. INTRODUCTION

The studies on the dichotomic behaviors have a long and rich history, the results in this area significantly contributing to the development of the asymptotic theory of dynamical systems (see [1–9, 11, 12, 14–20, 24–28, 31, 33–39, 41–51, 53–65, 68–72]). In the past decades, the great majority of the works on this topic were focused on *exponential dichotomies* of uniform or nonuniform type (see [1–4, 6, 11, 12, 17, 20, 25, 27, 31, 33, 38, 39, 41, 42, 46, 48–50, 55–64, 68–72]). For all that, as we pointed out in [24], in certain situations, a dynamical system may exhibit a splitting of the state space into (closed, invariant) stable and unstable subspaces, but with non-exponential rates in describing stability and instability. In this framework, some of the most representative asymptotic behaviors, which are not of an exponential nature, are those of polynomial type (see [5,7–9,18,19,24,29,30,51,52] and the references therein). Thus, we emphasize that, in the case of the dichotomic behaviors, in contrast with the concepts of exponential dichotomy, in the notions of *polynomial dichotomy* the rates of contraction and expansion are of polynomial type (see [5, 7–9, 18, 19, 24, 51]).

The concepts of nonuniform polynomial dichotomy were introduced (in slight different versions) by Barreira and Valls in [5] and respectively by Bento and Silva in [7, 8]. Dragičević defined in [18] a general notion of polynomial dichotomy with respect to a sequence of norms (for discrete-time nonautonomous dynamics) and completely characterized it via a discrete admissibility property. We stress that the notion of polynomial dichotomy with respect to a sequence of norms includes both the notions of polynomial

Received: 27.03.2022. In revised form: 04.07.2022. Accepted: 04.07.2022
1991 *Mathematics Subject Classification.* 34D05, 34D09, 93C05, 93D25, 39A05, 37D05, 34E10 .
Key words and phrases. *ordinary dichotomy, exponential dichotomy, polynomial dichotomy, discrete nonautonomous system, input-output system, admissibility, robustness.*
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dichotomy and respectively of nonuniform polynomial dichotomy as particular cases. In the case of continuous-time nonautonomous systems, Dragičević introduced in [19] the notion of polynomial dichotomy with respect to a family of norms and characterized it in terms of admissibility relative to an integral equation. Other notions of polynomial stabilities, instability and expansiveness were explored by Hai in [29, 30]. The importance of the polynomial behaviors is certified even more by the recent studies on generalized dichotomies (see Silva [65] and the references therein). It should be mentioned here that, over more than a decade, various studies on the polynomial behaviors of dynamical systems both in nonautonomous and variational case were coordinated by Megan (see [9, 51, 52] and the references therein). For a more detailed presentation of the history of this topic and connections between the dichotomy concepts we refer to Dragičević, Sasu and Sasu [24].

Among the most important methods in this area we mention the input-output techniques, or the so-called *admissibility methods*, which proved their effectiveness particularly in studying dichotomies and provided interesting answers to open problems regarding nonautonomous dynamics (see Aulbach and Minh [1], Barreira, Dragičević and Valls [2–4], Chicone and Latushkin [11], Chow and Leiva [12], Dragičević [17–19], Dragičević, Zhang and Zhou [25], Elaydi and Janglajew [27], Huy and Minh [33], Megan, Sasu and Sasu [38–40], Minh, Räbiger and Schnaubelt [42], Palmer [44–46], Pliss and Sell [50], Sasu and Sasu [55, 59, 61–63], Sasu [56], Sasu [57, 58], Sasu, Babuţia and Sasu [60], Silva [65], Zhang [68], Zhou and Zhang [70], Zhou, Lu and Zhang [71]). Even though technically these methods trace back to the pioneering works of Perron [47] and Li [35], for the historical origins of the *admissibility* notions and related methods we refer to the landmark works of Massera and Schäffer [36, 37] and Coffman and Schäffer [14] as well as to the books of Daleckĭi and Kreĭn [16], Coppel [15] and Henry [31]. The admissibility methods were presented for both nonautonomous and variational systems in the remarkable monograph of Chicone and Latushkin [11] from the perspective of the theory of evolution semigroups. For thorough presentations of the evolution of admissibility methods across the past decades we refer to the book of Barreira, Dragičević and Valls [4] and to the recent works Dragičević, Sasu and Sasu [20–22], Dragičević, Sasu, Sasu and Singh [23], Dragičević, Zhang and Zhou [25], Sasu and Sasu [61–63], Zhou and Zhang [70], Zhou, Lu and Zhang [71].

Despite the advances made in the admissibility theory so far, the studies on the polynomial behaviors via admissibility methods represented a challenging topic that required new and different approaches compared with those previously used in the literature (see Dragičević [18, 19], Hai [29, 30]). For instance, it should be noted that in the first studies devoted to admissibilities for polynomial dichotomies (see [18, 19]) the structures of the input-output systems were different from the classic ones. Thus, it turned out that the expression of the input-output operator has to be modified accordingly (see [18]). Consequently, the admissibility notions were distinct compared with those generally used when exploring a dichotomy of a nonautonomous system - see the approaches in [18] compared with those in [39,55,56,61] and respectively the method (and the integral equation considered) in [19] versus the admissibility concepts (and the corresponding integral equations) employed in [38, 42, 59, 60]. Similarly, nontrivial technical changes must be done when using admissibility conditions to study other polynomial behaviors of nonautonomous systems (such as polynomial stability, polynomial expansiveness), as it can be seen by comparing the approaches in [29, 30] with the methods (and the input-output systems) considered in [1, 21, 40, 42]. In this context, for the forthcoming studies, the natural question arises whether one can use the "classic structure" for the input-output systems when exploring a polynomial behavior and, if so, which would be the new requirements regarding the input or output spaces. Another question is whether the solvability of the control system should be unique or not. One of the goals in what follows will be to answer these questions in the case of (uniform) polynomial dichotomy.

The aim of this paper is to provide new characterizations for polynomial dichotomy of nonautonomous systems in infinite dimensional spaces by means of some admissibility conditions. We consider a discrete nonautonomous system (A) on the half-line and we associate to it an input-output system (C_A) (see Section 3 and the notations therein). Based on the results in [61], first we present conditions for ordinary and exponential dichotomy that rely on the solvability of the system (C_A), assuming that the initial stable subspace is complemented. Furthermore, we describe the dichotomy projections in terms of the stable subspaces and respectively of the initial unstable subspace. After that, we show that certain unique solvabilities of (C_A) ensure the complementarity of the stable subspace at the initial time. Consequently, we give new necessary and sufficient conditions for ordinary and exponential dichotomy via some unique solvabilities with respect to a fixed initial unstable subspace.

Next, we introduce a new admissibility method and two categories of characterizations for polynomial dichotomy. Given a discrete nonautonomous system (A), for every $h \in \mathbb{N}, h \geq 2$, we consider a discrete nonautonomous system (Q^h) whose coefficients are expressed in terms of the propagator associated to (A) at some well-chosen moments of time (see Section 4 and the notations therein). Then, besides the input-output system (C_A) initially studied, we also consider the input-output system (C_{Q^h}) associated to (Q^h). We obtain for the first time a characterization of polynomial dichotomy via double admissibilities relative to the input-output systems (C_A) and (C_{Q^h}), assuming that certain stable subspaces at the initial time are complemented. This is achieved via a deep analysis of the connections between the underlying families of projections and by applying a result recently obtained in [24]. Finally, we give a new characterization of polynomial dichotomy by means of two unique solvabilities of the associated input-output systems (C_A) and (C_{Q^h}) with respect to a fixed initial unstable subspace.

All the results are obtained in the general case, without any additional hypotheses on the systems coefficients and without assuming any growth type properties for the associated propagators. Furthermore, we emphasize that throughout our study, all the admissibility conditions imply (or are equivalent to) the existence of a *uniform* dichotomic behavior (ordinary, exponential or polynomial).

In the last section, we present an application of our admissibility results to the study of the robustness of polynomial dichotomy of discrete nonautonomous systems. We provide a new method that combines control type approaches and operator theory arguments, showing for the first time how the information regarding the polynomial behavior of a perturbed system can be recovered via suitable *double* admissibilities.

2. Preliminaries: dichotomy notions on the half-line

In this section we present the notations and the basic definitions of the dichotomy concepts treated in this paper and we recall several properties established in [24]. For more connections between notions and properties related to the admissibility methods we also refer to [18, 61].

Denote by $\mathbb{N} = \{1, 2, \dots\}$ and by $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $\Gamma = \{(m, k) \in \mathbb{N} \times \mathbb{N} : m \geq k\}$ and $\Gamma_0 = \{(m, k) \in \mathbb{N}_0 \times \mathbb{N}_0 : m \geq k\}$.

Let $(X, \|\cdot\|)$ be a Banach space and let $\mathcal{B}(X)$ denote the Banach algebra of all bounded linear operators on X equipped with the operator norm that will be also denoted by $\|\cdot\|$. We denote by I_d the identity operator on X.

