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Dedicated to Prof. Emeritus Mihail Megan on the occasion of his 75^{th} anniversary

Fixed points and the stability of the linear functional equations in a single variable

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ABSTRACT. In this paper we prove that an interesting result concerning the generalized Hyers-Ulam stability of the linear functional equation $g(\varphi(x)) = a(x) \bullet g(x)$ on a complete metric group, given in 2014 by S.M. Jung, D. Popa and M.T. Rassias, can be obtained using the fixed point technique. Moreover, we give a characterization of the functions that can be approximated with a given error, by the solution of the linear equation mention above. Our results are also related to a recent result of G.H. Kim and Th.M. Rassias concerning the stability of Psi functional equation.

1. INTRODUCTION

"When a solution of an equation differing slightly from a given one must be somehow near to the solution of the given equation?" is the question formulated in 1940 by S.M. Ulam [33] while giving a lecture at the University of Wisconsin, on the stability group homomorphisms. In a more precise formulation, its problem of stability reads as follows:

Let $(G_1, \circ)$ be a group, $(G_2, *)$ be a metric group with the metric $d(\cdot, \cdot)$ and $\varepsilon > 0$. Does there exists a $\delta > 0$ such that if $f : G_1 \to G_2$ satisfies

$$d(f(x \circ y), f(x) * f(y)) \leq \delta, \quad \text{for all } x, y \in G_1$$

there exists a homomorphism $h : G_1 \to G_2$ with

$$d(f(x), h(x)) \leq \varepsilon, \quad \text{for all } x \in G_1?$$

A first answer to Ulam's question was given by D. H. Hyers [22] in 1941 concerning the Cauchy functional equation. Afterwards different generalizations of that initial answer of Hyers were obtained. Hyers' theorem was generalized by Aoki [1] for additive mappings and by Th.M. Rassias [31] for approximately linear mappings, by considering an unbounded Cauchy difference. See also [17], [30] and [32]. Nowadays we speak about the concept of *Hyers-Ulam stability*.

A further generalization was obtained by P. Găvruţa [18] in 1994. See also [19] and [21] for more generalizations. The papers mentioned above use the direct method (of Hyers), i.e., the exact solution of the functional equation is explicitly constructed as a limit of a sequence, starting from the given approximate solution.

For other results and generalizations, see the books [5],[15], [23],[25] and their references.

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Among applications of the functional equations, we mention modeling in science and engineering (see [13]). An interesting application of the stability in the sense of Hyers-Ulam pointed out by D.H. Hyers, G. Isac and Th. M. Rassias is in the study of complementarity problems (see the book [23]). For other complementarity problems, see also the article of G. Isac [24].

On the other hand, J.A. Baker [2] used in 1991 the Banach fixed point theorem to give Hyers-Ulam stability results for a nonlinear functional equation.

In 2003, V. Radu [29] proposed a new method, successively developed in [6], to obtain the existence of the exact solutions and the error estimations, based on the fixed point alternative. For some other applications of the fixed point theorem in the generalized Hyers-Ulam stability see the papers [7], [8], [9], [11], [12], [14], [16], [20], [28].

Recently, J. Brzdęk, J. Chudziak & Z. Páles proved in [3] a general fixed point theorem for (not necessarily) linear operators and they used it to obtain Hyers-Ulam stability results for a class of functional equations in a single variable. A fixed point result of the same type was proved by J. Brzdęk & K. Ciepliński [4] in complete non-Archimedean metric spaces as well as in complete metric spaces. Also, they formulated an open problem concerning the uniqueness of the fixed point.

In the paper [10] we obtained a fixed point theorem for a class of operators with suitable properties, in very general conditions. Also, we showed that some recent results in [3] and [4] can be obtained as particular cases of our theorem. Moreover, by using our outcome, we gave affirmative answer to the open problem of J. Brzdęk & K. Ciepliński, formulated at the end of the paper [4]. We also showed that our main Theorem is an efficient tool for proving generalized Hyers-Ulam stability results of several functional equations in a single variable. To this end, we prove in this paper that an interesting result concerning generalized Hyers-Ulam-Rassias stability of a linear functional equation obtained in 2014 by S.M. Jung, D. Popa and M.T. Rassias in [26] is a particular case of a fixed point theorem given by us in [10]. Moreover, we give a characterization of the functions that can be approximated with a given error, by the solution of the previously mention linear equation.

