CARPATHIAN J. MATH.
Volume **38** (2022), No. 3,
Pages 809-816

Online version at `https://semnul.com/carpathian/`
Print Edition: ISSN 1584 - 2851; Online Edition: ISSN 1843 - 4401
DOI: https://doi.org/10.37193/CJM.2022.03.24

Dedicated to Prof. Emeritus Mihail Megan on the occasion of his 75^{th} anniversary

On uniform polynomial trichotomy of skew-evolution semiflows

CLAUDIA LUMINIŢA MIHIŢ[1]

ABSTRACT. The paper treats two concepts of uniform polynomial trichotomy for the skew-evolution semiflows in Banach spaces. We obtain the connection between them, a characterization for a property of uniform polynomial growth and a sufficient criteria for the uniform polynomial trichotomy.

1. INTRODUCTION

The study of the asymptotic behaviors of dynamical systems as stability, dichotomy and trichotomy has known an impressive development in the last decades. These properties are treated from various perspectives: exponential ([17], [18], [23]), polynomial ([4], [5], [8]-[10], [19], [21], [22]) or with growth rates ([3], [20]).

The polynomial behavior is approached for the first time by L. Barreira and C. Valls in [2], more exactly a concept of nonuniform polynomial dichotomy for evolution operators. Also, we remark the results obtained by P. V. Hai ([12]) for the polynomial stability, using techniques of Banach function spaces. Regarding more general properties of nonuniform polynomial trichotomy, in [1], the authors consider three concepts of polynomial trichotomy for evolution operators and emphasize interesting examples, respectively counterexamples.

The article of R. Datko ([7]) represented an important direction of research to obtain integral conditions for the asymptotic properties. In this sense, we mention the contributions of M. Megan, A. L. Sasu, B. Sasu ([14]-[16]) for the the exponential stability/instability of linear skew-product semiflows. Recently, R. Boruga (Toma) and M. Megan ([6]) have proved necessary and sufficient conditions of Datko-type for the polynomial dichotomy of evolution operators in the nonuniform case.

The trichotomy property is considered the most complex asymptotic property and it is intensive studied in a large number of papers: [11], [13], [27], [28] and the references therein. Different known tools are used to obtain qualitative results: in [24] and [26], the authors prove important criteria for the exponential trichotomy, using input-output techniques. Also, the exponential trichotomy is studied in discrete case by A. L. Sasu and B. Sasu in [25], with Zabczyk-type methods.

In this paper the uniform polynomial trichotomy in the classical sense and the uniform polynomial trichotomy are approached with invariant families of projectors. The relation between them is established and a sufficient condition of Datko type is given, using a property of uniform polynomial growth.

Received: 27.03.2022. In revised form: 14.07.2022. Accepted: 15.07.2022
1991 *Mathematics Subject Classification.* 34D05; 34D09.
Key words and phrases. *Polynomial trichotomy; skew-evolution semiflows.*

2. PRELIMINARIES

Let X be a real or complex Banach space and Θ a metric space. $\mathcal{B}(X)$ represents the Banach algebra of all bounded linear operators on X and the norms on X, respectively on $\mathcal{B}(X)$, will be denoted by $||\cdot||$. Also,

$$\Delta = \{(t,s) \in \mathbb{R}_+^2 : t \geq s\}, \quad T = \{(t,s,t_0) \in \mathbb{R}_+^3 : t \geq s \geq t_0\}$$

and $\Gamma = \Theta \times X$.

Definition 2.1. We say that a continuous mapping $\varphi : \Delta \times \Theta \to \Theta$ is *evolution semiflow* on Θ if:

(es_1) $\varphi(s,s,\theta) = \theta,$ for all $(s,\theta) \in \mathbb{R}_+ \times \Theta$;

(es_2) $\varphi(t,s,\varphi(s,t_0,\theta)) = \varphi(t,t_0,\theta),$ for all $(t,s,t_0,\theta) \in T \times \Theta$.

Definition 2.2. The mapping $\Phi : \Delta \times \Theta \to \mathcal{B}(X)$ is called *evolution cocycle* over the evolution semiflow φ if the following conditions hold:

(ec_1) $\Phi(s,s,\theta) = I$ (the identity operator on X), for all $(s,\theta) \in \mathbb{R}_+ \times \Theta$;

(ec_2) $\Phi(t,s,\varphi(s,t_0,\theta))\Phi(s,t_0,\theta) = \Phi(t,t_0,\theta),$ for all $(t,s,t_0,\theta) \in T \times \Theta$;

(ec_3) $(t,s,\theta) \mapsto \Phi(t,s,\theta)x$ is continuous for every $x \in X$.