Let $\{A(n)\}_{n\in\mathbb{N}} \subset \mathcal{B}(X)$. Consider the nonautonomous system

$$(A) \qquad x(n+1) = A(n)x(n), \quad n \in \mathbb{N}$$

and the associated evolution family $\Phi_A = \{\Phi_A(m, k)\}_{(m,k)\in\Gamma}$, i.e.

$$\Phi_A(m,k) = \begin{cases} A(m-1)\cdots A(k), & m > k \\ I_d, & m = k \end{cases}.$$

Definition 2.1. We say that (A) admits an *ordinary dichotomy* if there exist a sequence of projections $\{P(k)\}_{k\in\mathbb{N}}$ and $K \geq 1$ such that:

- (d_1) $\Phi_A(m,k)P(k) = P(m)\Phi_A(m,k)$, for all $(m,k) \in \Gamma$;
- (d_2) $\sup_{k\in\mathbb{N}} \|P(k)\| < \infty$;
- (d_3) $\Phi_A(m,k) : KerP(k) \to KerP(m)$ is invertible, for all $(m,k) \in \Gamma$;
- (d_4) $\|\Phi_A(m,k)x\| \leq K\,\|x\|$, for all $x \in RangeP(k)$ and all $(m,k) \in \Gamma$;
- (d_5) $\|\Phi_A(m,k)y\| \geq \dfrac{1}{K}\,\|y\|$, for all $y \in KerP(k)$ and all $(m,k) \in \Gamma$.

Definition 2.2. We say that (A) admits an *exponential dichotomy* if there exist a sequence of projections $\{P(k)\}_{k\in\mathbb{N}}$ and two constants $N \geq 1$, $\nu > 0$ such that the properties $(d_1) - (d_3)$ from Definition 2.1 are satisfied and in addition:

- (d_4') $\|\Phi_A(m,k)x\| \leq N\,e^{-\nu(m-k)}\,\|x\|$, for all $x \in RangeP(k)$ and all $(m,k) \in \Gamma$;
- (d_5') $\|\Phi_A(m,k)y\| \geq \dfrac{1}{N}\,e^{\nu(m-k)}\,\|y\|$, for all $y \in KerP(k)$ and all $(m,k) \in \Gamma$.

Definition 2.3. We say that (A) admits a *polynomial dichotomy* if there are a sequence of projections $\{P(k)\}_{k\in\mathbb{N}}$ and two constants $L \geq 1$, $\delta > 0$ such that the properties $(d_1) - (d_3)$ from Definition 2.1 are satisfied and in addition:

- $(\hat{d}_4)$ $\|\Phi_A(m,k)x\| \leq L\left(\dfrac{m}{k}\right)^{-\delta}\|x\|$, for all $x \in RangeP(k)$ and all $(m,k) \in \Gamma$;
- $(\hat{d}_5)$ $\|\Phi_A(m,k)y\| \geq \dfrac{1}{L}\left(\dfrac{m}{k}\right)^{\delta}\|y\|$, for all $y \in KerP(k)$ and all $(m,k) \in \Gamma$.

Remark 2.1. From Definitions 2.1-2.3 we immediately deduce that

- (i) if (A) admits a polynomial dichotomy, then (A) admits an ordinary dichotomy;
- (ii) if (A) admits an exponential dichotomy, then (A) admits a polynomial dichotomy.

In general, the converse implications do not hold true.

Example 2.1. Let $X = \mathbb{R}^2$ with $||(x_1, x_2)|| = \max\{|x_1|, |x_2|\}$. For $k \in \mathbb{N}$ and $(x_1, x_2) \in X$, let

$$A(k)(x_1, x_2) = (e^{\sin(k+1)-\sin k}x_1, e^{\cos k-\cos(k+1)}x_2) \tag{2.1}$$

and $P(k)(x_1, x_2) = (x_1, 0)$. Then (A) admits an ordinary dichotomy with respect to the sequence of projections $\{P(k)\}_{k\in\mathbb{N}}$, but (A) does not admit a polynomial dichotomy (for details see Example 2.1 in [24]).

This shows that in general the ordinary dichotomy does not imply the polynomial dichotomy.

Example 2.2. Let $X = \mathbb{R}^2$ with $||(x_1, x_2)|| = \max\{|x_1|, |x_2|\}$. For $k \in \mathbb{N}$ and $(x_1, x_2) \in X$, let

$$A(k)(x_1, x_2) = \left(\frac{k}{k+1}x_1, \frac{k+1}{k}x_2\right)$$

and $P(k)(x_1, x_2) = (x_1, 0)$. Then (A) admits a polynomial dichotomy with the projections $\{P(k)\}_{k\in\mathbb{N}}$, but (A) does not admit an exponential dichotomy (for details we refer to Example 2.2 in [24]).

This points out that in general the polynomial dichotomy does not imply the exponential dichotomy.

Regarding the structure of the projections for a polynomial dichotomy, the following result was established in Corollary 2.1 in [24]:

Proposition 2.1. *If (A) admits a polynomial dichotomy with respect to a sequence of projections $\{P(k)\}_{k\in\mathbb{N}}$, then there exists $r \in (1, \infty)$ such that for every $k \in \mathbb{N}$ we have*

$$\begin{aligned} RangeP(k) &= \{x \in X : \sup_{j\geq k} \|\Phi_A(j,k)x\| < \infty\} \\ &= \{x \in X : \lim_{j\to\infty} \Phi_A(j,k)x = 0\} \\ &= \{x \in X : \sum_{j=k}^{\infty} \|\Phi_A(j,k)x\|^p < \infty\}, \quad \forall p \in (r, \infty). \end{aligned}$$

In all that follows, for every $h \in \mathbb{N}, h \geq 2$, we consider

$$Q^h(n) : X \to X, \quad Q^h(n) = \Phi_A(h^n, h^{n-1}). \tag{2.2}$$

We associate with (A) the system

$$y(n+1) = Q^h(n)y(n), \quad n \in \mathbb{N} \tag*{(Q^h).}$$

Then the evolution family $\Phi_{Q^h} = \{\Phi_{Q^h}(m,k)\}_{(m,k)\in\Gamma}$ associated to (Q^h) satisfies:

$$\Phi_{Q^h}(m,k) = \Phi_A(h^{m-1}, h^{k-1}), \quad \forall (m,k) \in \Gamma. \tag{2.3}$$

The following characterization of polynomial dichotomy was obtained in [24] (see Theorem 3.2 therein):

Theorem 2.1. *The following assertions are equivalent:*

- (i) *the system (A) admits a polynomial dichotomy;*
- (ii) *(A) admits an ordinary dichotomy with respect to a sequence of projections $\{P(k)\}_{k\in\mathbb{N}}$ and for every $h \in \mathbb{N}$, $h \geq 2$, the system (Q^h) admits an exponential dichotomy with respect to the sequence of projections $\{P^h(k)\}_{k\in\mathbb{N}}$, given by*

$$P^h(k) = P(h^{k-1}), \quad \forall k \in \mathbb{N}; \tag{2.4}$$

- (iii) *(A) admits an ordinary dichotomy with respect to a sequence of projections $\{P(k)\}_{k\in\mathbb{N}}$ and there is $h \in \mathbb{N}$, $h \geq 2$ such that the system (Q^h) admits an exponential dichotomy with respect to the sequence of projections $\{P^h(k)\}_{k\in\mathbb{N}}$ that satisfy* (2.4).

3. Admissibility and Ordinary / Exponential Dichotomy

In this section we present new admissibility conditions for ordinary and exponential dichotomy that will motivate and support our study and methods developed in exploring polynomial dichotomies from the next section. Furthermore, we point out the connections between certain admissibility concepts that can be employed in order to study ordinary or exponential dichotomies. In the same time, we provide the structures of projections for ordinary and exponential dichotomy induced by suitable admissibilities and emphasize the relevance of the choice of the complement for the stable subspace at the initial time.

We maintain all the notations and the framework from Section 2.

Notations Let $\mathbb{I} \in \{\mathbb{N}, \mathbb{N}_0\}$. By $\ell^\infty(\mathbb{I}, X)$ we denote the space of all bounded sequences $s : \mathbb{I} \to X$ with the norm

$$\|s\|_\infty = \sup_{k\in\mathbb{I}} \|s(k)\|.$$

For each $p \in [1, \infty)$, denote by $\ell^p(\mathbb{I}, X)$ the space of all sequences $s : \mathbb{I} \to X$ with $\sum\limits_{k\in\mathbb{I}} \|s(k)\|^p < \infty$ equipped with the norm

$$\|s\|_p = \left(\sum_{k\in\mathbb{I}} \|s(k)\|^p\right)^{\frac{1}{p}}.$$

For $p \in [1, \infty]$, denote by

- $\ell_0^p(\mathbb{N}_0, X) := \{s \in \ell^p(\mathbb{N}_0, X) : s(0) = 0\}$
- $\ell_0^p(\mathbb{N}, X) = \{s \in \ell^p(\mathbb{N}, X) : s(1) = 0\}$.

We associate with (A) the control system

$$(C_A) \qquad \gamma(n+1) = A(n)\gamma(n) + s(n+1), \quad \forall n \in \mathbb{N}$$

with $s : \mathbb{N} \to X$ as input and $\gamma : \mathbb{N} \to X$ as output.

Definition 3.1. Let $p \in [1, \infty]$. We say that the pair $(\ell^\infty(\mathbb{N}, X), \ell_0^p(\mathbb{N}, X))$ is *admissible* for (C_A) if for every $s \in \ell_0^p(\mathbb{N}, X)$ there is $\gamma \in \ell^\infty(\mathbb{N}, X)$ such that (γ, s) satisfies (C_A).

Example 3.1. Let $X = \mathbb{R}^2$ with $||(x_1, x_2)|| = \max\{|x_1|, |x_2|\}$ and let $\{A(k)\}_{k\in\mathbb{N}}$ be the sequence of operators from Example 2.1, given by (2.1). Consider the corresponding nonautonomous system (A) and the associated control system (C_A).

