CARPATHIAN J. MATH.
Volume **38** (2022), No. 3,
Pages 817-835

Online version at https://semnul.com/carpathian/
Print Edition: ISSN 1584 - 2851; Online Edition: ISSN 1843 - 4401
DOI: https://doi.org/10.37193/CJM.2022.03.25

Dedicated to Prof. Emeritus Mihail Megan on the occasion of his 75^{th} anniversary

Invariant manifolds for difference equations with generalized trichotomies

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ABSTRACT. On an arbitrary Banach space, assuming that a linear nonautonomous difference equation $x_{m+1} = A_m x_m$ admits a very general type of trichotomy, we establish conditions for the existence of global Lipschitz invariant center manifolds of the perturbed equation $x_{m+1} = A_m x_m + f_m(x_m)$. Our results not only improve results already existing in the literature, but also include new cases.

1. INTRODUCTION

Let X be a Banach space and let $\mathcal{B}(X)$ be the space of linear bounded operators acting on X. Assume, for all $m \in \mathbb{Z}$, that $A_m \in \mathcal{B}(X)$ is an invertible operator and that $f_m \colon X \to X$ is a Lipschitz function such that $f_m(0) = 0$. We find sufficient conditions in order that the nonautonomous difference equation

$$x_{m+1} = A_m x_m + f_m(x_m),$$

has an invariant center Lipschitz manifolds, when the nonautonomous linear difference equation

$$x_{m+1} = A_m x_m$$

admits a generalized trichotomy.

Center manifolds are a very important tool in the study of stability and of bifurcations because they frequently permit the reduction of the dimension of the state space (see Carr [14], Henry [24], Guckenheimer and Holmes [21], Hale and Koçak [22] and Haragus and Iooss [23]). The first results on the existence of center manifolds were obtained in the sixties by Pliss [32] and by Kelley [25, 26]. After that many authors studied the problem and proved results about center manifolds. For autonomous differential equations in the finite dimensional case see Vanderbauwhede [39] (see also Vanderbauwhede and Gils [40]) and for autonomous differential equations in the infinite dimensional case see Vanderbauwhede and Iooss [41]. For nonautonomous invariant manifolds, and in particular to nonautonomous center manifolds, we suggest the paper by Aulbach and Wanner [2]. See also Chow, Liu and Yi [17, 16] for more details in the finite dimensional case and Sijbrand [38], Mielke [29], Chow and Lu [18, 19] and Chicone and Latushkin [15] for the infinite dimensional case.

The notion of (uniform) exponential trichotomy, introduced by Sacker and Sell [33], Aulbach [1] and Elaydi and Hájek [20], is inspired in the concept of (uniform) exponential dichotomy that goes back to the works of Perron [30, 31]. The definition of Sacker and Sell [33] is motivated by the autonomous case of a matrix having semisimple eigenvalues on the imaginary axis and because of that they only impose boundedness in the central

Received: 31.03.2022. In revised form: 26.07.2022. Accepted: 26.07.2022
1991 *Mathematics Subject Classification.* 37D10.
Key words and phrases. *Difference equations; Trichotomies; Invariant manifold .*

direction. On the other hand, Elaydi and Hájek [20] impose in both time directions an exponential decay on the central direction and thus their case is inherently nonautonomous.

However, the definition of exponential trichotomy is very stringent and several generalizations have appeared in the literature. On the one hand, Fenner and Pinto [27] introduced the so-called (h, k)-trichotomies, replacing the exponential growth rates by nonexponential growth rates, and, on the other hand, Barreira and Valls [5, 6] introduced the nonuniform exponential trichotomies that also depend on the initial time. The next step was given by Barreira and Valls [8, 9] with the introduction of the ρ-nonuniform exponential trichotomies that are both nonexponential and nonuniform, but the (h, k)-trichotomies of Fenner and Pinto [27] are not a particular case of the notion of ρ-nonuniform exponential trichotomy. For characterizations of exponential trichotomies we recommend Barreira, Dragičević and Valls [3, 4] and Sasu and Sasu [34, 35, 36, 37] and the references therein.

In [11] it was introduced, for linear differential equations, a very general type of trichotomies that include as particular cases all the notions of trichotomies mentioned above, as well as new cases. Despite of this generality, it was possible to prove the existence of central invariant Lipschitz manifolds for sufficiently small Lipschitz perturbations of the linear differential equations that admit this type of generalized trichotomy.

This paper is a discrete time counterpart of [11]. We are going to consider for linear difference equations the same general type of trichotomies. We only suppose that the linear equation admits an invariant splitting in three invariant subspaces and the norms of the linear evolution operator composed with the three different projections are bounded by general sequences that only depend on the initial and on the final time (see $(\mathbf{T1})$, $(\mathbf{T2})$ and $(\mathbf{T3})$). Despite of that we were able to obtain invariant manifolds provided that the Lipschitz constants of the perturbation are sufficiently small. Note that for dichotomies this has already been done in [12] and in [13] for differential and for difference equations, respectively.

The proof of the main theorem is based in the Lyapunov-Perron method (see [28, 30, 31]) that consists in the following:

- relate the solutions of the linear equation with the solutions of the perturbed equation using the variation of constants formula;
- construction of a suitable space of sequences of functions that is a complete metric space and the construction of a suitable contraction on this complete metric space;
- the application of Banach's fixed point theorem to the referred contraction gives a sequence of functions that is the only fixed point of the contraction and whose graphs are the invariant manifold.

The Lyapunov-Perron method was used by many authors, namely [6, 8, 12, 13]. In the mentioned papers the authors use two applications of the Banach's fixed point theorem, the first one to obtain the solutions of the perturbed equation along the stable/center direction and the other to obtain the solutions of the perturbed equation in the other directions. In this paper, as in [11], we use only one application of the Banach's fixed point theorem to obtain the solutions of the perturbed equation in all directions.

As particular case of our main result we improve the results obtained by Barreira and Valls [5, 10] for nonuniform exponential trichotomies. Moreover, we also obtain as particular cases new results for nonuniform $(\mathbb{a}, \mathbb{b}, \mathbb{c}, \mathbb{d})$-trichotomies.

The structure of the paper is as follows. In Section 2 we introduce the notation and preliminaries. The main theorem of the paper is stated in Section 3 and in Section 4 we apply our main result to particular cases of trichotomies. Finally, in Section 5, we prove the main theorem.

2. Notation and preliminaries

Let X be a Banach space and let $\mathcal{B}(X)$ be the Banach algebra of all bounded linear operators acting on X. Given a sequence $(A_m)_{m\in\mathbb{Z}}$ of invertible operators in $\mathcal{B}(X)$, we are going to consider the difference equation

$$x_{m+1} = A_m x_m, \quad m \in \mathbb{Z}, \tag{2.1}$$

and denote by $\Phi_{m,n}$ its evolution operator, i.e.,

$$\Phi_{m,n} = \begin{cases} A_{m-1}\cdots A_m & \text{if } m > n, \\ \mathrm{Id} & \text{if } m = n, \\ A_m^{-1}\cdots A_{n-1}^{-1} & \text{if } m < n. \end{cases}$$

We say that equation (2.1) admits an *invariant trichotomic splitting* if, for every $m \in \mathbb{Z}$, there are projections $P_m^{\mathrm{o}}, P_m^+, P_m^- \in \mathcal{B}(X)$ such that

- $(\mathbf{S1})$ $P_m^{\mathrm{o}} + P_m^+ + P_m^- = \mathrm{Id}$ for all $m \in \mathbb{Z}$;
- $(\mathbf{S2})$ $P_m^{\mathrm{o}} P_m^+ = 0$ for all $m \in \mathbb{Z}$;
- $(\mathbf{S3})$ $\Phi_{m,n} P_n^{\mathrm{o}} = P_m^{\mathrm{o}} \Phi_{m,n}$ for all $m, n \in \mathbb{Z}$;
- $(\mathbf{S4})$ $\Phi_{m,n} P_n^+ = P_m^+ \Phi_{m,n}$ for all $m, n \in \mathbb{Z}$;

From $(\mathbf{S1})$ and $(\mathbf{S2})$ it follows that

$$P_m^{\mathrm{o}} P_m^- = P_m^+ P_m^{\mathrm{o}} = P_m^+ P_m^- = P_m^- P_m^{\mathrm{o}} = P_m^- P_m^+ = 0 \ \text{ for all } m \in \mathbb{Z}$$

and from $(\mathbf{S1})$, $(\mathbf{S3})$ and $(\mathbf{S4})$ we obtain

$$\Phi_{m,n} P_n^- = P_m^- \Phi_{m,n} \ \text{ for all } m, n \in \mathbb{Z}.$$

Under these conditions we define, for every $m \in \mathbb{Z}$, the subspaces $E_m^{\mathrm{o}} = P_m^{\mathrm{o}}(X)$, $E_m^+ = P_m^+(X)$ and $E_m^- = P_m^-(X)$ and, as usual we identify $E_m^{\mathrm{o}} \times E_m^+ \times E_m^-$ and $E_m^{\mathrm{o}} \oplus E_m^+ \oplus E_m^- = X$ as the same vector space.

Consider $\alpha^{\mathrm{o}}\colon \mathbb{Z}^2 \to]0,+\infty[$, $\alpha^+ \colon \mathbb{Z}^2_{\geqslant} \to]0,+\infty[$ and $\alpha^- \colon \mathbb{Z}^2_{\leqslant} \to]0,+\infty[$, where

$$\mathbb{Z}^2_{\geqslant} = \left\{(m,n) \in \mathbb{Z}^2 \colon m \geqslant n\right\} \quad \text{and} \quad \mathbb{Z}^2_{\leqslant} = \left\{(m,n) \in \mathbb{Z}^2 \colon m \leqslant n\right\},$$

and denote $\alpha^{\mathrm{o}}(m,n)$, $\alpha^+(m,n)$ and $\alpha^-(m,n)$ by $\alpha^{\mathrm{o}}_{m,n}$, $\alpha^+_{m,n}$ and $\alpha^-_{m,n}$, respectively. We say that equation (2.1) admits a *generalized trichotomy with bounds* $(\alpha^{\mathrm{o}}, \alpha^+, \alpha^-)$ if it admits an invariant trichotomic splitting such that

- $(\mathbf{T1})$ $\|\Phi_{m,n} P_m^{\mathrm{o}}\| \leqslant \alpha^{\mathrm{o}}_{m,n}$ for all $(m,n) \in \mathbb{Z}^2$;
- $(\mathbf{T2})$ $\|\Phi_{m,n} P_m^+\| \leqslant \alpha^+_{m,n}$ for all $(m,n) \in \mathbb{Z}^2_{\geqslant}$;
- $(\mathbf{T3})$ $\|\Phi_{m,n} P_m^-\| \leqslant \alpha^-_{m,n}$ for all $(m,n) \in \mathbb{Z}^2_{\leqslant}$.

