CARPATHIAN J. MATH.
Volume **39** (2023), No. 1,
Pages 01 - 26

Online version at `https://semnul.com/carpathian/`
Print Edition: ISSN 1584 - 2851; Online Edition: ISSN 1843 - 4401
DOI: https://doi.org/10.37193/CJM.2023.01.01

In memoriam Professor Charles E. Chidume (1947- 2021)

Relaxed modified Tseng algorithm for solving variational inclusion problems in real Banach spaces with applications

ABUBAKAR ADAMU[1,2], POOM KUMAM[2,3*], DUANGKAMON KITKUAN[4] and ANANTACHAI PADCHAROEN[4]

ABSTRACT. In this paper, relaxed and relaxed inertial modified Tseng algorithms for approximating zeros of sum of two monotone operators whose zeros are fixed points or J-fixed points of some nonexpansive-type mappings are introduced and studied. Strong convergence theorems are proved in the setting of real Banach spaces that are uniformly smooth and 2-uniformly convex. Furthermore, applications of the theorems to the concept of J-fixed point, convex minimization, image restoration and signal recovery problems are also presented. In addition, some interesting numerical implementations of our proposed methods in solving image recovery and compressed sensing problems are presented. Finally, the performance of our proposed methods are compared with that of some existing methods in the literature.

1. INTRODUCTION

Let H be a real Hilbert space. Let $A : H \to H$ and $B : H \to 2^H$ be single valued and multi-valued operators, respectively. The variational inclusion problem (VIP) which is to

$$\text{find} \quad u \in H \quad \text{with} \quad 0 \in (A+B)u, \tag{1.1}$$

has attracted the interest of many authors over the years due to its numerous applications in solving problems arising from image restoration, signal recovery and machine learning.

Assuming existence of solution, one of the classical technique for approximating solutions of the VIP (1.1) involving maximal monotone operators A and B in the setting of real Hilbert spaces, is the forward-backward algorithm (FBA); which was introduced independently by Lions [34] and Passty [42] and studied extensively by many authors (see, e.g.; [21], [5], [22]). The FBA is an iterative procedure that starts at a point $x_1 \in H$ and generates iteratively a sequence $\{x_n\} \subset H$ by solving the recursive equation:

$$x_{n+1} = \big(I + \lambda_n B\big)^{-1}\big(I - \lambda_n A\big)x_n, \tag{1.2}$$

where $\{\lambda_n\}$ is a sequence of positive real numbers. Lions [34] proved that if the operator A is *α-inverse strongly monotone*, that is, there exists $\alpha > 0$ such that

$$\langle x - y, Ax - Ay\rangle \geq \alpha\|Ax - Ay\|^2, \ \forall x, y \in H,$$

and $\liminf \lambda_n > 0$ with $\limsup \lambda_n < 2\alpha$, then the sequence generated by (1.2) converges weakly to a solution of problem (1.1).

Several modifications of the FBA have been proposed by many authors using the idea of Halpern-type or viscosity-type approximation technique to obtain strong convergence of

Received: 25.10.2021. In revised form: 18.06.2022. Accepted: 24.06.2022
2010 *Mathematics Subject Classification.* 49M20, 47J25, 47H05, 47H09, 47H10, 47J25, 47J05, 47J20.
Key words and phrases. *relaxed, Tseng algorithm, zeros, J-fixed point, image restoration, signal recovery.*
Corresponding author: P. Kumam; poom.kum@kmutt.ac.th

the sequence generated by the modified versions of the FBAs to a solution of the VIP (1.1) (see, e.g.; [48], [3] [30],[29]).

Remark 1.1. It is worthy of mentioning that virtually *all* the modifications of the FBA require the operator A to be α-*inverse strongly monotone* (for the case of Hilbert spaces) or α-*inverse strongly accretive* (for the case of Banach spaces). As rightly noted by Tseng [50] this restrictions rules out some important applications, see, section 4 of [50].

To dispense with the α-inverse strong monotonicity assumption on A, using the idea of the extragradient method of Korpelevic [31] for monotone variational inequalities, Tseng [50] introduced the following algorithm in real Hilbert spaces:

$$\begin{cases} x_1 \in C; \\ y_n = (I + \lambda_n B)^{-1}(I - \lambda_n A)x_n; \\ x_{n+1} = P_C\big(y_n - \lambda_n(Ay_n - Ax_n)\big); \end{cases} \tag{1.3}$$

where $C \subset H$ is nonempty closed and convex such that $C \cap (A+B)^{-1}0 \neq \emptyset$, A is maximal monotone and Lipschitz continuous with constant $L > 0$ and B is maximal monotone. He proved weak convergence of the sequence generated by his algorithm to a solution of problem (1.1).