We consider a nonempty set X, a complete metric space (Y, d) and the mappings

$$\Lambda : \mathbb{R}_+^X \to \mathbb{R}_+^X \text{ and } \mathcal{T} : Y^X \to Y^X.$$

We recall that, for two sets M and N, N^M is the space of all mappings from M to N and if $(\delta_n)_{n\in\mathbb{N}}$ is a sequence of elements of $\mathbb{R}_+^X$, we write

$$\lim_{n\to\infty} \delta_n = 0 \quad \text{pointwise if} \quad \lim_{n\to\infty} \delta_n(x) = 0 \text{ for every } x \in X.$$

$\mathbb{R}_+$ stands for the set of all nonnegative numbers, i.e., $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{R}_+^* = (0, \infty)$.

Definition 1.1. [10] *We say that $\mathcal{T}$ is $\Lambda-$ contractive if for all $u, v \in Y^X$ and $\delta \in \mathbb{R}_+^X$ with*

$$d(u(x), v(x)) \le \delta(x), \ \forall\, x \in X,$$

it follows

$$d((\mathcal{T}u)(x), (\mathcal{T}v)(x)) \le (\Lambda\delta)(x), \ \forall\, x \in X.$$

In the paper [10] we obtained the following fixed point theorem:

Theorem 1.1. *We suppose that the operator $\mathcal{T}$ is $\Lambda-$*contractive, *where Λ satisfies the condition:*
(C_1) *for every sequence $(\delta_n)_{n\in\mathbb{N}}$ in $\mathbb{R}_+^X$ such that*

$$\lim_{n\to\infty} \delta_n = 0 \textit{ pointwise, it follows that } \lim_{n\to\infty} \Lambda\delta_n = 0 \textit{ pointwise.}$$

We suppose that $\varepsilon \in \mathbb{R}_+^X$ is a given function such that

$$(C_2) \qquad \varepsilon^*(x) := \sum_{k=0}^{\infty} \left(\Lambda^k \varepsilon\right)(x) < \infty,\ \forall x \in X.$$

We consider a mapping $f \in Y^X$ such that

$$(1.1) \qquad d((\mathcal{T} f)(x), f(x)) \leq \varepsilon(x),\ \forall x \in X.$$

Then, for every $x \in X$, the limit

$$(1.2) \qquad g(x) := \lim_{n\to\infty} (\mathcal{T}^n f)(x),$$

exists and the function g is the unique fixed point of $\mathcal{T}$ with the property

$$(1.3) \qquad d((\mathcal{T}^m f)(x), g(x)) \leq \sum_{k=m}^{\infty} \left(\Lambda^k \varepsilon\right)(t),\ x \in X,\ m \in \mathbb{N}.$$

Moreover, if we have

$$(C_3) \qquad \lim_{n\to\infty} \Lambda^n \varepsilon^* = 0 \textit{ pointwise},$$

then g is the unique fixed point of $\mathcal{T}$ with the property

$$(1.4) \qquad d(f(x), g(x)) \leq \varepsilon^*(x), \forall x \in X.$$

Theorem 1.1 generalizes a result of J. Brzdęk and K. Ciepliński [4] concerning the existence of the fixed points. Moreover, our theorem provides a positive answer to the open question raised by these authors concerning the uniqueness of the fixed point.

2. Stability of the functional equation $g(\varphi(x)) = a(x) \bullet g(x)$

We take a nonempty set X and a complete metric group $(G, \bullet, d)$ with the metric d invariant to the left translation, i.e.,

$$d(x \bullet y, x \bullet z) = d(y, z),\ \text{for all } x, y, z \in G.$$

We consider the given functions $\varphi : X \to X$ and $a : X \to G$.