Example 2.1. *Let $(\Bbbk_n)_{n\in\mathbb{Z}}$ be a sequence such that $\Bbbk_n \geqslant 1$ for every $n \in \mathbb{Z}$ and let*

$$P_n^{\mathrm{o},+}, P_n^{\mathrm{o},-}, P_n^+, P_n^- \colon \mathbb{R}^4 \to \mathbb{R}^4$$

be defined by

$$\begin{aligned} P_n^{\mathrm{o},+}(x_1,x_2,x_3,x_4) &= (0,0,x_3 + (\Bbbk_n - 1)x_4, 0), \\ P_n^{\mathrm{o},-}(x_1,x_2,x_3,x_4) &= ((1-\Bbbk_n)x_2, x_2, 0, 0), \\ P_n^{+}(x_1,x_2,x_3,x_4) &= (x_1 + (\Bbbk_n - 1)x_2, 0, 0, 0), \\ P_n^{-}(x_1,x_2,x_3,x_4) &= (0,0,(1-\Bbbk_n)x_4, x_4). \end{aligned}$$

It is clear that

$$P_n^{\mathrm{o},+} + P_n^{\mathrm{o},-} + P_n^+ + P_n^- = \mathrm{Id}$$

for every $n \in \mathbb{Z}$. Moreover,

$$\begin{cases} P_m^{o,+}P_n^{o,+} = P_n^{o,+} \\ P_m^{o,+}P_n^{o,-} = 0 \\ P_m^{o,+}P_n^{+} = 0 \end{cases} \quad \begin{cases} P_m^{o,-}P_n^{o,+} = 0 \\ P_m^{o,-}P_n^{o,-} = P_m^{o,-} \\ P_m^{o,-}P_n^{+} = 0 \\ P_m^{o,-}P_n^{+} = 0 \end{cases} \quad \begin{cases} P_m^{+}P_n^{o,+} = 0 \\ P_m^{+}P_n^{+} = P_n^{+} \\ P_m^{+}P_n^{+} = 0 \end{cases} \quad \begin{cases} P_m^{-}P_n^{o,+} = 0 \\ P_m^{-}P_n^{o,-} = P_m^{o,-} \\ P_m^{-}P_n^{+} = 0 \\ P_m^{-}P_n^{+} = P_m^{-} \end{cases}$$

and

$$P_m^{o,+}P_n^{-}(x_1,x_2,x_3,x_4) = (0,0,(\mathbb{k}_m - \mathbb{k}_n)x_4, 0)$$

and

$$P_m^{+}P_n^{o,-}(x_1,x_2,x_3,x_4) = ((\mathbb{k}_m - \mathbb{k}_n)x_2, 0,0,0)$$

for every $m,n \in \mathbb{Z}$.

If $(\mathbb{a}_n)_{n\in\mathbb{Z}}$, $(\mathbb{b}_n)_{n\in\mathbb{Z}}$, $(\mathbb{c}_n)_{n\in\mathbb{Z}}$ and $(\mathbb{d}_n)_{n\in\mathbb{Z}}$ are sequences of positive numbers, then

$$A_n = \frac{\mathbb{a}_n}{\mathbb{a}_{n+1}}P_n^{o,+} + \frac{\mathbb{c}_{n+1}}{\mathbb{c}_n}\frac{\mathbb{k}_n}{\mathbb{k}_{n+1}}P_{n+1}^{o,-} + \frac{\mathbb{d}_n}{\mathbb{d}_{n+1}}P_n^{+} + \frac{\mathbb{b}_{n+1}}{\mathbb{b}_n}\frac{\mathbb{k}_n}{\mathbb{k}_{n+1}}P_{n+1}^{-}$$

is an invertible operator (on $\mathbb{R}^4$) and

$$A_n^{-1} = \frac{\mathbb{a}_{n+1}}{\mathbb{a}_n}P_{n+1}^{o,+} + \frac{\mathbb{c}_n}{\mathbb{c}_{n+1}}\frac{\mathbb{k}_{n+1}}{\mathbb{k}_n}P_n^{o,-} + \frac{\mathbb{d}_{n+1}}{\mathbb{d}_n}P_{n+1}^{+} + \frac{\mathbb{b}_n}{\mathbb{b}_{n+1}}\frac{\mathbb{k}_{n+1}}{\mathbb{k}_n}P_n^{-}.$$

Clearly,

$$\Phi_{m,n} = \frac{\mathbb{a}_n}{\mathbb{a}_m}P_n^{o,+} + \frac{\mathbb{c}_m}{\mathbb{c}_n}\frac{\mathbb{k}_n}{\mathbb{k}_m}P_m^{o,-} + \frac{\mathbb{d}_n}{\mathbb{d}_m}P_n^{+} + \frac{\mathbb{b}_m}{\mathbb{b}_n}\frac{\mathbb{k}_n}{\mathbb{k}_m}P_m^{-}$$

and using the projections $P_n^{o} = P_n^{o,+} + P_n^{o,-}$, P_n^{+} and P_n^{-} we have

$$\begin{aligned} \Phi_{m,n}P_n^{o} &= \frac{\mathbb{a}_n}{\mathbb{a}_m}P_n^{o,+} + \frac{\mathbb{c}_m}{\mathbb{c}_n}\frac{\mathbb{k}_n}{\mathbb{k}_m}P_m^{o,-} = P_m^{o}\Phi_{m,n}, \\ \Phi_{m,n}P_n^{+} &= \frac{\mathbb{d}_n}{\mathbb{d}_m}P_n^{+} = P_m^{+}\Phi_{m,n}, \\ \Phi_{m,n}P_n^{-} &= \frac{\mathbb{b}_m}{\mathbb{b}_n}\frac{\mathbb{k}_n}{\mathbb{k}_m}P_m^{-} = P_m^{-}\Phi_{m,n}. \end{aligned}$$

Equipping $\mathbb{R}^4$ with the maximum norm, we have

$$\|P_n^{o,+}\| = \|P_n^{+}\| = \mathbb{k}_n \quad \textit{and} \quad \|P_n^{o,-}\| = \|P_n^{-}\| = \max\{1, \mathbb{k}_n - 1\} \leqslant \mathbb{k}_n$$

and this implies that

$$\begin{aligned} \|\Phi_{m,n}P_n^{o}\| &= \max\left\{\frac{\mathbb{a}_n}{\mathbb{a}_m}\|P_n^{o,+}\|, \frac{\mathbb{c}_m}{\mathbb{c}_n}\frac{\mathbb{k}_n}{\mathbb{k}_m}\|P_m^{o,-}\|\right\} \leqslant \max\left\{\frac{\mathbb{a}_n}{\mathbb{a}_m}, \frac{\mathbb{c}_m}{\mathbb{c}_n}\right\}\mathbb{k}_n, \\ \|\Phi_{m,n}P_n^{+}\| &= \frac{\mathbb{d}_n}{\mathbb{d}_m}\|P_n^{+}\| = \frac{\mathbb{d}_n}{\mathbb{d}_m}\mathbb{k}_n \\ \|\Phi_{m,n}P_n^{-}\| &= \frac{\mathbb{b}_m}{\mathbb{d}_n}\frac{\mathbb{k}_n}{\mathbb{k}_m}\|P_m^{-}\| \leqslant \frac{\mathbb{b}_m}{\mathbb{b}_n}\mathbb{k}_n. \end{aligned}$$

Thus, if

$$\frac{\mathbb{a}_n}{\mathbb{a}_m} \geqslant \frac{\mathbb{c}_m}{\mathbb{c}_n} \textit{ for every } (m,n) \in \mathbb{Z}^2_{\geqslant}, \tag{2.2}$$

we get

$$\|\Phi_{m,n}P_n^{o}\| \leqslant \begin{cases} \dfrac{\mathbb{a}_n}{\mathbb{a}_m}\mathbb{k}_n & \textit{if } m \geqslant n, \\ \dfrac{\mathbb{c}_m}{\mathbb{c}_n}\mathbb{k}_n & \textit{if } m \leqslant n. \end{cases}$$

Therefore, if (2.2) *holds, then we conclude that* (2.1) *admits a generalized trichotomy with bounds given by*

$$\alpha^{\mathrm{o}}_{m,n} = \begin{cases} \dfrac{\mathbb{a}_n}{\mathbb{a}_m}\mathbb{k}_n & \text{for all } (m,n) \in \mathbb{Z}^2_{\geqslant}, \\ \dfrac{\mathbb{c}_m}{\mathbb{c}_n}\mathbb{k}_n & \text{for all } (m,n) \in \mathbb{Z}^2_{\leqslant}, \end{cases}$$
$$\alpha^{+}_{m,n} = \frac{\mathbb{d}_n}{\mathbb{d}_m}\mathbb{k}_n \quad \text{for all } (m,n) \in \mathbb{Z}^2_{\geqslant},\tag{2.3}$$
$$\alpha^{-}_{m,n} = \frac{\mathbb{b}_m}{\mathbb{b}_n}\mathbb{k}_n \quad \text{for all } (m,n) \in \mathbb{Z}^2_{\leqslant}.$$

The trichotomies with these bounds are called nonuniform $(\mathbb{a}, \mathbb{b}, \mathbb{c}, \mathbb{d})$*-trichotomies.*

Example 2.2. *Let* $\rho\colon \mathbb{Z} \to \mathbb{R}$ *be a strictly increasing sequence such that*

$$\rho(0) = 0, \qquad \lim_{m\to-\infty} \rho(m) = -\infty \qquad \text{and} \qquad \lim_{m\to+\infty} \rho(m) = +\infty.$$

Taking in (2.3)

$$\mathbb{a}_n = \mathrm{e}^{-a\rho(n)}, \qquad \mathbb{b}_n = \mathrm{e}^{-b\rho(n)}, \qquad \mathbb{c}_n = \mathrm{e}^{-c\rho(n)}, \qquad \mathbb{d}_n = \mathrm{e}^{-d\rho(n)}$$

and

$$\mathbb{k}_n = K\,\mathrm{e}^{\varepsilon|\rho(n)|},$$

where $a, b, c, d \in \mathbb{R}$*,* $K \geqslant 1$ *and* $\varepsilon \geqslant 0$*, we obtain the following bounds*

$$\alpha^{\mathrm{o}}_{m,n} = \begin{cases} K\,\mathrm{e}^{a(\rho(m)-\rho(n))+\varepsilon|n|} & \text{for all } (m,n) \in \mathbb{Z}^2_{\geqslant}, \\ K\,\mathrm{e}^{c(\rho(n)-\rho(m))+\varepsilon|n|} & \text{for all } (m,n) \in \mathbb{Z}^2_{\leqslant}, \end{cases}$$
$$\alpha^{+}_{m,n} = K\,\mathrm{e}^{d(\rho(m)-\rho(n))+\varepsilon|n|} \quad \text{for all } (m,n) \in \mathbb{Z}^2_{\geqslant},\tag{2.4}$$
$$\alpha^{-}_{m,n} = K\,\mathrm{e}^{b(\rho(n)-\rho(m))+\varepsilon|n|} \quad \text{for all } (m,n) \in \mathbb{Z}^2_{\leqslant}.$$