Definition 2.3. If φ is an evolution semiflow on Θ and Φ is an evolution cocycle over the evolution semiflow φ, then the pair $C = (\varphi, \Phi)$ is called *skew-evolution semiflow* on Γ.

Definition 2.4. A continuous mapping $P : \mathbb{R}_+ \times \Theta \to \mathcal{B}(X)$, with

$$P^2(t,\theta) = P(t,\theta), \quad \text{for all} \quad (t,\theta) \in \mathbb{R}_+ \times \Theta,$$

is called *family of projectors* on X.

Definition 2.5. A family of projectors $P : \mathbb{R}_+ \times \Theta \to \mathcal{B}(X)$ is called *compatible* with the skew-evolution semiflow $C = (\varphi, \Phi)$ if:

$$P(t,\varphi(t,s,\theta))\Phi(t,s,\theta) = \Phi(t,s,\theta)P(s,\theta), \quad \text{for all } (t,s,\theta) \in \Delta \times \Theta.$$

Remark 2.1. If a family of projectors $P : \mathbb{R}_+ \times \Theta \to \mathcal{B}(X)$ is compatible with a skew-evolution semiflow $C = (\varphi, \Phi)$, then the following invariance property holds:

$$\Phi(t,s,\theta) Range\ P(s,\theta) \subseteq Range\ P(t,\varphi(t,s,\theta)), \quad \text{for all } (t,s,\theta) \in \Delta \times \Theta.$$

Definition 2.6. Let $P_1, P_2, P_3 : \mathbb{R}_+ \times \Theta \to \mathcal{B}(X)$ be three families of projectors on X. We say that $\mathcal{P} = \{P_1, P_2, P_3\}$ is a family of *supplementary* projectors if:

(s_1) $P_1(s,\theta) + P_2(s,\theta) + P_3(s,\theta) = I$;

(s_2) $P_i(s,\theta)P_j(s,\theta) = 0,$

for all $(s,\theta) \in \mathbb{R}_+ \times \Theta,\ i,j \in \{1,2,3\},\ i \neq j$.

3. THE MAIN RESULTS

Let $C = (\varphi, \Phi)$ be a skew-evolution semiflow on Γ and $\mathcal{P} = \{P_1, P_2, P_3\}$ a family of projectors supplementary and compatible with C.

Definition 3.7. We say that $(C, \mathcal{P})$ admits a *uniform polynomial trichotomy in the classical sense* if there exist $N \geq 1$, $\nu, \omega > 0$ and $s_0 > 0$ such that:

$(cupt_1)$ $u^\nu(||\Phi(u,r,\theta)P_1(r,\theta)x|| + ||\Phi(s,r,\theta)P_2(r,\theta)x||) \leq$
$\leq Ns^\nu(||\Phi(s,r,\theta)P_1(r,\theta)x|| + ||\Phi(u,r,\theta)P_2(r,\theta)x||)$;

$(cupt_2)$ $s^\omega||\Phi(u,r,\theta)P_3(r,\theta)x|| \leq Nu^\omega||\Phi(s,r,\theta)P_3(r,\theta)x||$;

$(cupt_3)$ $s^\omega||\Phi(s,r,\theta)P_3(r,\theta)x|| \leq Nu^\omega||\Phi(u,r,\theta)P_3(r,\theta)x||$, for all $(u, s, r) \in T$, $r \geq s_0$ and all $(\theta, x) \in \Gamma$.

Remark 3.1. The pair $(C, \mathcal{P})$ is uniformly polynomially trichotomic in the classical sense if and only if there are the constants $N \geq 1$, $\nu, \omega > 0$ and $s_0 > 0$ with:

$(cupt'_1)$ $u^\nu(||\Phi(u,s,\theta)P_1(s,\theta)x|| + ||P_2(s,\theta)x||) \leq$
$\leq Ns^\nu(||P_1(s,\theta)x|| + ||\Phi(u,s,\theta)P_2(s,\theta)x||)$;

$(cupt'_2)$ $s^\omega||\Phi(u,s,\theta)P_3(s,\theta)x|| \leq Nu^\omega||P_3(s,\theta)x||$;

$(cupt'_3)$ $s^\omega||P_3(s,\theta)x|| \leq Nu^\omega||\Phi(u,s,\theta)P_3(s,\theta)x||$,

for all $(u, s) \in \Delta$, $s \geq s_0$ and all $(\theta, x) \in \Gamma$.