Let $s = (s_1, s_2) \in \ell^1(\mathbb{N}, X)$. Let $\gamma : \mathbb{N} \to X$ be defined by

$$\gamma(n) = \big(\sum_{j=1}^{n} e^{\sin n - \sin j} s_1(j), -\sum_{j=n+1}^{\infty} e^{\cos j - \cos n} s_2(j)\big).$$

Then we note that

$$A(n)\gamma(n) = \big(\sum_{j=1}^{n} e^{\sin(n+1) - \sin j} s_1(j), -\sum_{j=n+1}^{\infty} e^{\cos j - \cos(n+1)} s_2(j)\big)$$

which easily implies that (γ, s) satisfies (C_A). Moreover, for every $n \in \mathbb{N}$ we have

$$\begin{aligned}\|\gamma(n)\| &\le \max\big\{\sum_{j=1}^{n} e^{\sin n - \sin j}|s_1(j)|, \sum_{j=n+1}^{\infty} e^{\cos j - \cos n}|s_2(j)|\big\}\\ &\le e^2\|s\|_1\end{aligned}$$

so $\gamma \in \ell^\infty(\mathbb{N}, X)$. In conclusion, the pair $(\ell^\infty(\mathbb{N}, X), \ell_0^1(\mathbb{N}, X))$ is admissible for (C_A).

For every $k \in \mathbb{N}$, we define the so-called *stable subspace*

$$X_s(k) = \{x \in X : \sup_{j \geq k} \|\Phi_A(j,k)x\| < \infty\}.$$

Remark 3.1. The subspace

$$X_s(1) = \{x \in X : \sup_{j \geq 1} \|\Phi_A(j,1)x\| < \infty\}$$

is usually called *the initial stable subspace*.

Definition 3.2. We say that a subspace $U \subset X$ is *complemented* in X if U is closed and there is a closed subspace $V \subset X$ such that $X = U \oplus V$.

Remark 3.2. In general, even if a (closed) subspace is complemented, it does not follow that its complement is unique.

For example, let $X = \mathbb{R}^2$ and $U = \mathbb{R} \times \{0\}$. For every $h > 0$, take

$$V_h = \{(t, ht) : t \in \mathbb{R}\}.$$

Then each V_h is a closed subspace in $\mathbb{R}^2$ and

$$\mathbb{R}^2 = U \oplus V_h, \quad \forall h > 0.$$

Remark 3.3. We recall that if $U \subset X$ is a closed subspace, generally it does not follow that it is complemented.

Remark 3.4. In the studies devoted to the detection of the dichotomies on the half-line a natural hypothesis is to assume that the stable subspace at the initial moment is (closed and) complemented regardless whether one studies a uniform or a nonuniform behavior (see Huy and Minh [33], Megan, Sasu and Sasu [38,39], Minh, Räbiger and Schnaubelt [42], Sasu and Sasu [55,59,61], Sasu [56], Sasu, Babuţia and Sasu [60] and the references therein).

First of all, based on our results in [61] we deduce criteria for dichotomy by means of the admissibility notion introduced in Definition 3.1 and point our several conclusions regarding the assumptions on the complementarity of the stable subspace at the initial time.

Thus, sufficient conditions for (ordinary and exponential) dichotomic behaviors in terms of the admissibility relative to (C_A) are given by:

Theorem 3.1. *Assume that $X_s(1)$ is complemented in X and let Y be a closed complement such that*

$$X = X_s(1) \oplus Y.$$

The following assertions hold:

(i) *if the pair $(\ell^\infty(\mathbb{N}, X), \ell_0^1(\mathbb{N}, X))$ is admissible for (C_A), then (A) admits an ordinary dichotomy with respect to a sequence of projections $\{P(k)\}_{k\in\mathbb{N}}$ such that*

$$RangeP(k) = X_s(k) \quad \text{and} \quad KerP(k) = \Phi_A(k,1)Y, \quad \forall k \in \mathbb{N}; \tag{3.1}$$

(ii) *if $p \in (1, \infty]$ and the pair $(\ell^\infty(\mathbb{N}, X), \ell_0^p(\mathbb{N}, X))$ is admissible for (C_A), then (A) admits an exponential dichotomy with respect to a sequence of projections $\{P(k)\}_{k\in\mathbb{N}}$ that satisfies* (3.1).

Proof. Consider the translated system

$$(\tilde{A}) \qquad y(n+1) = \tilde{A}(n)y(n), \quad n \in \mathbb{N}_0$$

where $\tilde{A}(n) = A(n+1)$, for all $n \in \mathbb{N}$. We denote by $\Phi_{\tilde{A}} = \{\Phi_{\tilde{A}}(m,j)\}_{(m,j)\in\Gamma_0}$ the associated evolution family (see [61], Section 3).

We associate to $(\tilde{A})$ the control system

$$(S_{\tilde{A}}) \qquad \varphi(n+1) = \tilde{A}(n)\varphi(n) + u(n+1), \quad \forall n \in \mathbb{N}_0.$$

Then, we note that, for any $p \in [1,\infty]$, the admissibility of $(\ell^\infty(\mathbb{N}, X), \ell_0^p(\mathbb{N}, X))$ for (C_A) is equivalent to the admissibility of $(\ell^\infty(\mathbb{N}_0, X), \ell_0^p(\mathbb{N}_0, X))$ for $(S_{\tilde{A}})$ (see Definition 3.3 in [61]). In addition, we observe that

$$\begin{aligned} \tilde{X}_s(0) &= \{x \in X : \sup_{k\in\mathbb{N}_0} \|\Phi_{\tilde{A}}(k,0)x\| < \infty\} \\ &= \{x \in X : \sup_{j\in\mathbb{N}} \|\Phi_A(j,1)x\| < \infty\} \\ &= X_s(1) \end{aligned}$$

and so $\tilde{X}_s(0)$ is (closed and) complemented in X.

Then, from Theorem 3.3 in [61] applied for $(\tilde{A})$ we obtain (i). Next, from Theorem 4.3 (i) in [61] applied for $(\tilde{A})$ we deduce (ii). □

A characterization of exponential dichotomy via admissibility relative to the system (C_A) is given by:

Theorem 3.2. *Let $p \in (1,\infty]$. The system (A) admits an exponential dichotomy if and only if $(\ell^\infty(\mathbb{N}, X), \ell_0^p(\mathbb{N}, X))$ is admissible for (C_A) and the subspace $X_s(1)$ is complemented in X.*

Proof. Using similar arguments as in the proof of Theorem 3.1, the conclusion follows from Theorem 4.3 (ii) in [61]. □

Remark 3.5. It is well known that when exploring the dichotomies on the whole line (via admissibility methods) there is no need to add hypotheses regarding the complementarity of the stable subspaces (see Aulbach and Minh [1], Palmer [45], Sasu and Sasu [62, 63]). But, in that case, the admissibility is based on a *unique solvability of the input-output system* and, essentially, this hypothesis is the key for the splitting of the state space into stable and unstable subspaces at every moment. In particular, it turns out (on the whole line) that an adequate *unique* admissibility provides the complementarity of all the stable subspaces.

In what follows, in order to drop the explicit assumption on the complementarity of the initial stable subspace, we consider sharper admissibility conditions. Then, as a consequence of the criteria above, we will show that we can also characterize exponential dichotomy on the half-line in terms of a suitable *unique* solvability of the system (C_A).

With this aim, we first need a technical result:

Theorem 3.3. *Let $p \in [1,\infty]$ and let $Y \subset X$ be a closed linear subspace of X. If for every $s \in \ell_0^p(\mathbb{N}, X)$ there exists a unique solution γ_s of (C_A) with $\gamma_s \in \ell^\infty(\mathbb{N}, X)$ and $\gamma_s(1) \in Y$, then the following properties hold:*

(i) *$X_s(1)$ is a closed linear subspace;*
(ii) *$X_s(1) \oplus Y = X$.*

Proof. We consider the space

$$\ell^\infty_Y(\mathbb{N},X) := \{\varphi \in \ell^\infty(\mathbb{N},X) : \varphi(1) \in Y\}.$$

Since Y is closed, we have that $\ell^\infty_Y(\mathbb{N},X)$ is a closed linear subspace of $\ell^\infty(\mathbb{N},X)$.

According to our hypothesis, it makes sense to define

$$\mathcal{I} : \ell^p_0(\mathbb{N},X) \to \ell^\infty_Y(\mathbb{N},X), \quad \mathcal{I}(s) = \gamma_s$$

where for each $s \in \ell^p_0(\mathbb{N},X)$, γ_s is the unique solution of (C_A) in $\ell^\infty(\mathbb{N},X)$ with $\gamma_s(1) \in Y$. We have that $\mathcal{I}$ is a linear operator. Furthermore, it is easy to prove that $\mathcal{I}$ is closed. This yields that $\mathcal{I}$ is bounded and so

(3.2) $$\|\mathcal{I}(s)\|_\infty \le \|\mathcal{I}\| \, \|s\|_p, \quad \forall s \in \ell^p_0(\mathbb{N},X).$$

(i) *Step 1.* We prove that there is $K > 0$ such that

(3.3) $$\|\Phi_A(k,1)x\| \le K\,\|x\|, \quad \forall x \in X_s(1), \forall k \in \mathbb{N}.$$

Let $x \in X_s(1)$. We take

$$s\colon \mathbb{N} \to X, \quad s(k) = \begin{cases} A(1)x, & k = 2 \\ 0, & k \ne 2 \end{cases}$$

and

$$\gamma\colon \mathbb{N} \to X, \quad \gamma(k) = \begin{cases} \Phi_A(k,1)x, & k \ge 2 \\ 0, & k = 1 \end{cases}.$$

We have that $s \in \ell^p_0(\mathbb{N},X)$ and since $x \in X_s(1)$ it yields that $\gamma \in \ell^\infty(\mathbb{N},X)$. Moreover, since $\gamma(1) = 0$ we have that $\gamma \in \ell^\infty_Y(\mathbb{N},X)$. An easy computation shows that (γ, s) satisfies (C_A), so

(3.4) $$\gamma = \mathcal{I}(s).$$

Then, from (3.2) and (3.4) we deduce that

(3.5) $$\|\gamma(k)\| \le \|\gamma\|_\infty \le \|\mathcal{I}\| \, \|s\|_p = \|\mathcal{I}\| \, \|A(1)\| \, \|x\|, \quad \forall k \ge 2.$$

Setting

$$K := \max\{\|\mathcal{I}\| \, \|A(1)\|, 1\}$$

from (3.5) it yields that (3.3) holds.