Note that for this bounds inequality (2.2) *is equivalent* $a + c \geqslant 0$*. This type of trichotomies is called nonuniform* ρ*-trichotomy (see* [7] *where, in our notation, the conditions* $0 \leqslant a < -b$ *and* $0 \leqslant c < -d$ *were also imposed).*

Example 2.3. *Taking* $\rho(n) = n$ *in* (2.4) *we obtain the nonuniform exponential trichotomies considered by Barreira and Valls in* [5] *with bounds given by*

$$\alpha^{\mathrm{o}}_{m,n} = \begin{cases} K\,\mathrm{e}^{a(m-n)+\varepsilon|n|} & \text{for all } (m,n) \in \mathbb{Z}^2_{\geqslant}, \\ K\,\mathrm{e}^{c(n-m)+\varepsilon|n|} & \text{for all } (m,n) \in \mathbb{Z}^2_{\leqslant}, \end{cases}$$
$$\alpha^{+}_{m,n} = K\,\mathrm{e}^{d(m-n)+\varepsilon|n|} \quad \text{for all } (m,n) \in \mathbb{Z}^2_{\geqslant},$$
$$\alpha^{-}_{m,n} = K\,\mathrm{e}^{b(n-m)+\varepsilon|n|} \quad \text{for all } (m,n) \in \mathbb{Z}^2_{\leqslant}.$$

As in the last example, inequality (2.2) *is equivalent to* $a + c \geqslant 0$*. Moreover, in* [5] *conditions* $0 \leqslant a < -b$ *and* $0 \leqslant c < -d$ *were also imposed.*

3. Main theorem

Suppose that (2.1) admits a generalized trichotomy with bounds $(\alpha^{\mathrm{o}}, \alpha^{+}, \alpha^{-})$ and consider the equation

$$x_{m+1} = A_m x_m + f_m(x_m), \quad m \in \mathbb{Z},\tag{3.5}$$

where $f_m\colon X \to X$ is a sequence of functions such that, for every $m \in \mathbb{Z}$,

(3.6) $A_m + f_m$ is invertible;

(3.7) $f_m(0) = 0$;

(3.8) $$\operatorname{Lip}(f_m) := \sup\left\{\frac{\|f_m(x) - f_m(y)\|}{\|x - y\|} : x, y \in X,\ x \neq y\right\} < +\infty.$$

From (3.8) if follows immediately that

(3.9) $$\|f_m(x) - f_m(y)\| \leqslant \operatorname{Lip}(f_m)\,\|x - y\| \quad \text{for all } x, y \in X \text{ and all } m \in \mathbb{Z}$$

and from (3.7) and (3.9) we obtain that

(3.10) $$\|f_m(x)\| \leqslant \operatorname{Lip}(f_m)\,\|x\| \quad \text{for all } x \in X \text{ and all } m \in \mathbb{Z}.$$

Since we are assuming that (2.1) admits a generalized trichotomy, given an initial condition $x_n = (x_n^{\mathrm{o}}, x_n^+, x_n^-) \in E_n^{\mathrm{o}} \times E_n^+ \times E_n^-$, the sequence $(x_m)_{m \in \mathbb{Z}}$ that satisfies

$$x_{m+1} = A_m x_m + f_m(x_m), \quad m \in \mathbb{Z}$$

is denoted by

$$\big(x_m^{\mathrm{o}}, x_m^+, x_m^-\big) = \big(x_m^{\mathrm{o}}(n, x_n), x_m^+(n, x_n), x_m^-(n, x_n)\big) \in E_m^{\mathrm{o}} \times E_m^+ \times E_m^-.$$

Then

(3.11) $$x_m^{\mathrm{o}} = \begin{cases} \Phi_{m,n} x_n^{\mathrm{o}} + \displaystyle\sum_{k=n}^{m-1} \Phi_{m,k+1} P_{k+1}^{\mathrm{o}} f_k(x_k^{\mathrm{o}}, x_k^+, x_k^-) & \text{if } m \geqslant n,\\ \Phi_{m,n} x_n^{\mathrm{o}} - \displaystyle\sum_{k=m}^{n-1} \Phi_{m,k+1} P_{k+1}^{\mathrm{o}} f_k(x_k^{\mathrm{o}}, x_k^+, x_k^-) & \text{if } m \leqslant n, \end{cases}$$

(3.12) $$x_m^{+} = \begin{cases} \Phi_{m,n} x_n^{+} + \displaystyle\sum_{k=n}^{m-1} \Phi_{m,k+1} P_{k+1}^{+} f_k(x_k^{\mathrm{o}}, x_k^+, x_k^-) & \text{if } m \geqslant n,\\ \Phi_{m,n} x_n^{+} - \displaystyle\sum_{k=m}^{n-1} \Phi_{m,k+1} P_{k+1}^{+} f_k(x_k^{\mathrm{o}}, x_k^+, x_k^-) & \text{if } m \leqslant n, \end{cases}$$

(3.13) $$x_m^{-} = \begin{cases} \Phi_{m,n} x_n^{-} + \displaystyle\sum_{k=n}^{m-1} \Phi_{m,k+1} P_{k+1}^{-} f_k(x_k^{\mathrm{o}}, x_k^+, x_k^-) & \text{if } m \geqslant n,\\ \Phi_{m,n} x_n^{-} - \displaystyle\sum_{k=m}^{n-1} \Phi_{m,k+1} P_{k+1}^{-} f_k(x_k^{\mathrm{o}}, x_k^+, x_k^-) & \text{if } m \leqslant n. \end{cases}$$

For every $k \in \mathbb{Z}$ we set

(3.14) $$\Psi_k(n, x_n) = \big(n + k, x_{n+k}^{\mathrm{o}}(n, x_n), x_{n+k}^+(n, x_n), x_{n+k}^-(n, x_n)\big)$$

The invariant manifolds that we are looking for will be given as the graphs of a sequence of functions. To find that sequence of functions we need to introduce a suitable space of sequences of functions. Given $D > 0$, let $\mathfrak{B}_D$ be the space of sequences $\varphi = (\varphi_n)_{n \in \mathbb{Z}}$ of functions

$$\varphi_n = (\varphi_n^+, \varphi_n^-)\colon E_n^{\mathrm{o}} \to E_n^+ \times E_n^-$$

such that

(3.15) $\varphi_n(0) = 0$ for all $n \in \mathbb{Z}$;

(3.16) $\left\|\varphi_n(\xi) - \varphi_n(\bar{\xi})\right\| \leqslant D\left\|\xi - \bar{\xi}\right\|$ for all $n \in \mathbb{Z}$ and all $\xi, \bar{\xi} \in E_n^{\mathrm{o}}$.

Clearly, by (3.15) and (3.16) we have

$$\|\varphi_n(\xi)\| \leqslant D\,\|\xi\| \quad \text{for all } n \in \mathbb{Z} \text{ and all } \xi \in E_n^{\mathrm{o}}. \tag{3.17}$$

For every $\varphi \in \mathfrak{B}_D$ we set

$$\begin{aligned}\Gamma_{n,\varphi} &= \{(\xi, \varphi_n(\xi)) : n \in \mathbb{Z},\ \xi \in E_n^{\mathrm{o}}\} \\ &= \left\{\left(\xi, \varphi_n^+(\xi), \varphi_n^-(\xi)\right) : n \in \mathbb{Z},\ \xi \in E_n^{\mathrm{o}}\right\}.\end{aligned}$$

Before stating the main theorem we need to introduce the following quantities:

$$\sigma := \sup_{(m,n)\in\mathbb{Z}^2_{\geqslant}} \left[\max\left\{\sum_{k=n}^{m-1} \frac{\alpha^{\mathrm{o}}_{m,k+1}\operatorname{Lip}(f_k)\alpha^{\mathrm{o}}_{k,n}}{\alpha^{\mathrm{o}}_{m,n}}, \sum_{k=n}^{m-1} \frac{\alpha^{\mathrm{o}}_{n,k+1}\operatorname{Lip}(f_k)\alpha^{\mathrm{o}}_{k,m}}{\alpha^{\mathrm{o}}_{n,m}}\right\}\right] \tag{3.18}$$

and

$$\omega := \sup_{n\in\mathbb{Z}} \left[\sum_{k=-\infty}^{n-1} \alpha^+_{n,k+1}\operatorname{Lip}(f_k)\alpha^{\mathrm{o}}_{k,n} + \sum_{k=n}^{+\infty} \alpha^-_{n,k+1}\operatorname{Lip}(f_k)\alpha^{\mathrm{o}}_{k,n}\right]. \tag{3.19}$$

Theorem 3.1. *Let X be a Banach space and suppose that equation* (2.1) *admits a generalized trichotomy with bounds $(\alpha^{\mathrm{o}}, \alpha^+, \alpha^-)$. Suppose that $f_m\colon X \to X$ satisfies* (3.6), (3.7) *and* (3.8). *If*

$$\lim_{m\to-\infty} \alpha^+_{n,m}\alpha^{\mathrm{o}}_{m,n} = \lim_{m\to+\infty} \alpha^-_{n,m}\alpha^{\mathrm{o}}_{m,n} = 0 \tag{3.20}$$

and

$$\sigma + \omega < 1/2, \tag{3.21}$$

then there exists $D \in\,]0,1[$ and a unique $\varphi \in \mathfrak{B}_D$ such that

$$\Psi_{m-n}\left(\Gamma_{n,\varphi}\right) = \Gamma_{m,\varphi} \ \text{ for all } m, n \in \mathbb{Z}. \tag{3.22}$$

Moreover,

$$\left\|\Psi_{m-n}\left(n,\xi,\varphi_n(\xi)\right) - \Psi_{m-n}\left(n,\xi,\varphi_n(\bar\xi)\right)\right\| \leqslant \frac{D}{\omega}\alpha^{\mathrm{o}}_{m,n}\left\|\xi - \bar\xi\right\| \tag{3.23}$$

for every $m, n \in \mathbb{Z}$ and every $\xi, \bar\xi \in E^{\mathrm{o}}_n$.

The proof of this theorem will be given in the last section of this paper.

4. Particular cases of the main theorem

Now we will apply our main theorem to nonuniform $(\mathbb{a}, \mathbb{b}, \mathbb{c}, \mathbb{d})$-trichotomies.