Example 3.1. We consider $X = \mathbb{R}^3$ endowed with the norm

$$||x|| = |x_1| + |x_2| + |x_3|,$$

$\Theta = \mathbb{R}_+$ and the evolution semiflow $\varphi : \Delta \times \Theta \to \Theta$,

$$\varphi(s,r,\theta) = \begin{cases} \ln\frac{s}{r} + \theta, & \text{if } s \geq r > 0 \\ \theta, & \text{if } s = r = 0. \end{cases}$$

In addition, we consider $P_i : \mathbb{R}_+ \times \Theta \to \mathcal{B}(X)$, $i = \overline{1,3}$, where $P_1(s,\theta)x = (x_1, 0, 0)$, $P_2(s,\theta)x = (0, x_2, 0)$, $P_3(s,\theta)x = (0, 0, x_3)$ and the mapping $\Phi : \Delta \times \Theta \to \mathcal{B}(X)$ given by $\Phi(s,r,\theta)x =$

$$= \begin{cases} \left(e^{-\int\limits_r^s \frac{A_1\left(\ln\frac{\xi}{r}+\theta\right)}{\xi}d\xi} x_1, e^{\int\limits_r^s \frac{A_1\left(\ln\frac{\xi}{r}+\theta\right)}{\xi}d\xi} x_2, \left(\frac{r}{s}\right)^\beta e^{2\int\limits_r^s \frac{A_2\left(\ln\frac{\xi}{r}+\theta\right)}{\xi}d\xi} x_3 \right), & \text{if } s \geq r > 0 \\ (x_1, x_2, x_3), & \text{if } s = r = 0, \end{cases}$$

where $x = (x_1, x_2, x_3) \in X$. Here A_1, A_2 are two continuous functions such that A_1 is decreasing, A_2 is nondecreasing and

$$\lim_{s\to+\infty} A_1(s) = \alpha, \qquad \lim_{s\to+\infty} A_2(s) = \beta.$$

After some computations, we obtain that $(C, \mathcal{P})$ is uniformly polynomially trichotomic in the classical sense with $N = 1$, $\nu = \alpha$ and $\omega = \beta$.

Definition 3.8. The pair $(C, \mathcal{P})$ is called *uniformly polynomially trichotomic* if there are $N \geq 1$, $\nu > 0$ and $s_0 > 0$ such that:

(upt_1) $u^\nu(||\Phi(u,r,\theta)P_1(r,\theta)x|| + ||\Phi(s,r,\theta)P_2(r,\theta)x||) \leq$
$\leq Ns^\nu(||\Phi(s,r,\theta)P_1(r,\theta)x|| + ||\Phi(u,r,\theta)P_2(r,\theta)x||)$;

(upt_2) $||\Phi(u,r,\theta)P_3(r,\theta)x|| \leq N||\Phi(s,r,\theta)P_3(r,\theta)x||$;

(upt_3) $||\Phi(s,r,\theta)P_3(r,\theta)x|| \leq N||\Phi(u,r,\theta)P_3(r,\theta)x||$,

for all $(u, s, r) \in T$, $r \geq s_0$ and all $(\theta, x) \in \Gamma$.

Remark 3.2. $(C, \mathcal{P})$ is uniformly polynomially trichotomic if and only if there exist $N \geq 1$, $\nu > 0$ and $s_0 > 0$ with:

(upt'_1) $u^\nu(||\Phi(u,s,\theta)P_1(s,\theta)x|| + ||P_2(s,\theta)x||) \leq$
$\leq Ns^\nu(||P_1(s,\theta)x|| + ||\Phi(u,s,\theta)P_2(s,\theta)x||)$;

(upt'_2) $||\Phi(u,s,\theta)P_3(s,\theta)x|| \leq N||P_3(s,\theta)x||$;

(upt'_3) $||P_3(s,\theta)x|| \leq N||\Phi(u,s,\theta)P_3(r,\theta)x||$,

for all $(u, s) \in \Delta$, $s \geq s_0$ and all $(\theta, x) \in \Gamma$.

Remark 3.3. We observe that if $(C, \mathcal{P})$ is uniformly polynomially trichotomic, then it is uniformly polynomially trichotomic in the classical sense. The converse is not accomplished, as we emphasize in the following example.