Remark 1.2. We remark here that the class of monotone operators that are Lipschitz continuous contain, properly, the class of monotone operators that are α-inverse strongly monotone, since every α-inverse strongly monotone operator is $\frac{1}{\alpha}$-Lipschitz continuous.

In the literature, for the special case when $A \equiv 0$, the *proximal point algorithm* (PPA) has been employed to solve the inclusion problem (1.1). Several acceleration strategies of the PPA via inertial extrapolation or relaxation has been employed by many authors to improve the performance of the PPA over the years (see, e.g.; [16], [18]). The following question naturally becomes of interest:

Question. Can the acceleration strategies of the PPA be employed for the FBA and its modified versions?

Recently, in 2021 Padcharoen et al. [41] proposed an inertial Tseng's-type algorithm for solving the inclusion problem (1.1) in the setting of real Hilbert spaces. They proved the following theorem:

Theorem 1.1. *Let H be a real Hilbert space. Let $A : H \to H$ be an L-Lipschitz continuous and monotone mapping and $B : H \to 2^H$ be a maximal monotone map. Assume that the solution set $(A + B)^{-1}0 \neq \emptyset$. Given $x_0, x_1 \in H$, let $\{x_n\}$ be a sequence defined by:*

$$\begin{cases} w_n = x_n + \alpha_n(x_n - x_{n-1}) \\ y_n = (I + \lambda_n B)^{-1}(I - \lambda_n A)w_n \\ x_{n+1} = y_n - \lambda_n(Ay_n - Aw_n), \end{cases} \tag{1.4}$$

where the control parameters satisfy some appropriate conditions. Then the sequence $\{x_n\}$ generated by (1.4) converges weakly to a solution of problem (1.1).

Just recently, as it has been done for the PPA, the relaxed and relaxed inertial versions of the Tseng's algorithm were introduced and studied by Cholamjiak et al. [24] in the setting of real Hilbert spaces. They proved the following Theorems:

Theorem 1.2 (Relaxed Tseng's Algorithm). *Let H be a real Hilbert space and let $A : H \to H$ be monotone and Lipschitz continuous and $B : H \to 2^H$ be maximal monotone. Suppose the solution set of the VIP (1.1) $(A+B)^{-1}0$ is nonempty. Let $x_1 \in H$, let $\{x_n\}$ be a sequence generated by*

$$
\begin{cases} y_n = (I + \lambda_n B)^{-1}(x_n - \lambda_n A x_n), \\ x_{n+1} = (1-\theta_n)x_n + \theta_n y_n + \theta_n \lambda_n (Ax_n - Ay_n), \\ \lambda_{n+1} = \min\left\{\lambda_n, \frac{\mu_n \|x_n - y_n\|}{\|Ax_n - By_n\|}\right\}, \end{cases} \tag{1.5}
$$

where $\lambda_1 > 0$, $\{\theta_n\} \subset [a,b] \subset (0,1)$, $\{\mu_n\} \subset [c,d] \subset (0,1)$. Then the sequence generated by (1.5) converges weakly to a solution of the VIP (1.1).

Theorem 1.3 (Relaxed Inertial Tseng's Algorithm). *Under the same hypothesis as in Theorem 1.2 above, given $x_0, x_1 \in H$, let $\{x_n\}$ be a sequence generated by*

$$
\begin{cases} w_n = x_n + \alpha(x_n - x_{n-1}), \\ y_n = (I + \lambda_n B)^{-1}(w_n - \lambda_n A w_n), \\ x_{n+1} = (1-\theta_n)w_n + \theta_n y_n + \theta_n \lambda_n (Aw_n - Ay_n), \\ \lambda_{n+1} = \min\left\{\lambda_n, \frac{\mu_n \|w_n - y_n\|}{\|Aw_n - By_n\|}\right\}, \end{cases} \tag{1.6}
$$

where $\lambda_0 > 0$, $\theta \in (0,1]$, $\mu \in (0,1)$, $\alpha \in [0,1)$ such that

$$
\frac{\theta(1-\mu^2)}{(2-\theta+\mu\theta)^2} + \frac{1-\theta}{\theta} > \frac{\alpha(1+\alpha)}{(1-\alpha)^2}.
$$

Then the sequence $\{x_n\}$ converges weakly to a solution of the VIP (1.1).