Corollary 4.1. *Let X be a Banach space and suppose that equation* (2.1) *admits a nonuniform $(\mathbb{a}, \mathbb{b}, \mathbb{c}, \mathbb{d})$-trichotomy and such that the sequences $\left(\dfrac{\mathbb{c}_m\mathbb{d}_m}{\mathbb{k}_m}\right)_{m\in\mathbb{Z}}$ and $(\mathbb{a}_m\mathbb{b}_m\mathbb{k}_m)_{m\in\mathbb{Z}}$ are increasing. Assume that $f_m\colon X \to X$, $m \in \mathbb{Z}$, satisfies* (3.6), (3.7) *and*

$$\operatorname{Lip}(f_m) \leqslant \frac{\delta\min\left\{\gamma_m, \dfrac{\mathbb{c}_{m+1}}{\mathbb{c}_m} - \dfrac{\mathbb{d}_m}{\mathbb{d}_{m+1}}\dfrac{\mathbb{k}_{m+1}}{\mathbb{k}_m}, \dfrac{\mathbb{b}_{m+1}}{\mathbb{b}_m}\dfrac{\mathbb{k}_{m+1}}{\mathbb{k}_m} - \dfrac{\mathbb{a}_m}{\mathbb{a}_{m+1}}\right\}}{\mathbb{k}^2_{m+1}} \tag{4.24}$$

with $\delta \in\,]0, 1/6[$ and $(\gamma_m)_{m\in\mathbb{Z}}$ a sequence of positive numbers such that

$$\max\left\{\sum_{k=-\infty}^{+\infty} \frac{\mathbb{a}_{k+1}}{\mathbb{a}_k\,\mathbb{k}_{k+1}}\gamma_k, \sum_{k=-\infty}^{+\infty} \frac{\mathbb{c}_k}{\mathbb{c}_{k+1}\,\mathbb{k}_{k+1}}\gamma_k\right\} \leqslant 1. \tag{4.25}$$

If

$$\lim_{m\to-\infty} \mathbb{c}_m\mathbb{d}_m\mathbb{k}_m = \lim_{m\to+\infty} \frac{\mathbb{k}_m}{\mathbb{a}_m\mathbb{b}_m} = 0, \tag{4.26}$$

then there exists $D \in]0,1[$ and a unique $\varphi \in \mathfrak{B}_D$ such that

$$\Psi_{m-n}\left(\Gamma_{n,\varphi}\right) = \Gamma_{m,\varphi} \text{ for all } m,n \in \mathbb{Z}.$$

Moreover,

$$\left\|\Psi_{m-n}\left(n,\xi,\varphi_n(\xi)\right) - \Psi_{m-n}\left(n,\xi,\varphi_n(\overline{\xi})\right)\right\| \leqslant \begin{cases} \dfrac{D}{\omega}\dfrac{\mathbb{a}_n}{\mathbb{a}_m}\mathbb{k}_n\left\|\xi-\overline{\xi}\right\| & \text{if } m \geqslant n, \\ \dfrac{D}{\omega}\dfrac{\mathbb{c}_m}{\mathbb{c}_n}\mathbb{k}_n\left\|\xi-\overline{\xi}\right\| & \text{if } m \leqslant n, \end{cases}$$

for every $m,n \in \mathbb{Z}$ and every $\xi,\overline{\xi} \in E_n^{\mathrm{o}}$.

Proof. For this type of bounds it is obvious that (3.20) is equivalent to (4.26). From (4.24) and (4.25) it follows for $(m,n) \in \mathbb{Z}^2_\geqslant$ that

$$\sum_{k=n}^{m-1} \frac{\alpha^{\mathrm{o}}_{m,k+1}\operatorname{Lip}(f_k)\alpha^{\mathrm{o}}_{k,n}}{\alpha^{\mathrm{o}}_{m,n}} = \sum_{k=n}^{m-1}\frac{\mathbb{a}_{k+1}}{\mathbb{a}_k}\mathbb{k}_{k+1}\operatorname{Lip}(f_k) \leqslant \sum_{k=n}^{m-1}\frac{\mathbb{a}_{k+1}}{\mathbb{a}_k\,\mathbb{k}_{k+1}}\delta\gamma_k \leqslant \delta$$

and for $(m,n) \in \mathbb{Z}^2_\leqslant$ that

$$\sum_{k=m}^{n-1} \frac{\alpha^{\mathrm{o}}_{m,k+1}\operatorname{Lip}(f_k)\alpha^{\mathrm{o}}_{k,n}}{\alpha^{\mathrm{o}}_{m,n}} = \sum_{k=m}^{n-1}\frac{\mathbb{c}_k}{\mathbb{c}_{k+1}}\mathbb{k}_{k+1}\operatorname{Lip}(f_k) \leqslant \sum_{k=m}^{n-1}\frac{\mathbb{c}_k}{\mathbb{c}_{k+1}\,\mathbb{k}_{k+1}}\delta\gamma_k \leqslant \delta,$$

which proves that $\sigma \leqslant \delta$. From (4.24) and (4.26) we have

$$\begin{aligned}
&\sum_{k=-\infty}^{n-1}\alpha^+_{n,k+1}\operatorname{Lip}(f_k)\alpha^{\mathrm{o}}_{k,n} + \sum_{k=n}^{+\infty}\alpha^-_{n,k+1}\operatorname{Lip}(f_k)\alpha^{\mathrm{o}}_{k,n} \\
&= \frac{\mathbb{k}_n}{\mathbb{c}_n\mathbb{d}_n}\sum_{k=-\infty}^{n-1}\mathbb{d}_{k+1}\mathbb{k}_{k+1}\mathbb{c}_k\operatorname{Lip}(f_k) + \mathbb{a}_n\mathbb{b}_n\mathbb{k}_n\sum_{k=n}^{+\infty}\frac{\mathbb{k}_{k+1}}{\mathbb{a}_k\mathbb{b}_{k+1}}\operatorname{Lip}(f_k) \\
&\leqslant \delta\frac{\mathbb{k}_n}{\mathbb{c}_n\mathbb{d}_n}\sum_{k=-\infty}^{n-1}\left[\frac{\mathbb{c}_{k+1}\mathbb{d}_{k+1}}{\mathbb{k}_{k+1}} - \frac{\mathbb{c}_k\mathbb{d}_k}{\mathbb{k}_k}\right] + \delta\mathbb{a}_n\mathbb{b}_n\mathbb{k}_n\sum_{k=n}^{+\infty}\left[\frac{1}{\mathbb{a}_k\mathbb{b}_k\mathbb{k}_k} - \frac{1}{\mathbb{a}_{k+1}\mathbb{b}_{k+1}\mathbb{k}_{k+1}}\right] \\
&= \delta - \delta\frac{\mathbb{k}_n}{\mathbb{c}_n\mathbb{d}_n}\lim_{k\to-\infty}\frac{\mathbb{c}_k\mathbb{d}_k}{\mathbb{k}_k} + \delta - \delta\mathbb{a}_n\mathbb{b}_n\mathbb{k}_n\lim_{k\to+\infty}\frac{1}{\mathbb{a}_{k+1}\mathbb{b}_{k+1}\mathbb{k}_{k+1}} \\
&= 2\delta
\end{aligned}$$

and thus $\omega \leqslant 2\delta$. Hence $\sigma + \omega \leqslant 3\delta < 1/2$ and from Theorem 3.1 the conclusions of this theorem follow. □

Now we consider the ρ-nonuniform exponential trichotomies. For that we need to introduce the following notation:

$$\overline{\rho}(m) = \rho(m+1) - \rho(m) \qquad \text{and} \qquad \overline{\overline{\rho}}(m) = |\rho(m+1)| - |\rho(m)|.$$

Corollary 4.2. *Let X be a Banach space and suppose that* (2.1) *admits ρ-nonuniform exponential trichotomy such that*

$$a + b + \varepsilon < 0 \qquad \text{and} \qquad c + d + \varepsilon < 0, \tag{4.27}$$

Assume that $f_m \colon X \to X$ satisfies (3.6), (3.7) *and*

$$\operatorname{Lip}(f_m) \leqslant \frac{\delta\min\left\{\gamma_m, \mathrm{e}^{-c\,\overline{\rho}(m)} - \mathrm{e}^{d\,\overline{\rho}(m)+\varepsilon\,\overline{\overline{\rho}}(m)}, \mathrm{e}^{-b\,\overline{\rho}(m)+\varepsilon\,\overline{\overline{\rho}}(m)} - \mathrm{e}^{a\,\overline{\rho}(m)}\right\}}{K^2\,\mathrm{e}^{2\varepsilon|\rho(m+1)|}},$$

where $\delta \in]0, 1/6[$, $\beta > 0$ *and*

$$\gamma_m = \frac{K\left|\mathrm{e}^{-\beta|\rho(m+1)|} - \mathrm{e}^{-\beta|\rho(m)|}\right|}{2\,\mathrm{e}^{\max\{-a,c\}\overline{\rho}(m)-\varepsilon|\rho(m+1)|}}.$$

Then there exists $D \in]0, 1[$ *and a unique* $\varphi \in \mathfrak{B}_D$ *such that*

$$\Psi_{m-n}(\Gamma_{n,\varphi}) = \Gamma_{m,\varphi} \text{ for all } m, n \in \mathbb{Z}.$$

Moreover, putting $\pi_{n,\xi} = (n, \xi, \varphi_n(\xi))$ *and* $\pi_{n,\overline{\xi}} = \left(n, \xi, \varphi_n(\overline{\xi})\right)$*, we have*

$$\left\|\Psi_{m-n}(\pi_{n,\xi}) - \Psi_{m-n}\left(\pi_{n,\overline{\xi}}\right)\right\| \leqslant \begin{cases} \dfrac{DK}{\omega}\,\mathrm{e}^{a(\rho(m)-\rho(n))+\varepsilon|\rho(n)|}\left\|\xi-\overline{\xi}\right\| & \text{if } m \geqslant n, \\ \dfrac{DK}{\omega}\,\mathrm{e}^{c(\rho(n)-\rho(m))+\varepsilon|\rho(n)|}\left\|\xi-\overline{\xi}\right\| & \text{if } m \leqslant n, \end{cases}$$

for every $m, n \in \mathbb{Z}$ *and every* $\xi, \overline{\xi} \in E^{\mathrm{o}}_n$.