Example 3.2. Let $X = \mathbb{R}^3$, $\mathcal{C}(\mathbb{R}_+, \mathbb{R}_+)$ be the set of all continuous functions $\theta : \mathbb{R}_+ \to \mathbb{R}_+$, endowed with the topology of uniform convergence on compact subsets of $\mathbb{R}_+$. Let Θ be the closure in $\mathcal{C}$ of the set $\{\theta_t,\ t \geq 0\}$, with $\theta_t(u) = \theta(t+u)$, $u \geq 0$.
Thus, the mapping $\varphi : \Delta \times \Theta \to \Theta$, $\varphi(s, r, \theta) = \theta_{\ln \frac{s+1}{r+1}}$. is an evolution semiflow on Θ.
We consider the evolution cocycle $\Phi : \Delta \times \Theta \to \mathcal{B}(X)$,

$$\Phi(s, r, \theta) = e^{c(s-r) - \int_r^s \theta\left(\ln \frac{\xi+1}{r+1}\right) d\xi} P_1(r, \theta) + e^{-c(s-r) + \int_r^s \theta\left(\ln \frac{\xi+1}{r+1}\right) d\xi} P_2(r, \theta) +$$

$$+ \left(\frac{\ln(s+1)}{\ln(r+1)}\right)^{\theta(0)-c} P_3(r, \theta),$$

where $\theta : \mathbb{R}_+ \to \mathbb{R}_+$ is a decreasing function with $\lim_{s \to +\infty} \theta(s) = l$, $0 < c < l$ and $P_i : \mathbb{R}_+ \times \Theta \to \mathcal{B}(X)$, $i = \overline{1, 3}$ are the families of projectors given by Example 3.1.

For all $s \geq r \geq 1$ and all $(\theta, x) \in \Gamma$ we have:

$$||\Phi(s, r, \theta) P_1(r, \theta)x|| \leq e^{c(s-r) - l(s-r)} ||P_1(r, \theta)x|| \leq \left(\frac{r}{s}\right)^{l-c} ||P_1(r, \theta)x|| =$$

$$= \left(\frac{r}{s}\right)^{\nu} ||P_1(r, \theta)x||,$$

where $\nu = l - c$;

$$||\Phi(s, r, \theta) P_2(r, \theta)x|| \geq e^{-c(s-r) + l(s-r)} ||P_2(r, \theta)x|| \geq \left(\frac{s}{r}\right)^{l-c} ||P_2(r, \theta)x|| =$$

$$= \left(\frac{s}{r}\right)^{\nu} ||P_2(r, \theta)x||;$$

$$r^{\theta(0)-c} ||\Phi(s, r, \theta) P_3(r, \theta)x|| = r^{\theta(0)-c} \left(\frac{\ln(s+1)}{\ln(r+1)}\right)^{\theta(0)-c} ||P_3(r, \theta)x|| \leq s^{\omega} ||P_3(r, \theta)x||,$$

where $\omega = \theta(0) - c$;

$$s^{\omega} ||\Phi(s, r, \theta) P_3(r, \theta)x|| \geq r^{\omega} ||P_3(r, \theta)x||.$$

So the pair $(C, \mathcal{P})$ is uniformly polynomially trichotomic in the classical sense with the constants $N = 1$, $\nu = l - c$, $\omega = \theta(0) - c$.
We suppose that $(C, \mathcal{P})$ is uniformly polynomially trichotomic. It follows that there is $\widetilde{N} \geq 1$ such that

$$||\Phi(s, r, \theta) P_3(r, \theta)x|| \leq \widetilde{N} ||P_3(r, \theta)x||, \text{ for all } (s, r) \in \Delta, r > 0, (\theta, x) \in \Gamma,$$

which implies

$$\left(\frac{\ln(s+1)}{\ln(r+1)}\right)^{\theta(0)-c} \leq \widetilde{N}.$$

For $r = e - 1$ and $s \to +\infty$ we obtain a contradiction.

In conclusion, $(C, \mathcal{P})$ is not uniformly polynomially trichotomic.

Remark 3.4. In contrast with the trichotomy notions in [24], [25], [26], [27], the concepts considered in the present manuscript in Definition 3.7 and Definition 3.8 are of weaker nature, in the sense that the asymptotic behavior does not assume any kind of reversibility of the evolution cocycle restricted to the ranges of the second and of the third family of projections. Furthermore, no type of invertibility property is considered within the trichotomy concepts studied in this paper.

Definition 3.9. We say that the pair $(C, \mathcal{P})$ has a *uniform polynomial growth* if there exist $M \geq 1, \omega, \varepsilon > 0$ and $s_0 > 0$ with:

(upg_1) $s^\omega(||\Phi(u,r,\theta)P_1(r,\theta)x|| + ||\Phi(s,r,\theta)P_2(r,\theta)x||) \leq$
$\leq Mu^\omega(||\Phi(s,r,\theta)P_1(r,\theta)x|| + ||\Phi(u,r,\theta)P_2(r,\theta)x||);$

(upg_2) $s^\varepsilon||\Phi(u,r,\theta)P_3(r,\theta)x|| \leq Mu^\varepsilon||\Phi(s,r,\theta)P_3(r,\theta)x||;$

(upg_3) $s^\varepsilon||\Phi(s,r,\theta)P_3(r,\theta)x|| \leq Mu^\varepsilon||\Phi(u,r,\theta)P_3(r,\theta)x||,$

for all $(u,s,r) \in T$, $r \geq s_0$ and all $(\theta, x) \in \Gamma$.