Step 2. We prove that $X_s(1)$ is closed.

Let $x \in \overline{X_s(1)}$. Then there is $(x_j)_{j\in\mathbb{N}} \subset X_s(1)$ with $x_j \underset{j\to\infty}{\longrightarrow} x$. Setting

$$\alpha := \sup_{j\in\mathbb{N}} \|x_j\|$$

from (3.3) we obtain

(3.6) $$\|\Phi_A(k,1)x_j\| \le \alpha K, \quad \forall k, j \in \mathbb{N}.$$

Letting $j \to \infty$ in relation (3.6) we deduce that

$$\|\Phi_A(k,1)x\| \le \alpha K, \quad \forall k \in \mathbb{N}$$

so $x \in X_s(1)$. This shows that $X_s(1)$ is closed.

(ii) We show that

(3.7) $$X_s(1) \oplus Y = X.$$

This will be done in two stages:

Step 1. We prove that

$$X_s(1) \cap Y = \{0\}. \tag{3.8}$$

Indeed, let $z \in X_s(1) \cap Y$. Now we take

$$u \colon \mathbb{N} \to X, \quad u(k) = \Phi_A(k,1)z.$$

We have that $u \in \ell^\infty_Y(\mathbb{N}, X)$ and $(u, 0)$ satisfies (C_A). This implies $u = \mathcal{J}(0) = 0$ which yields in particular that $z = u(1) = 0$. Hence, we have that (3.8) holds.

Step 2. We show that

$$X_s(1) + Y = X. \tag{3.9}$$

Let $x \in X$. We take

$$s : \mathbb{N} \to X, \quad s(k) = \begin{cases} -A(1)x, & k = 2 \\ 0, & k \neq 2 \end{cases}$$

and let $\gamma = \mathcal{J}(s)$. Then

$$\gamma(2) = A(1)(\gamma(1) - x) \tag{3.10}$$

and

$$\gamma(k) = A(k)\gamma(k), \quad \forall k \geq 2. \tag{3.11}$$

From (3.10) and (3.11) it follows that

$$\gamma(k) = \Phi_A(k,1)(\gamma(1) - x), \quad \forall k \geq 2. \tag{3.12}$$

Since $\gamma = \mathcal{J}(s)$ we have that $\gamma \in \ell^\infty_Y(\mathbb{N}, X)$. Then, using (3.12) we deduce that

$$\sup_{k \geq 2} \|\Phi_A(k,1)(\gamma(1) - x)\| \leq \|\gamma\|_\infty$$

which yields that

$$x_s := \gamma(1) - x \in X_s(1).$$

Furthermore, from $\gamma \in \ell^\infty_Y(\mathbb{N}, X)$ we have that $\gamma(1) \in Y$. Thus, we get that

$$x = -x_s + \gamma(1) \in X_s(1) + Y.$$

Thus, we have shown that (3.9) holds.

In conclusion, since $X_s(1)$ and Y are closed subspaces, from (3.8) and (3.9) it follows that (3.7) holds. □

Theorem 3.4. *Let $Y \subset X$ be a closed subspace of X. The following assertions hold:*

(i) if for every $s \in \ell^1_0(\mathbb{N}, X)$ there exists a unique solution γ_s of (C_A) with $\gamma_s \in \ell^\infty(\mathbb{N}, X)$ and $\gamma_s(1) \in Y$, then (A) admits an ordinary dichotomy with respect to a sequence of projections $\{P(k)\}_{k\in\mathbb{N}}$ such that

$$Range P(k) = X_s(k) \quad \text{and} \quad Ker P(k) = \Phi_A(k,1)Y, \quad \forall k \in \mathbb{N}; \tag{3.13}$$

(ii) if $p \in (1, \infty]$ and for every $s \in \ell^p_0(\mathbb{N}, X)$ there exists a unique solution γ_s of (C_A) with $\gamma_s \in \ell^\infty(\mathbb{N}, X)$ and $\gamma_s(1) \in Y$, then (A) admits an exponential dichotomy with respect to a sequence of projections $\{P(k)\}_{k\in\mathbb{N}}$ that satisfies (3.13).

Proof. (i) This follows from Theorem 3.3 and Theorem 3.1 (i).

(ii) This follows from Theorem 3.3 and Theorem 3.1 (ii). □

Proposition 3.1. *Let $p \in (1, \infty]$. If the system (A) admits an exponential dichotomy with respect to a sequence of projections $\{P(k)\}_{k\in\mathbb{N}}$, then for every $s \in \ell_0^p(\mathbb{N}, X)$ there exists a unique solution γ_s of (C_A) such that $\gamma_s \in \ell^\infty(\mathbb{N}, X)$ and $\gamma_s(1) \in KerP(1)$.*

Proof. Let $N \geq 1$ and $\nu > 0$ be given by Definition 2.2. Then, in particular

$$\|\Phi_A(m,1)y\| \geq \frac{1}{N}e^{\nu(m-1)}\|y\|, \quad \forall y \in KerP(1), \forall m \in \mathbb{N}. \tag{3.14}$$

Let $s \in \ell_0^p(\mathbb{N}, X)$. Since (A) admits an exponential dichotomy, from Theorem 3.2 we have that there exists $\gamma \in \ell^\infty(\mathbb{N}, X)$ such that (γ, s) satisfies (C_A).

Let $x = P(1)\gamma(1)$ and $y = (I_d - P(1))\gamma(1)$. We take now

$$\gamma_s : \mathbb{N} \to X, \quad \gamma_s(k) = \gamma(k) - \Phi_A(k,1)x.$$

Since $x \in RangeP(1)$, from Remark 2.1 (ii) and Proposition 2.1 it follows that

$$\sup_{k\geq 1} \|\Phi_A(k,1)x\| < \infty$$

which implies that $\gamma_s \in \ell^\infty(\mathbb{N}, X)$. Furthermore

$$\gamma_s(1) = y \in KerP(1)$$

and it is easy to verify that (γ_s, s) satisfies (C_A).

It remains to prove that γ_s is unique. Let $\tilde{\gamma}$ be a solution of (C_A) for the input s such that $\tilde{\gamma} \in \ell^\infty(\mathbb{N}, X)$ and $\tilde{\gamma}(1) \in KerP(1)$. We set $\varphi = \gamma_s - \tilde{\gamma}$ and then

$$\varphi(k+1) = A(k)\varphi(k), \quad \forall k \in \mathbb{N}. \tag{3.15}$$

From (3.15) it yields that

$$\varphi(k) = \Phi_A(k,1)\varphi(1), \quad \forall k \in \mathbb{N}. \tag{3.16}$$

Hence, from (3.14) and (3.16) we get that

$$\frac{1}{N}e^{\nu(m-1)}\|\varphi(1)\| \leq \|\Phi_A(m,1)\varphi(1)\| = \|\varphi(m)\| \leq \|\varphi\|_\infty, \quad \forall m \in \mathbb{N}$$

which implies

$$\|\varphi(1)\| \leq N\|\varphi\|_\infty e^{-\nu(m-1)}, \quad \forall m \in \mathbb{N}. \tag{3.17}$$

Letting $m \to \infty$ in (3.17) we obtain that $\varphi(1) = 0$. Hence from (3.16) it yields that $\varphi = 0$. Thus, we have shown that $\tilde{\gamma} = \gamma_s$, so γ_s is unique. □

We can give now the characterization of exponential dichotomy in terms of unique admissibility that points out the conclusions of our previous results:

Theorem 3.5. *Let $p \in (1, \infty]$. Then (A) admits an exponential dichotomy if and only if there is a closed subspace $Y \subset X$ such that for every $s \in \ell_0^p(\mathbb{N}, X)$ there exists a unique solution γ_s of (C_A) such that $\gamma_s \in \ell^\infty(\mathbb{N}, X)$ and $\gamma_s(1) \in Y$.*

Proof. To prove the direct implication, assume that (A) admits an exponential dichotomy with a sequence of projections $\{P(k)\}_{k\in\mathbb{N}}$. Then, by taking $Y = KerP(1)$, the conclusion follows from Proposition 3.1.

The converse implication follows from Theorem 3.4 (ii). □

4. Admissibility and Polynomial Dichotomies

The central aim of this section is to give new characterizations of the polynomial dichotomy of discrete nonautonomous systems on the half-line by means of admissibility properties. With this purpose we introduce here for the first time a new method that relies (roughly speaking) on a *double admissibility* relative to certain well-chosen input-output systems. On the one hand, we continue the study in the previous section, by applying the results therein, and, on the other hand, we bring into attention interesting properties and connections between the sequences of projections that describe the dichotomic behaviors. We provide here a detailed analysis of the hypotheses regarding the (unique) solvabilities of associated control systems versus suitable assumptions on the complementarity of initial stable subspaces.

We maintain all the notations and the framework from Sections 2 and 3. We consider a discrete system (A) and the associated control system (C_A).