Proof. It is easy to see that for this type of bounds (4.27) is equivalent to (4.26) and

$$\frac{\mathbb{a}_{m+1}}{\mathbb{a}_m} = \mathrm{e}^{-a\,\overline{\rho}(m)}, \quad \frac{\mathbb{b}_{m+1}}{\mathbb{b}_m} = \mathrm{e}^{-c\,\overline{\rho}(m)}, \quad \frac{\mathbb{c}_{m+1}}{\mathbb{c}_m} = \mathrm{e}^{-c\,\overline{\rho}(m)}, \quad \frac{\mathbb{d}_{m+1}}{\mathbb{d}_m} = \mathrm{e}^{-d\,\overline{\rho}(m)}$$

and

$$\frac{\mathbb{k}_{m+1}}{\mathbb{k}_m} = \mathrm{e}^{\varepsilon\,\overline{\overline{\rho}}(m)}.$$

This implies that

$$\frac{\mathbb{c}_{m+1}}{\mathbb{c}_m} - \frac{\mathbb{d}_m}{\mathbb{d}_{m+1}}\frac{\mathbb{k}_{m+1}}{\mathbb{k}_m} = \mathrm{e}^{-c\,\overline{\rho}(m)} - \mathrm{e}^{d\,\overline{\rho}(m)+\varepsilon\,\overline{\overline{\rho}}(m)}$$

and

$$\frac{\mathbb{b}_{m+1}}{\mathbb{b}_m}\frac{\mathbb{k}_{m+1}}{\mathbb{k}_m} - \frac{\mathbb{a}_m}{\mathbb{a}_{m+1}} = \mathrm{e}^{-b\,\overline{\rho}(m)+\varepsilon\,\overline{\overline{\rho}}(m)} - \mathrm{e}^{a\,\overline{\rho}(m)}.$$

Moreover,

$$\frac{\mathbb{a}_{k+1}}{\mathbb{a}_k\mathbb{k}_{k+1}}\gamma_k = \frac{\mathrm{e}^{-a\overline{\rho}(k)}}{K\,\mathrm{e}^{\varepsilon|\rho(k+1)|}}\frac{K\left|\mathrm{e}^{-\beta|\rho(k+1)|} - \mathrm{e}^{-\beta|\rho(k)|}\right|}{2\,\mathrm{e}^{\max\{-a,c\}\overline{\rho}(k)-\varepsilon|\rho(k+1)|}} \leqslant \frac{1}{2}\left|\mathrm{e}^{-\beta|\rho(k+1)|} - \mathrm{e}^{-\beta|\rho(k)|}\right|$$

and

$$\frac{\mathbb{c}_k}{\mathbb{c}_{k+1}\mathbb{k}_{k+1}}\gamma_k = \frac{\mathrm{e}^{c\overline{\rho}(k)}}{K\,\mathrm{e}^{\varepsilon|\rho(k+1)|}}\frac{K\left|\mathrm{e}^{-\beta|\rho(k+1)|} - \mathrm{e}^{-\beta|\rho(k)|}\right|}{2\,\mathrm{e}^{\max\{-a,c\}\overline{\rho}(k)-\varepsilon|\rho(k+1)|}} \leqslant \frac{1}{2}\left|\mathrm{e}^{-\beta|\rho(k+1)|} - \mathrm{e}^{-\beta|\rho(k)|}\right|.$$

Since

$$\begin{aligned}\sum_{k=-\infty}^{+\infty}\left|\mathrm{e}^{-\beta|\rho(k+1)|} - \mathrm{e}^{-\beta|\rho(k)|}\right| &= \sum_{k=0}^{+\infty}\mathrm{e}^{-\beta\rho(k)} - \mathrm{e}^{-\beta\rho(k+1)} + \sum_{k=-\infty}^{-1}\mathrm{e}^{\beta\rho(k+1)} - \mathrm{e}^{\beta\rho(k)} \\ &= \mathrm{e}^{-\beta\rho(0)} - \lim_{k\to+\infty}\mathrm{e}^{-\beta\rho(k+1)} + \mathrm{e}^{\beta\rho(0)} - \lim_{k\to-\infty}\mathrm{e}^{\beta\rho(k+1)} \\ &= 2,\end{aligned}$$

the theorem is proved. □

Corollary 4.3. *Let* X *be a Banach space and suppose that* (2.1) *admits a generalized trichotomy with bounds of the form* (2.4) *and such that* (4.27) *holds. Assume that* $f_m \colon X \to X$ *satisfies* (3.6), (3.7) *and*

$$\operatorname{Lip}(f_m) \leqslant \frac{\delta\min\left\{\dfrac{K\left|\mathrm{e}^{-\beta|m+1|} - \mathrm{e}^{-\beta|m|}\right|}{2\,\mathrm{e}^{\max\{-a,c\}-\varepsilon|m+1|}}, \mathrm{e}^{-c} - \mathrm{e}^{d+\varepsilon\nu(m)}, \mathrm{e}^{-b+\varepsilon\nu(m)} - \mathrm{e}^{a}\right\}}{K^2\,\mathrm{e}^{2\varepsilon|m+1|}}$$

with

$$\nu(m) = \begin{cases} 1 & \text{if } m \geqslant 0, \\ -1 & \text{if } m < 0, \end{cases}$$

$\delta \in]0, 1/6[$ *and* $\beta > 0$. *Then there exists* $D \in]0, 1[$ *and a unique* $\varphi \in \mathfrak{B}_D$ *such that*

$$\Psi_{m-n}(\Gamma_{n,\varphi}) = \Gamma_{m,\varphi} \text{ for all } m, n \in \mathbb{Z}.$$

Moreover, putting $\pi_{n,\xi} = (n, \xi, \varphi_n(\xi))$ *and* $\pi_{n,\overline{\xi}} = \left(n, \xi, \varphi_n(\overline{\xi})\right)$, *we have*

$$\left\|\Psi_{m-n}(\pi_{n,\xi}) - \Psi_{m-n}\left(\pi_{n,\overline{\xi}}\right)\right\| \leqslant \begin{cases} \dfrac{DK}{\omega} \mathrm{e}^{a(m-n)+\varepsilon|n|} \left\|\xi - \overline{\xi}\right\| & \text{if } m \geqslant n, \\ \dfrac{DK}{\omega} \mathrm{e}^{c(n-m)+\varepsilon|n|} \left\|\xi - \overline{\xi}\right\| & \text{if } m \leqslant n, \end{cases}$$

for every $m, n \in \mathbb{Z}$ *and every* $\xi, \overline{\xi} \in E_n^{\mathrm{o}}$.

This theorem improves Barreira and Valls [5, Theorem 2] (see also Barreira and Valls [10, Theorem 2]) because we obtain a better decay along the invariant manifold.

5. Proof of the main theorem

Lemma 5.1 ([11, Lemma 5.1]). *If* σ *and* ω *are positive numbers such that* $\sigma + \omega < 1/2$, *then there are* $C \in]1, 2[$ *and* $D \in]0, 1[$ *such that*

$$\sigma = \frac{C-1}{C(1+D)} \quad \text{and} \quad \omega = \frac{D}{C(1+D)}. \tag{5.28}$$

From now on the numbers C and D will be given by (5.28). Moreover, the number D in Theorem 3.1 is also given by (5.28).

To prove the main theorem we need to introduce another space of sequences of functions. Let $\mathfrak{A}_C$ be the space of sequences $x = (x_{m,n})_{m,n\in\mathbb{Z}}$ of functions

$$x_{m,n} \colon E_n^{\mathrm{o}} \to E_m^{\mathrm{o}}$$

such that

$$x_{m,n}(0) = 0 \text{ for all } m, n \in \mathbb{Z}; \tag{5.29}$$

$$x_{n,n}(\xi) = \xi \text{ for all } n \in \mathbb{Z} \text{ and all } \xi \in E_n^{\mathrm{o}}; \tag{5.30}$$

$$\left\|x_{m,n}(\xi) - x_{m,n}(\overline{\xi})\right\| \leqslant C\alpha_{m,n}^{\mathrm{o}} \left\|\xi - \overline{\xi}\right\| \text{ for all } m, n \in \mathbb{Z} \text{ and all } \xi, \overline{\xi} \in E_n^{\mathrm{o}}. \tag{5.31}$$

It is obvious that from (5.29) and (5.31) we have

$$\|x_{m,n}(\xi)\| \leqslant C\alpha_{m,n}^{\mathrm{o}} \|\xi\| \text{ for all } m, n \in \mathbb{Z} \text{ and all } \xi \in E_n^{\mathrm{o}}. \tag{5.32}$$

The space $\mathfrak{A}_C$ equipped with the metric defined, for every $x, y \in \mathfrak{A}_C$, by

$$d_1(x, y) = \sup\left\{\frac{\|x_{m,n}(\xi) - y_{m,n}(\xi)\|}{\alpha_{m,n}^{\mathrm{o}} \|\xi\|} : m, n \in \mathbb{Z},\ \xi \in E_n^{\mathrm{o}} \setminus \{0\}\right\} \tag{5.33}$$

is a complete metric space.

The space $\mathfrak{B}_D$ is also a complete metric space with the metric given, for all $\varphi, \psi \in \mathfrak{B}_D$, by

$$d_2(\varphi, \psi) = \sup\left\{\frac{\|\varphi_n(\xi) - \psi_n(\xi)\|}{\|\xi\|} : n \in \mathbb{Z},\ \xi \in E_n^{\mathrm{o}} \setminus \{0\}\right\}. \tag{5.34}$$

Therefore, the space $\mathfrak{C}_{C,D} = \mathfrak{A}_C \times \mathfrak{B}_D$ equipped with the metric defined, for every $(x, \varphi), (y, \psi) \in \mathfrak{C}_{C,D}$, by

$$d((x, \varphi), (y, \psi)) = d_1(x, y) + d_2(\varphi, \psi) \tag{5.35}$$

is a complete metric space.

For every $(x,\varphi) \in \mathfrak{C}_{C,D}$ and every $k,n \in \mathbb{Z}$ set

$$\begin{aligned} f_{x,\varphi,k,n}(\xi) &= f_k(x_{k,n}(\xi), \varphi_k(x_{k,n}(\xi))) \\ &= f_k(x_{k,n}(\xi), \varphi_k^+(x_{k,n}(\xi)), \varphi_k^-(x_{k,n}(\xi))). \end{aligned} \tag{5.36}$$

To prove (3.22) we need to show that there is a unique $(x,\varphi) \in \mathfrak{C}_{C,D}$ such that

$$\big(x_{m,n}(\xi), \varphi_m^+(x_{m,n}(\xi)), \varphi_m^-(x_{m,n}(\xi))\big)$$

is a solution of (3.5). By (3.11), (3.12) and (3.13) we need to prove that

$$x_{m,n}(\xi) = \begin{cases} \Phi_{m,n}\xi + \displaystyle\sum_{k=n}^{m-1} \Phi_{m,k+1} P_{k+1}^{\mathrm{o}} f_{x,\varphi,k,n}(\xi) & \text{if } m \geqslant n, \\ \Phi_{m,n}\xi - \displaystyle\sum_{k=m}^{n-1} \Phi_{m,k+1} P_{k+1}^{\mathrm{o}} f_{x,\varphi,k,n}(\xi) & \text{if } m \leqslant n, \end{cases} \tag{5.37}$$

$$\varphi_m^+(x_{m,n}(\xi)) = \begin{cases} \Phi_{m,n}\varphi_n^+(\xi) + \displaystyle\sum_{k=n}^{m-1} \Phi_{m,k+1} P_{k+1}^{+} f_{x,\varphi,k,n}(\xi) & \text{if } m \geqslant n, \\ \Phi_{m,n}\varphi_n^+(\xi) - \displaystyle\sum_{k=m}^{n-1} \Phi_{m,k+1} P_{k+1}^{+} f_{x,\varphi,k,n}(\xi) & \text{if } m \leqslant n, \end{cases} \tag{5.38}$$

and

$$\varphi_m^-(x_{m,n}(\xi)) = \begin{cases} \Phi_{m,n}\varphi_n^-(\xi) + \displaystyle\sum_{k=n}^{m-1} \Phi_{m,k+1} P_{k+1}^{-} f_{x,\varphi,k,n}(\xi) & \text{if } m \geqslant n \\ \Phi_{m,n}\varphi_n^-(\xi) - \displaystyle\sum_{k=m}^{n-1} \Phi_{m,k+1} P_{k+1}^{-} f_{x,\varphi,k,n}(\xi) & \text{if } m \leqslant n \end{cases} \tag{5.39}$$