We denote

$$A_n(x) := a\left(\varphi^{n-1}(x)\right) \bullet \ldots \bullet a(\varphi(x)) \bullet a(x), \quad x \in X,\ n \geq 1.$$

We have

$$A_n(\varphi(x)) = A_{n+1}(x) \bullet (a(x))^{-1}, \quad x \in X, n \geq 1,$$

and successive by

$$A_n\left(\varphi^m(x)\right) = A_{n+m}(x) \bullet \left(A_m(x)\right)^{-1}, \quad x \in X,\ m, n \geq 1.$$

In this section we discuss the generalized Hyers-Ulam-Rassias stability of the functional equation

$$(2.5) \qquad g(\varphi(x)) = a(x) \bullet g(x), x \in X,$$

where $g : X \to G$ is the unknown function.

The equation (2.5) is equivalent to

$$(2.6) \qquad (a(x))^{-1} \bullet g(\varphi(x)) = g(x), x \in X.$$

We remark also that

$$(2.7) \qquad g\left(\varphi^n(x)\right) = A_n(x) \bullet g(x), \quad x \in X, n \geq 1.$$

In what follows we will show that the main result of the paper [26] concerning the generalized Hyers-Ulam-Rassias stability of the equation (2.5) is a consequence of our Theorem 1.1. To this end, we start with the presentation of the main result from [26]:

Theorem 2.2. [26] *Let $\varepsilon : X \to \mathbb{R}_+$ be a given function with the property*

$$\varepsilon^*(x) := \sum_{k=0}^{\infty} \varepsilon\left(\varphi^k(x)\right) < \infty,\ \forall x \in X.$$

Then, for every function $f : X \to G$ satisfying the inequality

$$d(f(\varphi(x)), a(x) \bullet f(x)) \leq \varepsilon(x),\ \forall x \in X, \tag{2.8}$$

there exists a unique solution g of the equation (2.5) such that

$$d(f(x), g(x)) \leq \varepsilon^*(x), \forall x \in X. \tag{2.9}$$

This solution is given by the formula

$$g(x) := \lim_{n\to\infty} (A_n(x))^{-1} \bullet f\left(\varphi^n(x)\right). \tag{2.10}$$

We can easily see that the above theorem is a particular case of our fixed point result emphasized in the first section.

Proof. We take in Theorem 1.1

$$(\mathcal{T}u)(x) = (a(x))^{-1} \bullet u(\varphi(x)) \quad \text{and} \quad (\Lambda\delta)(x) = \delta(\varphi(x)).$$

So, it follows

$$d\left((\mathcal{T}u)(x), (\mathcal{T}v)(x)\right) = d(u(\varphi(x)), v(\varphi(x))) \leq (\Lambda\delta)(x)$$

if

$$d(u(x), v(x)) \leq \delta(x),$$

hence the operator $\mathcal{T}$ is $\Lambda-$ contractive in the sense of the Definition 1.1.

On the other hand, by using the invariance property to the left translation of the metric d and the assumption (2.8), we obtain that (1.1) holds.

Uniqueness of g results also from Theorem 1.1. In fact, we prove that Λ satisfies the hypothesis (C_3) :

$$\begin{aligned}\Lambda^n(\varepsilon^*(x)) &= \Lambda^n\left(\sum_{k=0}^{\infty}\varepsilon\left(\varphi^k(x)\right)\right) = \\ &= \sum_{k=0}^{\infty}\varepsilon\left(\varphi^{n+k}(x)\right) = \sum_{m=n}^{\infty}\varepsilon\left(\varphi^m(x)\right).\end{aligned}$$

Thus

$$\lim_{n\to\infty} \Lambda^n(\varepsilon^*(x)) = 0,\ x \in X.$$

□

In the second Theorem of this section we will give a characterization of the functions $f : X \to G$ that can be approximated with a given error, by a solution of the equation (2.5).