From Section 2 and Section 3 it yields that if one aims to characterize the polynomial dichotomy instead of the exponential dichotomy by means of the solvability of the input-output system (C_A), then the admissibility hypotheses considered in the preceding section must be changed. In what follows, we will show that this can be done via a suitable modification of the coefficients of the input-output system and by adding new admissibility conditions, besides certain admissibility assumptions relative to the system (C_A).

For every $h \in \mathbb{N}, h \geq 2$, we associate with (A) the discrete system

$$(Q^h) \qquad y(n+1) = Q^h(n)y(n), \quad n \in \mathbb{N}$$

where the coefficients are defined by relation (2.2). Further, we associate with each (Q^h) a control system

$$(C_{Q^h}) \qquad \gamma(n+1) = Q^h(n)\gamma(n) + s(n+1), \quad \forall n \in \mathbb{N}$$

where s is the input and γ is the solution (output).

Remark 4.1. We note that (C_{Q^h}) is equivalent to

$$\gamma(n+1) = \Phi_A(h^n, h^{n-1})\gamma(n) + s(n+1), \quad \forall n \in \mathbb{N}.$$

The first central result of this section is a characterization of the polynomial dichotomy by means of a double admissibility and it is given by:

Theorem 4.1. *The system (A) admits a polynomial dichotomy if and only if the following properties are satisfied:*

- (i) *$(\ell^\infty(\mathbb{N}, X), \ell_0^1(\mathbb{N}, X))$ is admissible for (C_A) and $X_s(1) := \{x \in X : \sup_{k\geq 1} \|\Phi_A(k,1)x\| < \infty\}$ is complemented in X;*
- (ii) *there are $p \in (1, \infty]$ and $h \in \mathbb{N}, h \geq 2$ such that $(\ell^\infty(\mathbb{N}, X), \ell_0^p(\mathbb{N}, X))$ is admissible for (C_{Q^h}) and $\tilde{X}_s^h(1) := \{x \in X : \sup_{k\geq 1} \|\Phi_{Q^h}(k,1)x\| < \infty\}$ is complemented in X.*

Proof. Necessity. Assume that (A) admits a polynomial dichotomy with respect to a sequence of projections $\{P(k)\}_{k\in\mathbb{N}}$ and let $L \geq 1, \delta > 0$ be constants given by Definition 2.3. Then we have

$$\|\Phi_A(m,k)x\| \leq L\left(\frac{m}{k}\right)^{-\delta}\|x\|, \quad \forall x \in Range P(k), \forall (m,k) \in \Gamma; \tag{4.1}$$

and

$$\|\Phi_A(m,k)y\| \geq \frac{1}{L}\left(\frac{m}{k}\right)^{\delta}\|y\|, \quad \forall y \in KerP(k), \forall (m,k) \in \Gamma. \tag{4.2}$$

We set

$$\alpha = \sup_{k\in\mathbb{N}} \|P(k)\|.$$

For every $(m,k) \in \Gamma$, we denote by $\Phi_A(m,k)_|^{-1}$ the inverse of $\Phi_A(m,k) : KerP(k) \to KerP(m)$.

From Proposition 2.1 we have that

$$X_s(1) = RangeP(1).$$

In particular, it yields that $X_s(1)$ is complemented.

Step 1. We show that $(\ell^\infty(\mathbb{N},X), \ell_0^1(\mathbb{N},X))$ is admissible for (C_A).

Let $s \in \ell_0^1(\mathbb{N},X)$. Let

$$\gamma : \mathbb{N} \to X, \quad \gamma(k) = \sum_{j=1}^{k} \Phi_A(k,j)P(j)s(j) - \sum_{j=k+1}^{\infty} \Phi_A(j,k)_|^{-1}(I_d - P(j))s(j).$$

Using (4.1) we deduce that

$$\begin{aligned} \sum_{j=1}^{k} \|\Phi_A(k,j)P(j)s(j)\| &\leq L\sum_{j=1}^{k}\left(\frac{k}{j}\right)^{-\delta}\|P(j)\|\|s(j)\| \\ &\leq L\alpha\sum_{j=1}^{k}\|s(j)\|, \quad \forall k \in \mathbb{N}. \end{aligned} \tag{4.3}$$

Using (4.2) we get that

$$\begin{aligned} \sum_{j=k+1}^{\infty} \|\Phi_A(j,k)_|^{-1}(I_d - P(j))s(j)\| &\leq L\sum_{j=k+1}^{\infty}\left(\frac{k}{j}\right)^{\delta}\|I_d - P(j)\|\|s(j)\| \\ &\leq L(1+\alpha)\sum_{j=k+1}^{\infty}\|s(j)\|, \quad \forall k \in \mathbb{N}. \end{aligned} \tag{4.4}$$

From (4.3) and (4.4) it follows that γ is well defined and

$$\|\gamma(k)\| \leq L(1+\alpha)\|s\|_1, \qquad \forall k \in \mathbb{N}. \tag{4.5}$$

From (4.5) it yields that $\gamma \in \ell^\infty(\mathbb{N},X)$. Furthermore, it is easy to verify that (γ, s) satisfies (C_A). So, we have shown that $(\ell^\infty(\mathbb{N},X), \ell_0^1(\mathbb{N},X))$ is admissible for (C_A).

Let now $p \in (1,\infty]$ and $h \in \mathbb{N}, h \geq 2$. Let (Q^h) be the system associated to (A) and let

$$\tilde{X}_s^h(1) = \{x \in X : \sup_{k\geq 1}\|\Phi_{Q^h}(k,1)x\| < \infty\}.$$

Step 2. We prove that $(\ell^\infty(\mathbb{N},X), \ell_0^p(\mathbb{N},X))$ is admissible for (C_{Q^h}) and $\tilde{X}_s^h(1)$ is complemented in X.

Since (A) admits a polynomial dichotomy, from Theorem 2.1 we have that the system (Q^h) admits an exponential dichotomy. Then, from Theorem 3.2 we deduce that $(\ell^\infty(\mathbb{N},X), \ell_0^p(\mathbb{N},X))$ is admissible for (C_{Q^h}) and $\tilde{X}_s^h(1)$ is complemented in X.

Sufficiency. Let Y be a closed subspace of X such that

$$X = X_s(1) \oplus Y. \tag{4.6}$$

From the hypothesis (i) and Theorem 3.1 (i) it follows that (A) admits an ordinary dichotomy with respect to a sequence of projections $\{P(k)\}_{k\in\mathbb{N}}$ whose ranges and kernels satisfy:

$$RangeP(k) = X_s(k) = \{x \in X : \sup_{j\geq k} \|\Phi_A(j,k)x\| < \infty\} \text{ and } KerP(k) = \Phi_A(k,1)Y, \tag{4.7}$$

for all $k \in \mathbb{N}$.

Let $K \geq 1$ be given by Definition 2.1. Then, in particular, we have

$$\|\Phi_A(k,1)y\| \geq \frac{1}{K}\|y\|, \quad \forall y \in Y, \forall k \in \mathbb{N}. \tag{4.8}$$

Let $p \in (1,\infty]$ and $h \in \mathbb{N}$, $h \geq 2$ be given by the hypothesis (ii). From Theorem 3.2 applied for (Q^h) we deduce that (Q^h) admits an exponential dichotomy with a sequence of projections $\{R(k)\}_{k\in\mathbb{N}}$.

By Remark 2.1 (ii) and Proposition 2.1 it follows

$$RangeR(1) = \tilde{X}_s^h(1) = \{x \in X : \lim_{k\to\infty} \Phi_{Q^h}(k,1)x = 0\}. \tag{4.9}$$

Step 1. We prove that

$$\tilde{X}_s^h(1) = X_s(1). \tag{4.10}$$

We recall that via (2.3) we have

$$\Phi_{Q^h}(k,1) = \Phi_A(h^{k-1},1), \quad \forall k \in \mathbb{N}. \tag{4.11}$$

Then, it yields that

$$X_s(1) \subset \tilde{X}_s^h(1). \tag{4.12}$$

Let now $x \in \tilde{X}_s^h(1)$. According to (4.6) there are $x_1 \in X_s(1)$ and $y \in Y$ such that $x = x_1 + y$. Using (4.8) and (4.11) we obtain

$$\begin{aligned} \frac{1}{K}\|y\| &\leq \|\Phi_A(h^{k-1},1)y\| = \|\Phi_{Q^h}(k,1)y\| \\ &\leq \|\Phi_{Q^h}(k,1)x\| + \|\Phi_{Q^h}(k,1)x_1\|, \quad \forall k \in \mathbb{N}. \end{aligned} \tag{4.13}$$

Since $x_1 \in X_s(1)$, from (4.12), we deduce that $x_1 \in \tilde{X}_s^h(1)$. Then, from (4.9) it follows

$$\lim_{k\to\infty} \Phi_{Q^h}(k,1)x = 0 \quad \text{and} \quad \lim_{k\to\infty} \Phi_{Q^h}(k,1)x_1 = 0. \tag{4.14}$$

Using (4.14) and letting $k \to \infty$ in (4.13), it yields that $y = 0$. This implies $x = x_1 \in X_s(1)$, which shows that

$$\tilde{X}_s^h(1) \subset X_s(1). \tag{4.15}$$

From (4.12) and (4.15) we have that (4.10) holds.