We consider $C = (\varphi, \Phi)$ a skew-evolution semiflow on Γ and $\lambda \in \mathbb{R}$.
We denote by $C_\lambda = (\varphi, \Phi_\lambda)$ *the shifted skew-evolution semiflow*, where

$$\Phi_\lambda(s,r,\theta) = \begin{cases} \left(\frac{s}{r}\right)^{-\lambda} \Phi(s,r,\theta), & \text{if } s \geq r > 0 \\ I, & \text{if } s = r = 0. \end{cases}$$

Proposition 3.1. *The pair $(C, \mathcal{P})$ has a uniform polynomial growth if and only if there exist $M \geq 1, \omega_1, \omega_2, \lambda > 0$ and $s_0 > 0$ with:*

(upg'_1) $u^{\omega_1}(||\Phi_\lambda(u,r,\theta)P_1(r,\theta)x|| + ||\Phi_{-\lambda}(s,r,\theta)P_2(r,\theta)x||) \leq$
$\leq Ms^{\omega_1}(||\Phi_\lambda(s,r,\theta)P_1(r,\theta)x|| + ||\Phi_{-\lambda}(u,r,\theta)P_2(r,\theta)x||);$

(upg'_2) $u^{\omega_2}||\Phi_\lambda(u,r,\theta)P_3(r,\theta)x|| \leq Ms^{\omega_2}||\Phi_\lambda(s,r,\theta)P_3(r,\theta)x||;$

(upg'_3) $u^{\omega_2}||\Phi_{-\lambda}(s,r,\theta)P_3(r,\theta)x|| \leq Ms^{\omega_2}||\Phi_{-\lambda}(u,r,\theta)P_3(r,\theta)x||,$

for all $(u,s,r) \in T$, $r \geq s_0$ and all $(\theta, x) \in \Gamma$.

Proof. Necessity. Suppose that the relations from Definition 3.9 are satisfied.
Thus, for all $(u,s,r) \in T$ with $r \geq s_0$ and all $(\theta, x) \in \Gamma$ we have:
(upg'_1)

$$\left(\frac{u}{r}\right)^{-\lambda}||\Phi(u,r,\theta)P_1(r,\theta)x|| + \left(\frac{s}{r}\right)^{\lambda}||\Phi(s,r,\theta)P_2(r,\theta)x|| \leq$$
$$\leq M\left(\frac{u}{s}\right)^{\omega}\left[\left(\frac{u}{r}\right)^{-\lambda}||\Phi(s,r,\theta)P_1(r,\theta)x|| + \left(\frac{s}{r}\right)^{\lambda}||\Phi(u,r,\theta)P_2(r,\theta)x||\right] =$$
$$= M\left(\frac{s}{u}\right)^{\lambda-\omega}(||\Phi_\lambda(s,r,\theta)P_1(r,\theta)x|| + ||\Phi_{-\lambda}(u,r,\theta)P_2(r,\theta)x||)$$

and for $\lambda = 2\omega$, $\omega_1 = \omega$, we deduce

$$u^{\omega_1}(||\Phi_\lambda(u,r,\theta)P_1(r,\theta)x|| + ||\Phi_{-\lambda}(s,r,\theta)P_2(r,\theta)x||) \leq$$
$$\leq Ms^{\omega_1}(||\Phi_\lambda(s,r,\theta)P_1(r,\theta)x|| + ||\Phi_{-\lambda}(u,r,\theta)P_2(r,\theta)x||);$$

(upg'_2)

$$||\Phi_\lambda(u,r,\theta)P_3(r,\theta)x|| = \left(\frac{u}{r}\right)^{-\lambda}||\Phi(u,r,\theta)P_3(r,\theta)x|| \leq$$
$$\leq M\left(\frac{s}{u}\right)^{\lambda-\varepsilon}||\Phi_\lambda(s,r,\theta)P_3(r,\theta)x|| \leq M\left(\frac{s}{u}\right)^{\omega_2}||\Phi_\lambda(s,r,\theta)P_3(r,\theta)x||,$$

where $\lambda = 2\varepsilon$, $\omega_2 = \varepsilon$;
(upg'_3)

$$||\Phi_{-\lambda}(u,r,\theta)P_3(r,\theta)x|| = \left(\frac{u}{r}\right)^{\lambda}||\Phi(u,r,\theta)P_3(r,\theta)x|| \geq$$
$$\geq \frac{1}{M}\left(\frac{u}{s}\right)^{\lambda-\varepsilon}||\Phi_{-\lambda}(s,r,\theta)P_3(r,\theta)x|| \geq \frac{1}{M}\left(\frac{u}{s}\right)^{\omega_2}||\Phi_{-\lambda}(s,r,\theta)P_3(r,\theta)x||.$$