We denote by

$$\mathcal{E}_\varphi = \left\{\varepsilon \in \mathbb{R}_+^X,\ \lim_{n\to\infty} \varepsilon\left(\varphi^n(x)\right) = 0, \forall x \in X\right\}.$$

Theorem 2.3. *The following statements are equivalent:*
(i) *There exists a unique solution* g *of (2.5) such that*

$$d(f(x),g(x)) \leq \varepsilon(x), \forall x \in X.$$

(ii) $\quad d\left(f\left(\varphi^{n}(x)\right), A_{n}(x) \bullet f(x)\right) \leq \varepsilon(x)+\varepsilon\left(\varphi^{n}(x)\right),\ x \in X, n \geq 1.$
(iii) *there exists* $\quad \delta \in \mathcal{E}_{\varphi} \quad$ *such that*

$$d\left(f\left(\varphi^{n}(x)\right), A_{n}(x) \bullet f(x)\right) \leq \varepsilon(x)+\delta\left(\varphi^{n}(x)\right),\ x \in X, n \geq 1.$$

Proof. $(i) \Rightarrow (ii)$. We have, by using (2.7)

$$\begin{aligned} d\left(f\left(\varphi^{n}(x)\right), A_{n}(x) \bullet f(x)\right) & \leq d\left(f\left(\varphi^{n}(x)\right), g\left(\varphi^{n}(x)\right)\right)+d\left(g\left(\varphi^{n}(x)\right), A_{n}(x) \bullet f(x)\right) \\ & \leq \varepsilon\left(\varphi^{n}(x)\right)+d\left(A_{n}(x) \bullet g(x), A_{n}(x) \bullet f(x)\right) \\ & = \varepsilon\left(\varphi^{n}(x)\right)+\varepsilon(x). \end{aligned}$$

$(ii) \Rightarrow (iii)$. We take in (ii) $\delta=\varepsilon$.

$(iii) \Rightarrow (i)$. In (iii) with $\varphi^{m}(x)$ instead of x, we have

$$d\left(f\left(\varphi^{n+m}(x)\right), A_{n}\left(\varphi^{m}(x)\right) \bullet f\left(\varphi^{m}(x)\right)\right) \leq \varepsilon\left(\varphi^{m}(x)\right)+\delta\left(\varphi^{n+m}(x)\right),$$

which means

$$d\left(f\left(\varphi^{n+m}(x)\right), A_{n+m}(x) \bullet\left(A_{m}(x)\right)^{-1} \bullet f\left(\varphi^{m}(x)\right)\right) \leq \varepsilon\left(\varphi^{m}(x)\right)+\delta\left(\varphi^{n+m}(x)\right),$$

hence

$$d\left(\left(A_{n+m}(x)\right)^{-1} \bullet f\left(\varphi^{m+n}(x)\right),\left(A_{m}(x)\right)^{-1} \bullet f\left(\varphi^{m}(x)\right)\right) \leq \varepsilon\left(\varphi^{m}(x)\right)+\delta\left(\varphi^{n+m}(x)\right).$$

It follows that the sequence

$$\left\{\left(A_{n}(x)\right)^{-1} \bullet f\left(\varphi^{n}(x)\right)\right\}_{n \geq 1}$$

is a Cauchy sequence. Since $(G, \bullet, d)$ is complete, it results that there exists

$$g(x):=\lim _{n \rightarrow \infty}\left(A_{n}(x)\right)^{-1} \bullet f\left(\varphi^{n}(x)\right), x \in X.$$

We have

$$g(\varphi(x))=a(x) \bullet \lim _{n \rightarrow \infty}\left(A_{n+1}(x)\right)^{-1} \bullet f\left(\varphi^{n+1}(x)\right)=a(x) \bullet g(x), x \in X,$$

hence g is a solution of (2.5) and

$$d\left(g(x),\left(A_{m}(x)\right)^{-1} \bullet f\left(\varphi^{m}(x)\right)\right) \leq \varepsilon\left(\varphi^{m}(x)\right), x \in X, m \geq 1.$$