In addition, from (4.6) and (4.10) we get that

$$X = \tilde{X}_s^h(1) \oplus Y. \tag{4.16}$$

Next, for every $k \in \mathbb{N}, k \geq 2$, we consider the subspace

$$\tilde{X}_s^h(k) = \{x \in X : \sup_{j \geq k} \|\Phi_{Q^h}(j,k)x\| < \infty\}.$$

Using relation (4.16), from the hypothesis (ii) and Theorem 3.1 (ii) we also deduce that (Q^h) admits an exponential dichotomy with respect to a sequence of projections $\{P^h(k)\}_{k\in\mathbb{N}}$, where for each $k \in \mathbb{N}$, their ranges and kernels are:

$$RangeP^h(k) = \tilde{X}_s^h(k) \quad \text{and} \quad KerP^h(k) = \Phi_{Q^h}(k,1)Y = \Phi_A(h^{k-1},1)Y. \tag{4.17}$$

Using (4.7) and (4.17) we have that

$$KerP^h(k) = \Phi_A(h^{k-1},1)Y = KerP(h^{k-1}), \quad \forall k \in \mathbb{N}. \tag{4.18}$$

Step 2. We prove that

$$RangeP^h(k) = RangeP(h^{k-1}), \quad \forall k \in \mathbb{N}. \tag{4.19}$$

Since from (2.3) we have

$$\Phi_{Q^h}(j,k) = \Phi_A(h^{j-1},h^{k-1}), \quad \forall (j,k) \in \Gamma,$$

using (4.7) and (4.17) we deduce that

$$\begin{aligned} RangeP(h^{k-1}) &= X_s(h^{k-1}) = \{x \in X : \sup_{j \geq h^{k-1}} \|\Phi_A(j,h^{k-1})x\| < \infty\} \\ &\subset \{x \in X : \sup_{j\geq k} \|\Phi_A(h^{j-1},h^{k-1})x\| < \infty\} \\ &= \{x \in X : \sup_{j \geq k} \|\Phi_{Q^h}(j,k)x\| < \infty\} \\ &= \tilde{X}_s^h(k) = RangeP^h(k), \qquad \forall k \in \mathbb{N}. \end{aligned} \tag{4.20}$$

Let $x \in RangeP^h(k)$. We take $u = P(h^{k-1})x$ and $v = (I_d - P(h^{k-1}))x$. From (4.20) we have that $u \in RangeP^h(k)$. This implies that

$$v = x - u \in RangeP^h(k). \tag{4.21}$$

But, from (4.18) we have that

$$v \in KerP(h^{k-1}) = KerP^h(k). \tag{4.22}$$

From (4.21) and (4.22) we get that $v = 0$. This implies that $x = u$, so

$$RangeP^h(k) \subset RangeP(h^{k-1}). \tag{4.23}$$

From (4.20) and (4.23) it yields that (4.19) holds.

From (4.18) and (4.19) we deduce that

$$P^h(k) = P(h^{k-1}), \quad \forall k \in \mathbb{N}. \tag{4.24}$$

From (4.24), by applying Theorem 2.1 we obtain that (A) admits a polynomial dichotomy. $\square$

As a consequence of the characterization of polynomial dichotomy by means of admissibility (given by Theorem 4.1) we obtain in what follows a version of Theorem 3.5 for the case of polynomial dichotomy. More precisely, we show that we can use suitable unique solvabilities when exploring a polynomial behavior. But, first we need a technical result:

Proposition 4.1. *If the system (A) admits a polynomial dichotomy with respect to a sequence of projections $\{P(k)\}_{k\in\mathbb{N}}$, then for every $s \in \ell_0^1(\mathbb{N}, X)$ there exists a unique solution w_s of (C_A) such that $w_s \in \ell^\infty(\mathbb{N}, X)$ and $w_s(1) \in KerP(1)$.*

Proof. Assume that (A) admits a polynomial dichotomy with respect to a sequence of projections $\{P(k)\}_{k\in\mathbb{N}}$.

From Theorem 4.1 it follows that $(\ell^\infty(\mathbb{N}, X), \ell_0^1(\mathbb{N}, X))$ is admissible for (C_A).

Let $s \in \ell_0^1(\mathbb{N}, X)$. Then via a similar construction as in the proof of Proposition 3.1 we first obtain that there exists a solution w_s of (C_A) such that $w_s \in \ell^\infty(\mathbb{N}, X)$ and $w_s(1) \in KerP(1)$. So it remains to check only the uniqueness of w_s.

Let $L \geq 1, \delta > 0$ be constants given by Definition 2.3. Then in particular we have

$$\|\Phi_A(m,1)y\| \geq \frac{m^\delta}{L}\|y\|, \quad \forall y \in KerP(1), \forall m \in \mathbb{N}. \tag{4.25}$$

Suppose that $\tilde{w} \in \ell^\infty(\mathbb{N}, X)$ is another solution of (C_A) with $\tilde{w}(1) \in KerP(1)$. Then, we set $\psi = w_s - \tilde{w}$. Hence $\psi \in \ell^\infty(\mathbb{N}, X)$, $\psi(1) \in KerP(1)$ and

$$\psi(m) = \Phi_A(m,1)\psi(1), \quad \forall m \in \mathbb{N}. \tag{4.26}$$

Then, from (4.25) and (4.26) we get that

$$\|\psi(1)\| \leq Lm^{-\delta}\|\Phi_A(m,1)\psi(1)\| \leq Lm^{-\delta}\|\psi\|_\infty, \quad \forall m \in \mathbb{N}. \tag{4.27}$$

Letting $m \to \infty$, from (4.27) it follows that $\psi(1) = 0$ and so $\psi = 0$. This shows that w_s is unique and the proof is complete. □

The second central result of this section provides a characterization of polynomial dichotomy by means of two (unique) solvabilities:

Theorem 4.2. *The system (A) admits a polynomial dichotomy if and only if there is a closed subspace $Y \subset X$ such that the following properties are satisfied:*

- (i) *for every $s \in \ell_0^1(\mathbb{N}, X)$ there exists a unique solution γ of (C_A) with $\gamma \in \ell^\infty(\mathbb{N}, X)$ and $\gamma(1) \in Y$;*
- (ii) *there are $p \in (1, \infty]$ and $h \in \mathbb{N}, h \geq 2$ such that for every $s \in \ell_0^p(\mathbb{N}, X)$ there exists a unique solution w of (C_{Q^h}) with $w \in \ell^\infty(\mathbb{N}, X)$ and $w(1) \in Y$.*

Proof. Necessity. Assume that (A) admits a polynomial dichotomy with respect to a sequence of projections $\{P(k)\}_{k\in\mathbb{N}}$. From Theorem 2.1 and from the proof of $(i) \Longrightarrow (ii)$ in Theorem 3.1 in [24] we have that (A) admits an ordinary dichotomy with respect to $\{P(k)\}_{k\in\mathbb{N}}$ and, in addition, if $h \in \mathbb{N}, h \geq 2$, then (Q^h) admits an exponential dichotomy with respect to the projections $\{P^h(k)\}_{k\in\mathbb{N}}$ such that

$$P^h(k) = P(h^{k-1}), \quad \forall k \in \mathbb{N}. \tag{4.28}$$

Let $Y = KerP(1)$. Then, the assertion (i) follows from Proposition 4.1.

Let $p \in (1, \infty]$. From (4.28) we note that

$$KerP^h(1) = KerP(1) = Y.$$

Then, the assertion (ii) follows from Proposition 3.1 applied for (Q^h).

Sufficiency. From (i) we obtain on the one hand that $(\ell^\infty(\mathbb{N}, X), \ell_0^1(\mathbb{N}, X))$ is admissible for (C_A). On the other hand, from Theorem 3.3 we deduce that $X_s(1) := \{x \in X : \sup\limits_{k\geq 1} \|\Phi_A(k,1)x\| < \infty\}$ is complemented in X.

From (ii) it follows that there are $p \in (1, \infty]$ and $h \in \mathbb{N}, h \geq 2$ such that $(\ell^\infty(\mathbb{N}, X), \ell_0^p(\mathbb{N}, X))$ is admissible for (C_{Q^h}). Furthermore, by Theorem 3.3 applied for (Q^h) it yields that $\tilde{X}_s^h(1) := \{x \in X : \sup\limits_{k\geq 1} \|\Phi_{Q^h}(k,1)x\| < \infty\}$ is complemented in X.

Finally, from Theorem 4.1 we conclude that (A) admits a polynomial dichotomy. $\square$

Remark 4.2. The methods developed to study ordinary, exponential and polynomial dichotomies on the half-line and the criteria obtained in this paper show that the assumptions regarding the complementarity of the stable subspaces at the initial time are natural and represent important starting points for any study on dichotomic behaviors via admissibilities.

Remark 4.3. All the results are obtained for the strong concepts of dichotomies i.e. for dichotomy notions in which the sequences of projections are bounded (the property (d_2) from Definition 2.3 holds).

Remark 4.4. We emphasize that our study was done in the most general case, without any additional requirements on the systems coefficients (such as their uniform boundeness or invertibilities) and there isn't any assumption regarding (exponential or polynomial) growth properties for the associated propagators.

Remark 4.5. Even if we do not impose any additional hypotheses on the initial system, we stress that the admissibility conditions formulated herein always imply that the initial system exhibits a *uniform* dichotomic behavior: ordinary, polynomial or exponential.

5. Robustness property of polynomial dichotomy

One of the most interesting topics in the asymptotic theory of dynamical systems is to explore whether an asymptotic behavior persists when the system is subjected to linear perturbations, i.e. to study its *robustness* (see [11,12,15,32,37,43,50,66] and also the recent works [3, 4, 6, 10, 20, 49, 58, 63, 69, 70, 72] and the references therein). There are various approaches in exploring robustness properties, some of the most representative being on the one hand those relying on direct estimates - that simply allow one to determine the explicit asymptotic behavior of the propagator of the perturbed system - and, on the other hand, those based on specific admissibility tools in which the asymptotic behavior of the perturbed system is implicitly deduced via admissibility criteria. The second category of methods is often stronger (and requires combined arguments of functional analysis and control) as in some cases it provides not only the robustness property, but also a radius within which the "size" of the perturbation should fit (see [10, 11, 32, 58, 63, 66] and the references therein). We note that, for the special case of nonuniform polynomial dichotomic behaviors, recent robustness results have been obtained in [18, 19, 65].