Lemma 5.2. *Let $(x,\varphi) \in \mathfrak{C}_{C,D}$. Suppose that for every $m,n \in \mathbb{Z}$ and every $\xi \in E_n^{\mathrm{o}}$ equation* (5.37) *holds. Then the following properties are equivalent:*

a) for every $m,n \in \mathbb{Z}$ and every $\xi \in E_n^{\mathrm{o}}$, identities (5.38) *and* (5.39) *hold;*
b) for every $n \in \mathbb{Z}$ and every $\xi \in E_n^{\mathrm{o}}$,

$$\varphi_n^+(\xi) = \sum_{k=-\infty}^{n-1} \Phi_{n,k+1} P_{k+1}^{+} f_{x,\varphi,k,n}(\xi) \tag{5.40}$$

and

$$\varphi_n^-(\xi) = -\sum_{k=n}^{+\infty} \Phi_{n,k+1} P_{k+1}^{-} f_{x,\varphi,k,n}(\xi). \tag{5.41}$$

Proof. First we prove that the series in equations (5.40) and (5.41) are convergent. By (5.36), (3.10), (3.17) and (5.32) we have

$$\begin{aligned} \|f_{x,\varphi,k,n}(\xi)\| &= \|f_k(x_{k,n}(\xi), \varphi_k(x_{k,n}(\xi)))\| \\ &\leqslant \operatorname{Lip}(f_k)\, \|x_{k,n}(\xi) + \varphi_k(x_{k,n}(\xi))\| \\ &\leqslant \operatorname{Lip}(f_k)\, (\|x_{k,n}(\xi)\| + \|\varphi_k(x_{k,n}(\xi))\|) \\ &\leqslant \operatorname{Lip}(f_k)\, (\|x_{k,n}(\xi)\| + D\, \|x_{k,n}(\xi)\|) \\ &\leqslant C(1+D) \operatorname{Lip}(f_k) \alpha_{k,n}^{\mathrm{o}}\, \|\xi\| \end{aligned} \tag{5.42}$$

for every $k, n \in \mathbb{Z}$ and every $\xi \in E_n^{\mathrm{o}}$. Thus by ($\mathbf{T2}$), (5.42) and (3.19) we have

$$\begin{aligned}\sum_{k=-\infty}^{n-1} \left\|\Phi_{n,k+1}P_{k+1}^{+} f_{x,\varphi,k,n}(\xi)\right\| &\leqslant \sum_{k=-\infty}^{n-1} \left\|\Phi_{n,k+1}P_{k+1}^{+}\right\| \, \|f_{x,\varphi,k,n}(\xi)\| \\ &\leqslant C(1+D)\,\|\xi\| \sum_{k=-\infty}^{n-1} \alpha_{n,k+1}^{+} \operatorname{Lip}(f_k)\alpha_{k,n}^{\mathrm{o}} \\ &\leqslant C(1+D)\,\|\xi\|\,\omega\end{aligned}$$

and from ($\mathbf{T3}$), (5.42) and (3.19) it follows that

$$\begin{aligned}\sum_{k=n}^{+\infty} \left\|\Phi_{n,k+1}P_{k+1}^{-} f_{x,\varphi,k,n}(\xi)\right\| &\leqslant \sum_{k=n}^{+\infty} \left\|\Phi_{n,k+1}P_{k+1}^{-}\right\| \, \|f_{x,\varphi,k,n}(\xi)\| \\ &\leqslant C(1+D)\,\|\xi\| \sum_{k=n}^{+\infty} \alpha_{n,k+1}^{-} \operatorname{Lip}(f_k)\alpha_{k,n}^{\mathrm{o}} \\ &\leqslant C(1+D)\,\|\xi\|\,\omega,\end{aligned}$$

and this proves that the series in equations (5.40) and (5.41) are convergent.

Now we prove that $a) \Rightarrow b)$. From (5.38), for every $m < n$, it follows that

$$\begin{aligned}\varphi_n^{+}(\xi) &= \Phi_{n,m}\varphi_m^{+}(x_{m,n}(\xi)) + \sum_{k=m}^{n-1} \Phi_{n,m}\Phi_{m,k+1}P_{k+1}^{+} f_{x,\varphi,k,n}(\xi) \\ &= \Phi_{n,m}P_m^{+}\varphi_m^{+}(x_{m,n}(\xi)) + \sum_{k=m}^{n-1} \Phi_{n,k+1}P_{k+1}^{+} f_{x,\varphi,k,n}(\xi).\end{aligned}$$

Since by ($\mathbf{T2}$), (3.17) and (5.32) we have

$$\left\|\Phi_{n,m}P_m^{+}\varphi_m^{+}(x_{m,n}(\xi))\right\| \leqslant \alpha_{n,m}^{+} D \,\|x_{m,n}(\xi)\| \leqslant CD\alpha_{n,m}^{+}\alpha_{m,n}^{\mathrm{o}} \,\|\xi\|\,,$$

from (3.20) we conclude that $\lim\limits_{m\to-\infty} \Phi_{n,m}P_m^{+}\varphi_m^{+}(x_{m,n}(\xi)) = 0$ and this implies that

$$\varphi_n^{+}(\xi) = \sum_{k=-\infty}^{n-1} \Phi_{n,k+1}P_{k+1}^{+} f_{x,\varphi,k,n}(\xi).$$

Similarly, from (5.39) we have

$$\begin{aligned}\varphi_n^{-}(\xi) &= \Phi_{n,m}\varphi_m^{-}(x_{m,n}(\xi)) - \sum_{k=n}^{m-1} \Phi_{n,m}\Phi_{m,k+1}P_{k+1}^{-} f_{x,\varphi,k,n}(\xi) \\ &= \Phi_{n,m}P_m^{-}\varphi_m^{-}(x_{m,n}(\xi)) - \sum_{k=n}^{m-1} \Phi_{n,k+1}P_{k+1}^{-} f_{x,\varphi,k,n}(\xi)\end{aligned}$$

for every $m \geqslant n$. Because

$$\left\|\Phi_{n,m}P_m^{-}\varphi_m^{-}(x_{m,n}(\xi))\right\| \leqslant \alpha_{n,m}^{-} D \,\|x_{m,n}(\xi)\| \leqslant CD\alpha_{n,m}^{-}\alpha_{m,n}^{\mathrm{o}} \,\|\xi\|\,,$$

by (3.20) we conclude that $\lim\limits_{m\to+\infty} \Phi_{n,m}P_m^{-}\varphi_m^{-}(x_{m,n}(\xi)) = 0$. Thus (5.41) holds.

Now we will prove that $b) \Rightarrow a)$. From (5.40) we get

$$\Phi_{m,n}\varphi_n^{+}(\xi) = \sum_{k=-\infty}^{n-1} \Phi_{m,n}\Phi_{n,k+1}P_{k+1}^{+} f_{x,\varphi,k,n}(\xi) = \sum_{k=-\infty}^{n-1} \Phi_{m,k+1}P_{k+1}^{+} f_{x,\varphi,k,n}(\xi)$$

and this implies that for $m \geqslant n$ we have

$$\begin{aligned}
&\Phi_{m,n}\varphi_n^+(\xi) + \sum_{k=n}^{m-1} \Phi_{m,k+1}P_{k+1}^+ f_{x,\varphi,k,n}(\xi) \\
&= \sum_{k=-\infty}^{m-1} \Phi_{m,k+1}P_{k+1}^+ f_{x,\varphi,k,n}(\xi) \\
&= \sum_{k=-\infty}^{m-1} \Phi_{m,k+1}P_{k+1}^+ f_k(x_{k,n}(\xi), \varphi_k(x_{k,n}(\xi))) \\
&= \sum_{k=-\infty}^{m-1} \Phi_{m,k+1}P_{k+1}^+ f_k(x_{k,m}(x_{m,n}(\xi)), \varphi_k(x_{k,m}(x_{m,n}(\xi)))) \\
&= \varphi_m^+(x_{m,n}(\xi))
\end{aligned}$$

and for $m \leqslant n$ we obtain

$$\begin{aligned}
&\Phi_{m,n}\varphi_n^+(\xi) - \sum_{k=m}^{n-1} \Phi_{m,k+1}P_{k+1}^+ f_{x,\varphi,k,n}(\xi) \\
&= \sum_{k=-\infty}^{m-1} \Phi_{m,k+1}P_{k+1}^+ f_{x,\varphi,k,n}(\xi) \\
&= \sum_{k=-\infty}^{m-1} \Phi_{m,k+1}P_{k+1}^+ f_k(x_{k,n}(\xi), \varphi_k(x_{k,n}(\xi))) \\
&= \sum_{k=-\infty}^{m-1} \Phi_{m,k+1}P_{k+1}^+ f_k(x_{k,m}(x_{m,n}(\xi)), \varphi_k(x_{k,m}(x_{m,n}(\xi)))) \\
&= \varphi_m^+(x_{m,n}(\xi)).
\end{aligned}$$

Similarly, from (5.41) we get

$$\Phi_{m,n}\varphi_n^-(\xi) = -\sum_{k=n}^{+\infty} \Phi_{m,n}\Phi_{n,k+1}P_{k+1}^- f_{x,\varphi,k,n}(\xi) = -\sum_{k=n}^{+\infty} \Phi_{m,k+1}P_{k+1}^- f_{x,\varphi,k,n}(\xi)$$

and this implies that for $m \geqslant n$ we have

$$\begin{aligned}
&\Phi_{m,n}\varphi_n^-(\xi) + \sum_{k=n}^{m-1} \Phi_{m,k+1}P_{k+1}^- f_{x,\varphi,k,n}(\xi) \\
&= -\sum_{k=m}^{+\infty} \Phi_{m,k+1}P_{k+1}^- f_{x,\varphi,k,n}(\xi) \\
&= -\sum_{k=m}^{+\infty} \Phi_{m,k+1}P_{k+1}^- f_k(x_{k,n}(\xi), \varphi_k(x_{k,n}(\xi))) \\
&= -\sum_{k=m}^{+\infty} \Phi_{m,k+1}P_{k+1}^- f_k(x_{k,m}(x_{m,n}(\xi)), \varphi_k(x_{k,m}(x_{m,n}(\xi)))) \\
&= \varphi_m^-(x_{m,n}(\xi))
\end{aligned}$$

and for $m \leqslant n$ we obtain

$$\begin{aligned}
&\Phi_{m,n}\varphi_n^-(\xi) - \sum_{k=m}^{n-1} \Phi_{m,k+1}P_{k+1}^- f_{x,\varphi,k,n}(\xi)\\
&= - \sum_{k=m}^{+\infty} \Phi_{m,k+1}P_{k+1}^- f_{x,\varphi,k,n}(\xi)\\
&= - \sum_{k=m}^{+\infty} \Phi_{m,k+1}P_{k+1}^- f_k(x_{k,n}(\xi), \varphi_k(x_{k,n}(\xi)))\\
&= - \sum_{k=m}^{+\infty} \Phi_{m,k+1}P_{k+1}^- f_k(x_{k,m}(x_{m,n}(\xi)), \varphi_k(x_{k,m}(x_{m,n}(\xi))))\\
&= \varphi_m^-(x_{m,n}(\xi))
\end{aligned}$$

and the lemma is proved. □

On the space $\mathfrak{C}_{C,D}$ consider the operator T^{o} that assigns to each $(x,\varphi) \in \mathfrak{C}_{C,D}$ the sequence $T^{\mathrm{o}}(x,\varphi) = \left(T^{\mathrm{o}}_{m,n}(x,\varphi)\right)_{(m,n)\in\mathbb{Z}^2}$ of functions $T^{\mathrm{o}}_{m,n}(x,\varphi)\colon E_n^{\mathrm{o}} \to X$ defined by

$$\left[T^{\mathrm{o}}_{m,n}(x,\varphi)\right](\xi) = \begin{cases} \Phi_{m,n}\xi + \displaystyle\sum_{k=n}^{m-1} \Phi_{m,k+1}P_{k+1}^{\mathrm{o}} f_{x,\varphi,k,n}(\xi) & \text{if } m \geqslant n,\\ \Phi_{m,n}\xi - \displaystyle\sum_{k=m}^{n-1} \Phi_{m,k+1}P_{k+1}^{\mathrm{o}} f_{x,\varphi,k,n}(\xi) & \text{if } m \leqslant n. \end{cases}$$