By (iii) it follows that

$$d\left(\left(A_{n}(x)\right)^{-1} \bullet f\left(\varphi^{n}(x)\right), f(x)\right) \leq \varepsilon(x)+\delta\left(\varphi^{n}(x)\right)$$

and by letting n go to infinity, we obtain

$$d(f(x),g(x)) \leq \varepsilon(x), \forall x \in X.$$

We prove now the uniqueness of g. To this end, let us consider a solution $h: X \rightarrow G$ of the equation (2.5), satisfying the relation

$$d(h(x),f(x)) \leq \varepsilon(x), \forall x \in X.$$

By replacing x by $\varphi^{m}(x)$, we have

$$d(h\left(\varphi^{m}(x)\right), f\left(\varphi^{m}(x)\right)) \leq \varepsilon\left(\varphi^{m}(x)\right), \forall x \in X.$$

Having in mind that $h\left(\varphi^{m}(x)\right)=A_{m}(x) \bullet h(x)$, it follows

$$d\left(h(x),\left(A_{m}(x)\right)^{-1} \bullet f\left(\varphi^{m}(x)\right)\right) \leq \varepsilon\left(\varphi^{m}(x)\right), x \in X.$$

Letting m go to infinity, we obtain

$$d(h(x), g(x)) = 0, \forall x \in X.$$

□

As a direct application of the Theorem 2.3 we will obtain the following result concerning the characterization of the functions $f : \mathbb{R}_+^* \to \mathbb{R}$ that can be approximated with a given error, by the solutions of Digamma functional equation

$$g(x+1) = g(x) + \frac{1}{x},\ x \in \mathbb{R}_+^*. \tag{2.11}$$

Corollary 2.1. *The following statements are equivalent:*
(i) *There exists a unique solution* g *of (2.11) such that*

$$|f(x) - g(x)| \le \varepsilon(x), \forall x \in \mathbb{R}_+^*.$$

$$(ii) \qquad \left| f(x+n) - f(x) - \sum_{k=0}^{n-1} \frac{1}{x+k} \right| \le \varepsilon(x) + \varepsilon(x+n),\ x \in \mathbb{R}_+^*, n \ge 1.$$

(iii) *There exists*

$$\delta \in \mathcal{E}_\varphi := \left\{ \varepsilon : X \to \mathbb{R}_+,\ \lim_{n\to\infty} \varepsilon\left(x+n\right) = 0, \forall x \in \mathbb{R}_+^* \right\}$$

so that

$$\left| f(x+n) - f(x) - \sum_{k=0}^{n-1} \frac{1}{x+k} \right| \le \varepsilon(x) + \delta(x+n),\ x \in \mathbb{R}_+^*, n \ge 1.$$

Proof. The result follows immediately by taking in Theorem 2.3, $X = \mathbb{R}_+^*$, $(G, \bullet) = (\mathbb{R}, +)$, d the Euclidean metric on $\mathbb{R}$, $\varphi(x) = x + 1$, $a(x) = \frac{1}{x}, x \in \mathbb{R}_+^*$. □

We give below a more general result obtained in the same way, which is in connection with the recent paper of G.H. Kim and Th. M. Rassias [27].

Corollary 2.2. *Let* p *be a positive real number. The following statements are equivalent:*
(i) *There exists a unique solution* g *of the functional equation*

$$g(x+p) = g(x) + a(x),\ x \in \mathbb{R}_+^*$$

such that

$$|f(x) - g(x)| \le \varepsilon(x), (\forall)\ x \in \mathbb{R}_+^*.$$

$$(ii) \qquad \left| f(x+np) - f(x) - \sum_{k=0}^{n-1} a(x+kp) \right| \le \varepsilon(x) + \varepsilon(x+np),\ x \in \mathbb{R}_+^*, n \ge 1.$$

(iii) *There exists*

$$\delta \in \mathcal{E}_\varphi := \left\{ \varepsilon : X \to \mathbb{R}_+,\ \lim_{n\to\infty} \varepsilon\left(x+np\right) = 0, \forall x \in \mathbb{R}_+^* \right\}$$

so that

$$\left| f(x+np) - f(x) - \sum_{k=0}^{n-1} a(x+kp) \right| \le \varepsilon(x) + \delta(x+np),\ x \in \mathbb{R}_+^*, n \ge 1.$$

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