In this section we present an application of our input-output criteria obtained in Section 4 to the study of the robustness of polynomial dichotomy of discrete nonautonomous systems. We provide a new method of exploring the persistence of polynomial dichotomy when subjected to perturbations, that is based on control type approaches and operator theory arguments. We show for the first time how one can recover the information regarding the polynomial behavior of a perturbed system by means of some *double admissibilities* relative to associated control systems.

Let $\{A(n)\}_{n\in\mathbb{N}}$ and $\{B(n)\}_{n\in\mathbb{N}}$ be two sequences in $\mathcal{B}(X)$. We consider the nonautonomous systems

$$(A) \qquad x(n+1) = A(n)x(n) \quad n \in \mathbb{N}$$

and

$$(B) \qquad x(n+1) = B(n)x(n), \quad n \in \mathbb{N}.$$

Let Φ_A and Φ_B be the evolution families associated with (A) and (B), respectively. It is easy to verify that

$$\Phi_B(m,n) = \Phi_A(m,n) + \sum_{j=n}^{m-1} \Phi_A(m,j+1)(B(j)-A(j))\Phi_B(j,n), \tag{5.1}$$

for all $(m,n) \in \Gamma, m \geq n+1$.

We recall first a classic result:

Lemma 5.1. (*Discrete Gronwall's Lemma) Let $n \in \mathbb{N}$ and $\alpha > 0$. Let $(u_m)_{m\geq n}$ and $(v_m)_{m\geq n}$ be two nonnegative sequences satisfying $u_n \leq \alpha$ and*

$$u_m \leq \alpha + \sum_{j=n}^{m-1} v_j u_j, \quad \forall m \geq n+1.$$

Then

$$u_m \leq \alpha\, e^{\sum\limits_{j=n}^{m-1} v_j}, \quad \forall m \geq n+1.$$

Remark 5.1. For interesting generalizations of the discrete Gronwall's Lemma we refer to Clark [13] and Zhou and Zhang [67].

Lemma 5.2. *Assume that there are $D, K, a > 0$ such that*

$$\|\Phi_A(m,n)\| \leq D\left(\frac{m}{n}\right)^a, \quad \forall (m,n) \in \Gamma \tag{5.2}$$

and

$$\|A(n) - B(n)\| \leq \frac{K}{n+1}, \quad \forall n \in \mathbb{N}. \tag{5.3}$$

Then there is $b > a$ such that

$$\|\Phi_B(m,n)\| \leq D\left(\frac{m}{n}\right)^b, \quad \forall (m,n) \in \Gamma.$$

Proof. Let $n \in \mathbb{N}$. From relations (5.1)-(5.3) we deduce that

$$\|\Phi_B(m,n)\| \leq D\left(\frac{m}{n}\right)^a + DK \sum_{j=n}^{m-1} \left(\frac{m}{j+1}\right)^a \frac{1}{j+1} \|\Phi_B(j,n)\|, \quad \forall m \geq n+1$$

which implies

$$\left(\frac{n}{m}\right)^a \|\Phi_B(m,n)\| \leq D + \sum_{j=n}^{m-1} \frac{DK}{j+1}\left(\frac{n}{j}\right)^a \|\Phi_B(j,n)\|, \quad \forall m \geq n+1. \tag{5.4}$$

Taking

$$u_m := \left(\frac{n}{m}\right)^a \|\Phi_B(m,n)\| \quad \text{and} \quad v_m = \frac{DK}{m+1}, \quad \forall m \geq n$$

we observe that $u_n = 1 \leq D$. Then, from (5.4) and Lemma 5.1 it follows that

$$u_m \leq De^{\sum_{j=n}^{m-1} \frac{DK}{j+1}}, \quad \forall m \geq n+1. \tag{5.5}$$

Since

$$\sum_{j=n}^{m-1} \frac{1}{j+1} < \ln \frac{m}{n}, \quad \forall m \geq n+1 \tag{5.6}$$

from relations (5.5) and (5.6) we obtain

$$u_m \leq D\left(\frac{m}{n}\right)^{DK}, \quad \forall m \geq n+1. \tag{5.7}$$

Setting $b = a + DK$ from (5.7) we get that

$$\|\Phi_B(m,n)\| \leq D\left(\frac{m}{n}\right)^b, \quad \forall m \geq n. \tag{5.8}$$

Since $n \in \mathbb{N}$ was arbitrary and b does not depend on n, from (5.8) we obtain the conclusion. □

Theorem 5.1. *Assume that (A) admits a polynomial dichotomy and that there exist $D, a > 0$ such that*

$$\|\Phi_A(m,n)\| \leq D\left(\frac{m}{n}\right)^a, \quad \forall (m,n) \in \Gamma. \tag{5.9}$$

Furthermore, suppose that there exist $c > 0$ and $\rho > 1$ such that

$$\|A(n) - B(n)\| \leq \frac{c}{(n+1)^\rho}, \quad \forall n \in \mathbb{N}. \tag{5.10}$$

Then, provided that c is sufficiently small, (B) admits a polynomial dichotomy.

Proof. Assume that (A) admits a polynomial dichotomy with respect to a sequence of projections $\{P(k)\}_{k\in\mathbb{N}}$ and set $Y = KerP(1)$.

Consider

$$\ell_Y^\infty(\mathbb{N}, X) := \{\lambda \in \ell^\infty(\mathbb{N}, X) : \lambda(1) \in Y\}.$$

Since Y is closed, we have that $\ell_Y^\infty(\mathbb{N}, X)$ is a closed subspace of $\ell^\infty(\mathbb{N}, X)$.

We consider the control systems

$$\gamma(n+1) = A(n)\gamma(n) + s(n+1), \quad \forall n \in \mathbb{N} \tag{C_A}$$

and

$$\gamma(n+1) = B(n)\gamma(n) + s(n+1), \quad \forall n \in \mathbb{N}. \tag{C_B}$$

Step 1. We prove that for every $s \in \ell^1(\mathbb{N}, X)$ there exists a unique solution γ of (C_B) with $\gamma \in \ell_Y^\infty(\mathbb{N}, X)$.

From (5.9) we have in particular that

$$\|A(n)\| \leq D2^a, \quad \forall n \in \mathbb{N}. \tag{5.11}$$

For every $\lambda \in \ell^\infty_Y(\mathbb{N}, X)$ we consider the sequence

$$h_\lambda : \mathbb{N} \to X, \quad h_\lambda(n) = \begin{cases} \lambda(n) - A(n-1)\lambda(n-1), & n \geq 2 \\ 0, & n = 1 \end{cases}.$$

Using (5.11) we obtain that $h_\lambda \in \ell^\infty_0(\mathbb{N}, X)$.

We consider the subspace

$$\mathcal{D} := \{\gamma \in \ell^\infty_Y(\mathbb{N}, X) : \ h_\gamma \in \ell^1_0(\mathbb{N}, X)\}$$

and the operator

$$\mathcal{H} : \mathcal{D} \to \ell^1_0(\mathbb{N}, X), \quad \mathcal{H}(\gamma) = h_\gamma.$$

It is easy to see that $\mathcal{H}$ is a closed linear operator. Set

$$\|\cdot\|_{\mathcal{H}} : \mathcal{D} \to \mathbb{R}_+, \quad \|\gamma\|_{\mathcal{H}} = \|\gamma\|_\infty + \|h_\gamma\|_1$$

and then $(\mathcal{D}, \|\cdot\|_{\mathcal{H}})$ is a Banach space. Moreover, by hypothesis and Proposition 4.1 we have that for every $s \in \ell^1_0(\mathbb{N}, X)$ there is a unique $\gamma \in \ell^\infty_Y(\mathbb{N}, Z)$ with $h_\gamma = s$. This shows that $\mathcal{H}$ is invertible.

For every $\gamma \in \mathcal{D}$ we take

$$t_\gamma : \mathbb{N} \to X, \quad t_\gamma(n) = \begin{cases} \gamma(n) - B(n-1)\gamma(n-1), & n \geq 2 \\ 0, & n = 1 \end{cases}.$$

Using (5.10) we deduce that

$$\|t_\gamma(n) - h_\gamma(n)\| \leq \|A(n-1) - B(n-1)\| \, \|\gamma\|_\infty \leq \frac{c\,\|\gamma\|_\infty}{n^\rho}, \quad \forall n \geq 2 \tag{5.12}$$

Then, from (5.12) we get that

$$\|t_\gamma(n)\| \leq \|t_\gamma(n) - h_\gamma(n)\| + \|h_\gamma(n)\| \leq \frac{c}{n^\rho}\|\gamma\|_\infty + \|h_\gamma(n)\|, \quad \forall n \geq 2. \tag{5.13}$$

Since $h_\gamma \in \ell^1_0(\mathbb{N}, X)$, by (5.13) it follows that $t_\gamma \in \ell^1_0(\mathbb{N}, X)$. Then, it makes sense to define the (linear) operator

$$\mathcal{T} : (\mathcal{D}, \|\cdot\|_{\mathcal{H}}) \to \ell^1_0(\mathbb{N}, X), \quad \mathcal{T}(\gamma) = t_\gamma.$$

Setting

$$\alpha := \sum_{n=1}^\infty \frac{1}{n^\rho}$$

from (5.13) we obtain that

$$\|\mathcal{T}(\gamma)\|_1 \leq \alpha c\|\gamma\|_\infty + \|h_\gamma\|_1 \leq (1 + \alpha c)\|\gamma\|_{\mathcal{H}}, \quad \forall \gamma \in \mathcal{D}.$$

This shows that $\mathcal{T}$ is bounded.