Lemma 5.3. *If $(x,\varphi) \in \mathfrak{C}_{C,D}$, then $T^{\mathrm{o}}(x,\varphi) \in \mathfrak{A}_C$.*

Proof. Let $(x,\varphi) \in \mathfrak{C}_{C,D}$. By definition $\left[T^{\mathrm{o}}_{m,n}(x,\varphi)\right](\xi) \in E_m^{\mathrm{o}}$ for every $m,n \in \mathbb{Z}$ and every $\xi \in E_n^{\mathrm{o}}$. From (5.29), (3.15) and (3.7) we have $\left[T^{\mathrm{o}}_{m,n}(x,\varphi)\right](0) = 0$ for every $m,n \in \mathbb{Z}$. Also by definition $\left[T^{\mathrm{o}}_{n,n}(x,\varphi)\right](\xi) = \xi$ for every $n \in \mathbb{Z}$ and every $\xi \in E_n^{\mathrm{o}}$. Hence $T(x,\varphi)$ satisfies (5.29) and (5.30).

To finish the proof we must prove that $T(x,\varphi)$ satisfies (5.31). From (5.36), (3.9), (3.16) and (5.31) it follows that

$$\begin{aligned}
&\left\|f_{x,\varphi,k,n}(\xi) - f_{x,\varphi,k,n}(\bar\xi)\right\|\\
&= \left\|f_k(x_{k,n}(\xi), \varphi_k(x_{k,n}(\xi))) - f_k(x_{k,n}(\bar\xi), \varphi_k(x_{k,n}(\bar\xi)))\right\|\\
&\leqslant \operatorname{Lip}(f_k)\left[\left\|x_{k,n}(\xi) - x_{k,n}(\bar\xi)\right\| + \left\|\varphi_k(x_{k,n}(\xi)) - \varphi_k(x_{k,n}(\bar\xi))\right\|\right]\\
&\leqslant \operatorname{Lip}(f_k)(1+D)\left\|x_{k,n}(\xi) - x_{k,n}(\bar\xi)\right\|\\
&\leqslant C(1+D)\operatorname{Lip}(f_k)\alpha^{\mathrm{o}}_{m,n}\left\|\xi - \bar\xi\right\|
\end{aligned} \tag{5.43}$$

and this implies by (**T1**), (3.18) and (5.28) that

$$\begin{aligned}
&\left\| \left[T^{\mathrm{o}}_{m,n}(x,\varphi)\right](\xi) - \left[T^{\mathrm{o}}_{m,n}(x,\varphi)\right](\overline{\xi})\right\| \\
&\leqslant \|\Phi_{m,n}P_n\| \left\|\xi - \overline{\xi}\right\| + \sum_{k=n}^{m-1} \left\|\Phi_{m,k+1}P^{\mathrm{o}}_{k+1}\right\| \left\|f_{x,\varphi,k,n}(\xi) - f_{x,\varphi,k,n}(\overline{\xi})\right\| \\
&\leqslant \alpha^{\mathrm{o}}_{m,n}\left\|\xi - \overline{\xi}\right\| + C(1+D)\left\|\xi - \overline{\xi}\right\| \sum_{k=n}^{m-1} \alpha^{\mathrm{o}}_{m,k+1}\operatorname{Lip}(f_k)\alpha^{\mathrm{o}}_{k,n} \\
&\leqslant \alpha^{\mathrm{o}}_{m,n}\left\|\xi - \overline{\xi}\right\| + C(1+D)\sigma\alpha^{\mathrm{o}}_{m,n}\left\|\xi - \overline{\xi}\right\| \\
&= C\alpha^{\mathrm{o}}_{m,n}\left\|\xi - \overline{\xi}\right\|
\end{aligned}$$

for every $m \geqslant n$ and every $\xi \in E^{\mathrm{o}}_n$. Similarly, for every $m \leqslant n$ and every $\xi \in E^{\mathrm{o}}_n$ we also have

$$\begin{aligned}
&\left\| \left[T^{\mathrm{o}}_{m,n}(x,\varphi)\right](\xi) - \left[T^{\mathrm{o}}_{m,n}(x,\varphi)\right](\overline{\xi})\right\| \\
&\leqslant \|\Phi_{m,n}P_n\| \left\|\xi - \overline{\xi}\right\| + \sum_{k=m}^{n-1} \left\|\Phi_{m,k+1}P^{\mathrm{o}}_{k+1}\right\| \left\|f_{x,\varphi,k,n}(\xi) - f_{x,\varphi,k,n}(\overline{\xi})\right\| \\
&\leqslant \alpha^{\mathrm{o}}_{m,n}\left\|\xi - \overline{\xi}\right\| + C(1+D)\left\|\xi - \overline{\xi}\right\| \sum_{k=m}^{n-1} \alpha^{\mathrm{o}}_{m,k+1}\operatorname{Lip}(f_k)\alpha^{\mathrm{o}}_{k,n} \\
&\leqslant \alpha^{\mathrm{o}}_{m,n}\left\|\xi - \overline{\xi}\right\| + C(1+D)\sigma\alpha^{\mathrm{o}}_{m,n}\left\|\xi - \overline{\xi}\right\| \\
&= C\alpha^{\mathrm{o}}_{m,n}\left\|\xi - \overline{\xi}\right\|
\end{aligned}$$

and this finishes the proof. □

Now, let $T^{\pm}$ be the operator that assigns to every $(x,\varphi) \in \mathfrak{C}_{C,D}$ the sequence $T^{\pm}(x,\varphi) = (T^{\pm}_n(x,\varphi))_{n\in\mathbb{Z}}$ of functions $T^{\pm}_n(x,y)\colon E^{\mathrm{o}}_n \to X$ defined by

$$\begin{aligned}
\left[T^{\pm}_n(x,\varphi)\right](\xi) &= \left(\left[T^{+}_n(x,\varphi)\right](\xi), \left[T^{-}_n(x,\varphi)\right](\xi)\right) \\
&= \left(\sum_{k=-\infty}^{n-1} \Phi_{n,k+1}P^{+}_{k+1}f_{x,\varphi,k,n}(\xi), -\sum_{k=n}^{+\infty} \Phi_{n,k+1}P^{-}_{k+1}f_{x,\varphi,k,n}(\xi)\right).
\end{aligned}$$

Lemma 5.4. *If $(x,\varphi) \in \mathfrak{C}_{C,D}$, then $T^{\pm}(x,\varphi) \in \mathfrak{B}_D$.*

Proof. Let $(x,\varphi) \in \mathfrak{C}_{C,D}$. It is obvious from the definition that $[T^{+}_n(x,\varphi)](\xi) \in E^{+}_n$ and $[T^{+}_n(x,\varphi)](\xi) \in E^{-}_n$ for every $n \in \mathbb{Z}$ and every $\xi \in E^{\mathrm{o}}_n$. From (5.29), (3.15) and (3.7) we have $[T^{\pm}_n(x,\varphi)](0) = 0$ for every $n \in \mathbb{Z}$, i.e., $T(x,\varphi)$ satisfies (3.15).

To finish the proof we must prove that $T(x,\varphi)$ satisfies (3.16). Since from by (**T2**) and (5.43) we obtain

$$\begin{aligned}
&\left\| \left[T^{+}_n(x,\varphi)\right](\xi) - \left[T^{+}_n(x,\varphi)\right](\overline{\xi})\right\| \\
&\leqslant \sum_{k=-\infty}^{n-1} \left\|\Phi_{n,k+1}P^{+}_{k+1}\right\| \left\|f_{x,\varphi,k,n}(\xi) - f_{x,\varphi,k,n}(\overline{\xi})\right\| \\
&\leqslant C(1+D)\left\|\xi - \overline{\xi}\right\| \sum_{k=-\infty}^{n-1} \alpha^{+}_{n,k+1}\operatorname{Lip}(f_k)\alpha^{\mathrm{o}}_{k,n}
\end{aligned}$$

and from (**T3**) and (5.43) we have

$$\begin{aligned}\left\|\left[T_n^-(x,\varphi)\right](\xi)-\left[T_n^-(x,\varphi)\right](\bar\xi)\right\| &\leqslant \sum_{k=n}^{+\infty}\left\|\Phi_{n,k+1}P_{k+1}^-\right\|\left\|f_{x,\varphi,k,n}(\xi)-f_{x,\varphi,k,n}(\bar\xi)\right\|\\ &\leqslant C(1+D)\left\|\xi-\bar\xi\right\|\sum_{k=n}^{+\infty}\alpha_{n,k+1}^-\operatorname{Lip}(f_k)\alpha_{k,n}^{\mathrm{o}},\end{aligned}$$

it follows by (3.19) and (5.28) that

$$\begin{aligned}&\left\|\left[T_n^\pm(x,\varphi)\right](\xi)-\left[T_n^\pm(x,\varphi)\right](\xi)\right\|\\ &\leqslant\left\|\left[T_n^+(x,\varphi)\right](\xi)-\left[T_n^+(x,\varphi)\right](\xi)\right\|+\left\|\left[T_n^-(x,\varphi)\right](\xi)-\left[T_n^-(x,\varphi)\right](\xi)\right\|\\ &\leqslant C(1+D)\left\|\xi-\bar\xi\right\|\left[\sum_{k=-\infty}^{n-1}\alpha_{n,k+1}^+\operatorname{Lip}(f_k)\alpha_{k,n}^{\mathrm{o}}+\sum_{k=n}^{+\infty}\alpha_{n,k+1}^-\operatorname{Lip}(f_k)\alpha_{k,n}^{\mathrm{o}}\right]\\ &\leqslant C(1+D)\omega\left\|\xi-\bar\xi\right\|\\ &=C\left\|\xi-\bar\xi\right\|,\end{aligned}$$

which finishes the proof. □

Lemma 5.5. *Let* $(x,\varphi),(y,\psi)\in\mathfrak{C}_{C,D}$. *Then*

$$d(T^{\mathrm{o}}(x,\varphi),T^{\mathrm{o}}(y,\psi))\leqslant\sigma\left[(1+D)d_1(x,y)+Cd_2(\varphi,\psi)\right] \tag{5.44}$$

and

$$d(T^{\pm}(x,\varphi),T^{\pm}(y,\psi))\leqslant\omega\left[(1+D)d_1(x,y)+Cd_2(\varphi,\psi)\right]. \tag{5.45}$$