Sufficiency. We prove that the inequalities from Definition 3.9 hold.
For all $(u,s,r) \in T$, $r \geq s_0$ and all $(\theta, x) \in \Gamma$ we deduce:
(upg_1)

$$||\Phi(u,r,\theta)P_1(r,\theta)x|| + ||\Phi(s,r,\theta)P_2(r,\theta)x|| =$$

$$= \left(\frac{u}{r}\right)^{\lambda} ||\Phi_\lambda(u,r,\theta)P_1(r,\theta)x|| + \left(\frac{s}{r}\right)^{-\lambda} ||\Phi_{-\lambda}(s,r,\theta)P_2(r,\theta)x|| \leq$$

$$\leq M\left(\frac{s}{u}\right)^{\omega_1}\left[\left(\frac{u}{r}\right)^{\lambda} ||\Phi_\lambda(s,r,\theta)P_1(r,\theta)x|| + \left(\frac{s}{r}\right)^{-\lambda} ||\Phi_{-\lambda}(u,r,\theta)P_2(r,\theta)x||\right] \leq$$

$$\leq M\left(\frac{u}{s}\right)^{\lambda-\omega_1} (||\Phi(s,r,\theta)P_1(r,\theta)x|| + ||\Phi(u,r,\theta)P_2(r,\theta)x||) \leq$$

$$\leq M\left(\frac{u}{s}\right)^{\omega} (||\Phi(s,r,\theta)P_1(r,\theta)x|| + ||\Phi(u,r,\theta)P_2(r,\theta)x||),$$

where

$$\omega = \begin{cases} \lambda - \omega_1, & \text{if } \lambda > \omega_1 \\ 1, & \text{if } \lambda \leq \omega_1; \end{cases}$$

(upg_2)

$$||\Phi(u,r,\theta)P_3(r,\theta)x|| = \left(\frac{u}{r}\right)^{\lambda} ||\Phi_\lambda(u,r,\theta)P_3(r,\theta)x|| \leq$$

$$\leq M\left(\frac{u}{s}\right)^{\lambda-\omega_2} ||\Phi(s,r,\theta)P_3(r,\theta)x|| \leq M\left(\frac{u}{s}\right)^{\varepsilon} ||\Phi(s,r,\theta)P_3(r,\theta)x||,$$

where

$$\varepsilon = \begin{cases} \lambda - \omega_2, & \text{if } \lambda > \omega_2 \\ 1, & \text{if } \lambda \leq \omega_2; \end{cases}$$

(upg_3)

$$||\Phi(u,r,\theta)P_3(r,\theta)x|| = \left(\frac{u}{r}\right)^{-\lambda} ||\Phi_{-\lambda}(u,r,\theta)P_3(r,\theta)x|| \geq$$

$$\geq \frac{1}{M}\left(\frac{s}{u}\right)^{\lambda-\omega_2} ||\Phi(s,r,\theta)P_3(r,\theta)x|| \geq \frac{1}{M}\left(\frac{s}{u}\right)^{\varepsilon} ||\Phi(s,r,\theta)P_3(r,\theta)x||.$$

So the pair $(C, \mathcal{P})$ has a uniform polynomial growth. □

Theorem 3.1. *Let $(C, \mathcal{P})$ be a pair with a uniform polynomial growth. If there exist $D \geq 1$ and $s_0 > 0$ with:*

$$(i) \quad \int_t^{+\infty} \frac{||\Phi(\tau,s,\theta)P_1(s,\theta)x||}{\tau} d\tau + \int_s^u \frac{||\Phi(\xi,s,\theta)P_2(s,\theta)x||}{\xi} d\xi \leq$$
$$\leq D(||\Phi(t,s,\theta)P_1(s,\theta)x|| + ||\Phi(u,s,\theta)P_2(s,\theta)x||);$$

$$(ii) \quad \int_t^u \frac{||\Phi(\tau,s,\theta)P_3(s,\theta)x||}{\tau} d\tau \leq D||\Phi(t,s,\theta)P_3(s,\theta)x||;$$

$$(iii) \quad \int_s^u \frac{||\Phi(\tau,s,\theta)P_3(s,\theta)x||}{\tau} d\tau \leq D||\Phi(u,s,\theta)P_3(s,\theta)x||,$$

for all $(u,t,s) \in T$, $s \geq s_0$ and all $(\theta, x) \in \Gamma$, then $(C, \mathcal{P})$ is uniformly polynomially trichotomic.