In addition, relation (5.12) implies that

$$\|(\mathcal{T} - \mathcal{H})(\gamma)\|_1 \leq \alpha c \,\|\gamma\|_\infty \leq \alpha c \,\|\gamma\|_{\mathcal{H}}, \quad \forall \gamma \in \mathcal{D}.$$

This yields that

$$\|\mathcal{T} - \mathcal{H}\| \leq \alpha c. \tag{5.14}$$

From (5.14) it follows that if

$$c < \frac{1}{\alpha \|\mathcal{H}^{-1}\|} \tag{5.15}$$

then $\mathcal{T}$ is invertible. This shows that for every $s \in \ell_0^1(\mathbb{N}, X)$ there is a unique $\gamma \in \ell_Y^\infty(\mathbb{N}, X)$ with $t_\gamma = s$, which means that (γ, s) satisfies (C_B). Thus the proof of *Step 1* is complete.

As in the previous sections, for $h \in \mathbb{N}, h \geq 2$, we consider

$$Q_A^h(n) : X \to X, \quad Q_A^h(n) = \Phi_A(h^n, h^{n-1})$$

and the system

$$y(n+1) = Q_A^h(n)y(n), \quad n \in \mathbb{N}. \tag{Q_A^h}$$

We associate with (Q_A^h) the control system

$$\gamma(n+1) = Q_A^h(n)\gamma(n) + s(n+1), \quad \forall n \in \mathbb{N}. \tag{$C_{Q_A^h}$}$$

Similarly, for $h \in \mathbb{N}, h \geq 2$, we take

$$Q_B^h(n) : X \to X, \quad Q_B^h(n) = \Phi_B(h^n, h^{n-1})$$

and consider the system

$$y(n+1) = Q_B^h(n)y(n), \quad n \in \mathbb{N}. \tag{Q_B^h}$$

We associate with (Q_B^h) the control system

$$\gamma(n+1) = Q_B^h(n)\gamma(n) + s(n+1), \quad \forall n \in \mathbb{N}. \tag{$C_{Q_B^h}$}$$

Step 2. Let $h \geq 2$. We prove that for every $s \in \ell_0^\infty(\mathbb{N}, X)$ there exists a unique solution w of $(C_{Q_B^h})$ with $w \in \ell_Y^\infty(\mathbb{N}, X)$.

First, we note that from (5.9) we have that

$$\|Q_A^h(n)\| \leq Dh^a, \quad \forall n \in \mathbb{N}. \tag{5.16}$$

Then, it makes sense to consider the linear operator

$$\mathcal{V} : \ell_Y^\infty(\mathbb{N}, X) \to \ell_0^\infty(\mathbb{N}, X), \quad (\mathcal{V}(\lambda))(n) = \begin{cases} \lambda(n) - Q_A^h(n-1)\lambda(n-1), & n \geq 2 \\ 0, & n = 1 \end{cases}.$$

Using (5.16) we obtain that $\mathcal{V}$ is well-defined and bounded.

From the proof of the necessity part of Theorem 4.2 it yields that for every $s \in \ell_0^\infty(\mathbb{N}, X)$ there exists a unique solution w of $(C_{Q_A^h})$ with $w \in \ell_Y^\infty(\mathbb{N}, X)$. This shows that $\mathcal{V}$ is invertible.

From (5.10) and Lemma 5.2 it follows that there is $b > a$ such that

$$\|\Phi_B(m, n)\| \leq D\left(\frac{m}{n}\right)^b, \quad \forall (m, n) \in \Gamma. \tag{5.17}$$

In particular, from (5.17) we get that

$$\|Q_B^h(n)\| \leq Dh^b, \quad \forall n \in \mathbb{N}. \tag{5.18}$$

Then, from (5.1), (5.9), (5.10) and (5.17) we deduce that

$$
\begin{aligned}
\|Q_B^h(n) - Q_A^h(n)\| &= \|\Phi_B(h^n, h^{n-1}) - \Phi_A(h^n, h^{n-1})\| \\
&\leq \sum_{j=h^{n-1}}^{h^n-1} \|\Phi_A(h^n, j+1)\| \, \|B(j) - A(j)\| \, \|\Phi_B(j, h^{n-1})\| \\
&\leq cD^2 \sum_{j=h^{n-1}}^{h^n-1} \left(\frac{h^n}{j+1}\right)^a \frac{1}{(j+1)^\rho} \left(\frac{j}{h^{n-1}}\right)^b \\
&\leq cD^2 \sum_{j=h^{n-1}}^{h^n-1} \frac{1}{j+1} \left(\frac{h^n}{j+1}\right)^b \left(\frac{j}{h^{n-1}}\right)^b \\
&< cD^2 h^b \sum_{j=h^{n-1}}^{h^n-1} \frac{1}{j+1} \\
&< cD^2 h^b \ln h, \qquad \forall n \in \mathbb{N}.
\end{aligned}
\tag{5.19}
$$

From (5.18) it follows that it makes sense to consider

$$
\mathcal{W} : \ell_Y^\infty(\mathbb{N}, X) \to \ell_0^\infty(\mathbb{N}, X), \quad (\mathcal{W}(\lambda))(n) = \begin{cases} \lambda(n) - Q_B^h(n-1)\lambda(n-1), & n \geq 2 \\ 0, & n = 1 \end{cases}
$$

which is a bounded linear operator.

Setting

$$
\beta := D^2 h^b \ln h
$$

from (5.19) it yields that

$$
\|(\mathcal{W} - \mathcal{V})(\lambda)(n)\| \leq \|Q_B^h(n-1) - Q_A^h(n-1)\| \, \|\lambda(n-1)\| < c\beta \|\lambda\|_\infty, \tag{5.20}
$$

for all $n \geq 2$ and all $\lambda \in \ell_Y^\infty(\mathbb{N}, X)$. From (5.20) we get that

$$
\|\mathcal{W} - \mathcal{V}\| \leq c\beta. \tag{5.21}
$$

So, if

$$
c < \frac{1}{\beta \|\mathcal{V}^{-1}\|} \tag{5.22}
$$

then from (5.21) it follows that $\mathcal{W}$ is invertible. This implies that for every $s \in \ell_0^\infty(\mathbb{N}, X)$ there is a unique $w \in \ell_Y^\infty(\mathbb{N}, X)$ with $\mathcal{W}(w) = s$. This means that (w, s) satisfies $(C_{Q_B^h})$ and so the proof of the second step is completed.

From *Step 1* and *Step 2* and relations (5.15) and (5.22) it follows that it is sufficient to choose

$$
c < \min\left\{\frac{1}{\alpha\|\mathcal{H}^{-1}\|}, \frac{1}{\beta\|\mathcal{V}^{-1}\|}\right\}
$$

and then the hypotheses (i) and (ii) of Theorem 4.2 are fulfilled. By Theorem 4.2 we obtain that (B) admits a polynomial dichotomy. $\square$

6. Conclusions

Our study has been devoted to admissibility methods for exploring dichotomies of discrete nonautonomous systems on the half-line in the most general case. We have described the dichotomic behaviors by means of some admissibility notions with respect to associated input-output systems for which the input sequences belong to some ℓ^p-spaces, with $p \in [1, \infty]$ and the output sequences are bounded.

We have given new conditions for ordinary and exponential dichotomy and we have obtained new characterizations for polynomial dichotomy. While the ordinary and exponential dichotomies have been detected via a solvability of an input-output system, we have shown that the polynomial dichotomy can be described by means of the solvability of *two input-output systems*: the one used for the first two dichotomy notions and another (well-chosen) one. Thus, we have provided two categories of characterizations for polynomial dichotomy by means of some suitable double admissibilities. Our methods are developed here for the first time, the major tool in our approach relying on connections between certain families of projections that describe the dichotomic behaviors (see the constructions in the proof of Theorem 4.1).

In parallel, we have presented a deep analysis on two technical requirements that occur in the studies regarding the dichotomies on the half line. One concerns the complementarity of the initial stable subspace and the other one the uniqueness of the solution in the admissibility notions. In this context, we have given criteria for (ordinary, exponential, polynomial) dichotomy assuming the complementarity of the stable subspace(s) at the initial time and this was done by working with non-unique solvabilities. After that, we have shown that if some unique solvabilities are imposed relative to a fixed closed subspace, then the initial stable subspaces are complemented. Consequently, we have obtained characterizations for dichotomies on the half-line in terms of some unique solvabilities with respect to a fixed initial unstable subspace.

As an application of our results we have shown that the polynomial dichotomy persists under small perturbations. With this purpose we have presented a new input-output method that points out interesting connections between some solvabilities of the control systems associated to initial system and those of the control system associated to the perturbed system (see the approach given in the proof of Theorem 5.1). The techniques developed here combine tools from functional analysis, operator theory and control.

Acknowledgments. D. D. was supported in part by Croatian Science Foundation under the project IP-2019-04-1239 and by the University of Rijeka under the projects uniri-prirod-18-9 and uniri-pr-prirod-19-16. B. S. was partially supported under the project "Consolidarea capacităţii instituţionale a Universităţii de Vest din Timişoara în domeniul cercetării ştiinţifice de excelenţă" FDI 0246. This work is partially supported by the Romanian Ministry of Research, Innovation and Digitization, CNCS - UEFISCDI, grant PN-III-P4-PCE-2021-1279, within PNCDI III.

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