Proof. By (3.9), (3.16), (5.33), (5.34) and (5.32) we have

$$\begin{aligned}&\|f_{x,\varphi,k,n}(\xi)-f_{y,\psi,k,n}(\xi)\|\\ &=\|f_k(x_{k,n}(\xi),\varphi_k(x_{k,n}(\xi)))-f_k(y_{k,n}(\xi),\psi_k(y_{k,n}(\xi)))\|\\ &\leqslant\operatorname{Lip}(f_k)\left[\|x_{k,n}(\xi)-y_{k,n}(\xi)\|+\|\varphi_k(x_{k,n}(\xi))-\psi_k(y_{k,n}(\xi))\|\right]\\ &\leqslant\operatorname{Lip}(f_k)\left[(1+D)\|x_{k,n}(\xi)-y_{k,n}(\xi)\|+\|\varphi_k(y_{k,n}(\xi))-\psi_k(y_{k,n}(\xi))\|\right]\\ &\leqslant\operatorname{Lip}(f_k)\left[(1+D)\alpha_{k,n}^{\mathrm{o}}d_1(x,y)\|\xi\|+d_2(\varphi,\psi)\|y_{k,n}(\xi)\|\right]\\ &\leqslant\operatorname{Lip}(f_k)\left[(1+D)\alpha_{k,n}^{\mathrm{o}}d_1(x,y)\|\xi\|+d_2(\varphi,\psi)C\alpha_{k,n}^{\mathrm{o}}\|\xi\|\right]\\ &=\operatorname{Lip}(f_k)\alpha_{k,n}^{\mathrm{o}}\|\xi\|\left[(1+D)d_1(x,y)+Cd_2(\varphi,\psi)\right].\end{aligned} \tag{5.46}$$

From (**T2**), (5.46) and (3.18), for every $(m,n)\in\mathbb{Z}^2_\geqslant$, it follows that

$$\begin{aligned}&\left\|\left[T_{m,n}^{\mathrm{o}}(x,\varphi)\right](\xi)-\left[T_{m,n}^{\mathrm{o}}(x,\varphi)\right](\xi)\right\|\\ &\leqslant\sum_{k=n}^{m-1}\left\|\Phi_{m,k+1}P_{k+1}^{\mathrm{o}}\right\|\|f_{x,\varphi,k,n}(\xi)-f_{y,\psi,k,n}(\xi)\|\\ &\leqslant\|\xi\|\left[(1+D)d_1(x,y)+Cd_2(\varphi,\psi)\right]\sum_{k=n}^{m-1}\alpha_{m,k+1}^{\mathrm{o}}\operatorname{Lip}(f_k)\alpha_{k,n}^{\mathrm{o}}\\ &\leqslant\alpha_{m,n}^{\mathrm{o}}\|\xi\|\,\sigma\left[(1+D)d_1(x,y)+Cd_2(\varphi,\psi)\right]\end{aligned}$$

and for every $(m,n) \in \mathbb{Z}^2_{\leqslant}$ we obtain

$$\begin{aligned}
&\left\| \left[T^{\mathrm{o}}_{m,n}(x,\varphi)\right](\xi) - \left[T^{\mathrm{o}}_{m,n}(x,\varphi)\right](\xi) \right\| \\
&\leqslant \sum_{k=m}^{n-1} \left\|\Phi_{m,k+1}P^{\mathrm{o}}_{k+1}\right\| \left\|f_{x,\varphi,k,n}(\xi) - f_{y,\psi,k,n}(\xi)\right\| \\
&\leqslant \|\xi\| \left[(1+D)d_1(x,y) + Cd_2(\varphi,\psi)\right] \sum_{k=m}^{n-1} \alpha^{\mathrm{o}}_{m,k+1} \operatorname{Lip}(f_k) \alpha^{\mathrm{o}}_{k,n} \\
&\leqslant \alpha^{\mathrm{o}}_{m,n} \|\xi\| \, \sigma \left[(1+D)d_1(x,y) + Cd_2(\varphi,\psi)\right]
\end{aligned}$$

and this proves that (5.44).

On the other hand, using (**T2**) and (5.46) we get

$$\begin{aligned}
&\left\| \left[T^{+}_{n}(x,\varphi)\right](\xi) - \left[T^{+}_{n}(x,\varphi)\right](\xi) \right\| \\
&\leqslant \sum_{k=-\infty}^{n-1} \left\|\Phi_{n,k+1}P^{+}_{k+1}\right\| \left\|f_{x,\varphi,k,n}(\xi) - f_{y,\psi,k,n}(\xi)\right\| \\
&\leqslant \|\xi\| \left[(1+D)d_1(x,y) + Cd_2(\varphi,\psi)\right] \sum_{k=-\infty}^{n-1} \alpha^{+}_{n,k+1} \operatorname{Lip}(f_k) \alpha^{\mathrm{o}}_{k,n}
\end{aligned}$$

and using (**T3**) and (5.46) we obtain

$$\begin{aligned}
&\left\| \left[T^{-}_{n}(x,\varphi)\right](\xi) - \left[T^{-}_{n}(x,\varphi)\right](\xi) \right\| \\
&\leqslant \sum_{k=n}^{+\infty} \left\|\Phi_{n,k+1}P^{-}_{k+1}\right\| \left\|f_{x,\varphi,k,n}(\xi) - f_{y,\psi,k,n}(\xi)\right\| \\
&\leqslant \|\xi\| \left[(1+D)d_1(x,y) + Cd_2(\varphi,\psi)\right] \sum_{k=n}^{+\infty} \alpha^{-}_{n,k+1} \operatorname{Lip}(f_k) \alpha^{\mathrm{o}}_{k,n}.
\end{aligned}$$

This implies that

$$\begin{aligned}
&\left\| \left[T^{\pm}_{n}(x,\varphi)\right](\xi) - \left[T^{\pm}_{n}(x,\varphi)\right](\xi) \right\| \\
&\leqslant \left\| \left[T^{+}_{n}(x,\varphi)\right](\xi) - \left[T^{+}_{n}(x,\varphi)\right](\xi) \right\| + \left\| \left[T^{-}_{n}(x,\varphi)\right](\xi) - \left[T^{-}_{n}(x,\varphi)\right](\xi) \right\| \\
&\leqslant \|\xi\| \left[(1+D)d_1(x,y) + Cd_2(\varphi,\psi)\right] \\
&\qquad \left[\sum_{k=-\infty}^{n-1} \alpha^{+}_{n,k+1} \operatorname{Lip}(f_k) \alpha^{\mathrm{o}}_{k,n} + \sum_{k=n}^{+\infty} \alpha^{-}_{n,k+1} \operatorname{Lip}(f_k) \alpha^{\mathrm{o}}_{k,n} \right] \\
&\leqslant \|\xi\| \, \omega \left[(1+D)d_1(x,y) + Cd_2(\varphi,\psi)\right]
\end{aligned}$$

which proves (5.45). □

Define the operator $T\colon \mathfrak{C}_{C,D} \to \mathfrak{C}_{C,D}$ by

$$T(x,\varphi) = \left(T^{\mathrm{o}}(x,\varphi), T^{\pm}(x,\varphi)\right).$$

Lemma 5.6. *The operator $T\colon \mathfrak{C}_{C,D} \to \mathfrak{C}_{C,D}$ is a contraction.*

Proof. From last lemma and (5.35) it follows, for every $(x,\varphi),(y,\psi) \in \mathfrak{C}_{C,D}$, that

$$\begin{aligned}
d(T(x,\varphi), T(y,\psi)) &= d_1(T^{\mathrm{o}}(x,\varphi), T^{\mathrm{o}}(y,\psi)) + d_2(T^{\pm}(x,\varphi), T^{\pm}(y,\psi)) \\
&\leqslant (\sigma+\omega)\left[Cd_1(x,y) + (1+D)d_2(\varphi,\psi)\right] \\
&\leqslant (\sigma+\omega)\max\{C,(1+D)\}\left[d_1(x,y) + d_2(\varphi,\psi)\right] \\
&= (\sigma+\omega)\max\{C,(1+D)\}\, d((x,\varphi),(y,\psi)).
\end{aligned}$$

However by (3.21) and Lemma 5.1 we can conclude that

$$(\sigma + \omega) \max \{C, (1 + D)\} < 1$$

and this finishes the proof. □

Now we are in conditions to prove the main theorem.

Proof Theorem 3.1. In Lemma 5.6 we proved that T is a contraction. Since $\mathfrak{C}_{C,D}$ is a complete metric space, by Banach Fixed Point Theorem, T as a unique fixed point $(x, \varphi) \in \mathfrak{C}_{C,D}$ that satisfies (5.37), (5.40) and (5.41). By Lemma (5.2) the fixed point (x, φ) satisfies (5.37), (5.38) and (5.39) and this proves that (3.22) holds.

To prove (3.23) we use (3.14), (3.16), (5.31) and (5.28) to get

$$\begin{aligned}
&\left\| \Psi_{m-n} \left(n, \xi, \varphi_n(\xi)\right) - \Psi_{m-n} \left(n, \xi, \varphi_n(\bar{\xi})\right) \right\| \\
&\leqslant \left\| (m-n, x_{m,n}(\xi), \varphi_m(x_{m,n}(\xi))) - \left(m-n, x_{m,n}(\bar{\xi}), \varphi_m(x_{m,n}(\bar{\xi}))\right) \right\| \\
&\leqslant \left\| x_{m,n}(\xi) - x_{m,n}(\bar{\xi}) \right\| + \left\| \varphi_m(x_{m,n}(\xi)) - \varphi_m(x_{m,n}(\bar{\xi})) \right\| \\
&\leqslant (1+D) \left\| x_{m,n}(\xi) - x_{m,n}(\bar{\xi}) \right\| \\
&\leqslant C(1+D)\alpha^{\circ}_{m,n} \left\| \xi - \bar{\xi} \right\| \\
&= \frac{D}{\omega} \alpha^{\circ}_{m,n} \left\| \xi - \bar{\xi} \right\|
\end{aligned}$$

for every $m, n \in \mathbb{Z}$ and every $\xi \in E^{\circ}_n$ and the theorem is proved. □

Acknowledgements

This work was partially supported by Fundação para a Ciência e Tecnologia though Centro de Matemática e Aplicações da Universidade da Beira Interior (CMA-UBI, project UID/MAT/00212/2020.

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