Proof. Using similar arguments with those in the proof of Theorem 2.2. in [19], from (i) we deduce that there are $N \geq 1$ and $\nu > 0$ such that

(3.1) $u^{\nu}(||\Phi(u,s,\theta)P_1(s,\theta)x|| + ||P_2(s,\theta)x||) \leq Ns^{\nu}(||P_1(s,\theta)x|| + ||\Phi(u,s,\theta)P_2(s,\theta)x||)$,

for all $(u,s) \in \Delta$, $s \geq s_0$ and all $(\theta,x) \in \Gamma$.

We consider $u \geq ts_0 \geq t$, $(\theta,x) \in \Gamma$ and $\tau \in \left[\frac{u}{s_0}, u\right]$. Thus,

$$\ln s_0 ||\Phi(u,s,\theta)P_3(s,\theta)x|| = \int_{\frac{u}{s_0}}^{u} \frac{||\Phi(u,\tau,\varphi(\tau,s,\theta))\Phi(\tau,s,\theta)P_3(s,\theta)x||}{\tau} d\tau \leq$$

$$\leq M \int_{\frac{u}{s_0}}^{u} \left(\frac{u}{\tau}\right)^{\varepsilon} \frac{||\Phi(\tau,s,\theta)P_3(s,\theta)x||}{\tau} d\tau \leq M s_0^{\varepsilon} \int_{t}^{u} \frac{||\Phi(\tau,s,\theta)P_3(s,\theta)x||}{\tau} d\tau \leq$$

$$\leq MDs_0^{\varepsilon}||\Phi(t,s,\theta)P_3(s,\theta)x||.$$

If $u \in [t, ts_0]$, then

$$||\Phi(u,s,\theta)P_3(s,\theta)x|| \leq M\left(\frac{u}{t}\right)^{\varepsilon}||\Phi(t,s,\theta)P_3(s,\theta)x|| \leq Ms_0^{\varepsilon}||\Phi(t,s,\theta)P_3(s,\theta)x||.$$

We denote $L = \max\left\{Ms_0^{\varepsilon}, \frac{MDs_0^{\varepsilon}}{\ln s_0}\right\}$ and we obtain

$$||\Phi(u,s,\theta)P_3(s,\theta)x|| \leq L||\Phi(t,s,\theta)P_3(s,\theta)x||,$$

for all $(u,t,s) \in T$, $s \geq s_0$ and all $(\theta,x) \in \Gamma$ and for $t = s$ it yields

$$||\Phi(u,s,\theta)P_3(s,\theta)x|| \leq L||P_3(s,\theta)x||, \tag{3.2}$$

for all $(u,s) \in \Delta$, $s \geq s_0$ and all $(\theta,x) \in \Gamma$.

Similarly, it follows

$$||P_3(s,\theta)x|| \leq L||\Phi(u,s,\theta)P_3(s,\theta)x||, \tag{3.3}$$

for all $(u,s) \in \Delta$, $s \geq s_0$ and all $(\theta,x) \in \Gamma$.

We consider $\widetilde{N} = \max\{N, L\}$ and from the relations (3.1), (3.2), (3.3) and Remark 3.2 it yields that $(C, \mathcal{P})$ is uniformly polynomially trichotomic. □

REFERENCES

[1] Babuţia, M. G.; Seimeanu, N. M. Connections between some concepts of polynomial trichotomy for noninvertible evolution operators in Banach spaces. *Int. J. Differ. Equ.* **2015**, Art. ID 241402, 7 pp.

[2] Barreira, L.; Valls, C. Polynomial growth rates. *Nonlinear Anal.* **71** (2009), 5208–5219.

[3] Bento, A. J. G.; Lupa, N.; Megan, M.; Silva, C. M. Integral conditions for nonuniform μ-dichotomy on the half-line. *Discrete Contin. Dyn. Syst. Ser. B* **22** (2017), no. 8, 3063–3077.

[4] Bento, A. J. G.; Silva, C. Stable manifolds for nonuniform polynomial dichotomies. *J. Funct. Anal.* **257** (2009), no. 1, 122–148.

[5] Boruga, R.; Megan, M. On uniform polynomial dichotomy in Banach spaces. *Bul. Ştiinţ. Univ. Politeh. Timiş. Ser. Mat. Fiz.* **63** (2018), 32–40.

[6] Boruga, R.; Megan, M. Datko type characterizations for nonuniform polynomial dichotomy. *Carpathian J. Math.* **37** (2021), no. 1, 45–51.

[7] Datko, R. Uniform asymptotic stability of evolutionary processes in Banach space. *SIAM J. Math. Anal.* 3 (1972), 428–445.

[8] Dragičević, D. Admissibility and nonuniform polynomial dichotomies. *Math. Nachr.* **293** (2020), no. 2, 226–243.

[9] Dragičević, D. Admissibility and polynomial dichotomies for evolution families. *Commun. Pure Appl. Anal.* **19** (2020), no. 3, 1321–1336.

[10] Dragičević, D.; Sasu, A. L.; Sasu, B. On polynomial dichotomies of discrete nonautonomous systems on the half-line. *Carpathian J. Math.* **38** (2022), no. 3, 663–680.

[11] Elaydi, S.; Hajek, O. Exponential trichotomy of differential systems. *J. Math. Anal. Appl.* **129** (1988), no. 2, 362–374.
[12] Hai, P. V. Polynomial stability of evolution cocycles and Banach function spaces. *Bull. Belg. Math. Soc. Simon Stevin* **26** (2019), no. 2, 299–314.
[13] Lupa, N.; Megan, M. Generalized exponential trichotomies for abstract evolution operators on the real line. *J. Funct. Spaces Appl.* **2013**, Art. ID 409049, 8 pp.
[14] Megan, M.; Sasu, A. L.; Sasu, B. Stabilizability and controllability of systems associated to linear skew-product semiflows. *Rev. Mat. Complut.* **15** (2002), no. 2, 599–618.
[15] Megan, M.; Sasu, A. L.; Sasu, B. On uniform exponential stability of linear skew-product semiflows in Banach spaces. *Bull. Belg. Math. Soc. Simon Stevin* **9** (2002), no. 1, 143–154.
[16] Megan, M.; Sasu, A. L.; Sasu, B. Exponential instability of linear skew-product semiflows in terms of Banach function spaces. *Results Math.* **45** (2004), no. 3-4, 309–318.
[17] Megan, M.; Stoica, C.; Buliga, L. On asymptotic behaviours for linear skew-evolution semiflows in Banach spaces. *Carpathian J. Math.* **23** (2007), 117–125.
[18] Megan, M.; Stoica, C. Discrete asymptotic behaviors for skew-evolution semiflows on Banach spaces. *Carpathian J. Math.* **24** (2008), 348–355.
[19] Mihiţ, C. L.; Lăpădat, M. On uniform polynomial dichotomy of skew-evolution semiflows on the half-line. *Bul. Ştiinţ. Univ. Politeh. Timiş. Ser. Mat. Fiz.* **62** (2017), 54–61.
[20] Popa, I. L.; Ceauşu, T.; Megan, M. Characterizations of the (h, k, μ, ν)-trichotomy for linear time-varying systems. *Math. Methods Appl. Sci.* **40** (2017), no. 17, 6172–6177.
[21] Rămneanţu, M. L.; Ceauşu, T.; Megan, M. On nonuniform polynomial dichotomy of evolution operators in Banach spaces. *Int. J. Pure Appl. Math.* **75** (2012), 305–318.
[22] Rămneanţu, M. L.; Megan, M.; Ceauşu, T. Polynomial instability of evolution operators in Banach spaces. *Carpathian J. Math.* **29** (2013), 77–83.
[23] Sasu, A. L.; Megan, M.; Sasu, B. On Rolewicz-Zabczyk techniques in the stability theory of dynamical systems. *Fixed Point Theory* **13** (2012), no. 1, 205–236.
[24] Sasu, A. L.; Sasu, B. Integral equations and exponential trichotomy of skew-product flows. *Adv. Difference Equ.* **2011**, Art. ID 918274, 18 pp.
[25] Sasu, A. L.; Sasu, B. A Zabczyk type method for the study of the exponential trichotomy of discrete dynamical systems. *Appl. Math. Comput.* 245 (2014), 447–461.
[26] Sasu, A. L.; Sasu, B. Admissibility and exponential trichotomy of dynamical systems described by skew-product flows. *J. Differential Equations* **260** (2016), no. 2, 1656–1689.
[27] Sasu, B.; Sasu, A. L. Nonlinear criteria for the existence of the exponential trichotomy in infinite dimensional spaces. *Nonlinear Anal.* **74** (2011), no. 15, 5097–5110.
[28] Stoica, C.; Megan, M. On (h, k)-trichotomy for skew-evolution semiflows in Banach spaces. *Stud. Univ. Babeş-Bolyai Math.* **56**(2011), no. 4, 147–156.

[1] "Aurel Vlaicu" University of Arad
Department of Mathematics and Computer Science
Elena Drăgoi Street no.2, 310330 Arad, Romania
Email address: claudia.mihit@uav.ro