Remark 1.3. Interested readers may see, for example, any of the following papers for a motivation about relaxation of an algorithm [9], [26], [1].

Extension of the VIP (1.1) to Banach spaces more general than Hilbert spaces is currently of interest to many authors (see, e.g.; [44], [12], [4], [28] and the references there in). In 2019, Shehu [46] extended the Theorem of Tseng [50] to real Banach spaces. He proved the following theorem:

Theorem 1.4. *Let E be a uniformly smooth and 2-uniformly convex real Banach space. Let $A : E \to E^*$ be a monotone and L-Lipschitz continuous mapping and $B : E \to 2^{E^*}$ be a maximal monotone mapping. Suppose the solution set $(A+B)^{-1}0 \neq \emptyset$. Let $\{x_n\}$ be a sequence defined by:*

$$
\begin{cases} x_1 \in E, \\ y_n = (J + \lambda)^{-1}(Jx_n - \lambda_n A x_n), \\ w_n = J^{-1}(Jy_n - \lambda_n(Ay_n - Ax_n)), \\ x_{n+1} = J^{-1}(\alpha_n J x_1 + (1-\alpha_n)Jw_n), \end{cases} \tag{1.7}
$$

where the control parameters satisfy some appropriate conditions. Then the sequence $\{x_n\}$ generated by (1.7) converges strongly to a solution of problem (1.1).

Remark 1.4. We remark here that in all the Theorems of Tseng [50], Padcharoen et al. [41] and Cholamjiak et al. [24] *weak* convergence Theorems were established which is *not* desirable in applications.

Now, having said all this, our interest is in the following generalization of the VIP (1.1):

$$
\text{find } u \in E \text{ such that } 0 \in (A+B)u \text{ and } Tu = u; \tag{1.8}
$$

where T is a nonexpansive-type operator. Problem (1.8) was studied by Takahashi et al. [47] in the setting of real Hilbert spaces. Recently, problem (1.8) is making some waves in the literature (see, e.g.; [15], [44], [23], [2]).

Motivated by remark 1.4 and the growing interest in problem (1.8), it is our purpose in this paper to resolve the concern raised in remark 1.4 and contribute our quota to the study of problem (1.8) by introducing relaxed and relaxed inertial modified Tseng algorithms in Banach spaces that are 2-uniformly convex and uniformly smooth that will converge *strongly* solutions of problem (1.8). Furthermore, applications of our theorems to J-fixed points, convex minimization, image recovery and compressed sensing problems will be presented. Finally, we compare the performance of our proposed methods with existing methods in solving image recovery and compressed sensing problems.

2. Preliminaries

Let E be a real normed space and let $J : E \to 2^{E^*}$ be the *normalized duality map* (see, e.g.; [6] for the explicit definition of J and its properties on certain Banach spaces). The following functional $\phi : E \times E \to \mathbb{R}$ defined on a smooth real Banach space by

$$\phi(x, y) := \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \ \forall\, x, y \in E, \tag{2.9}$$

will be needed in our estimations in the sequel. For any $x, y, z \in E$ and $\tau \in [0, 1]$ using the definition of ϕ, one can easily deduce the following (see, e.g.; Nilsrakoo and Saejung, [39]):

- D1: $(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2$,
- D2: $\phi(x, J^{-1}(\tau Jy + (1-\tau)Jz) \leq \tau\phi(x, y) + (1-\tau)\phi(x, z)$,
- D3: $\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle z - x, Jy - Jz\rangle$,

where J and J^{-1} are the duality maps on E and E^*.

Definition 2.1. A mapping $S : C \to E$ is called *relatively nonexpansive* if the set of its *asymptotic fixed points* equals the set of its fixed points and $\phi_p(\varpi, S\nu) \leq \phi_p(\varpi, \nu)$ for any $\varpi \in F(S)$ and $\nu \in C$.

Definition 2.2. Let E be a smooth, strictly convex and reflexive real Banach space and let C be a nonempty, closed and convex subset of E. Following Alber [6], the *generalized projection map*, $\Pi_C : E \to C$ is defined by

$$\Pi_C(u) = \underset{v\in C}{inf}\,\phi(v, u), \ \forall\, u \in E.$$

Clearly, in a real Hilbert space, the generalized projection Π_C coincides with the metric projection P_C from E onto C.

Definition 2.3. Let E be a reflexive, strictly convex and smooth real Banach space and let $B : E \to 2^{E^*}$ be a maximal monotone operator. Then for any $\lambda > 0$ and $u \in E$, there exists a unique element $u_\lambda \in E$ such that $Ju \in (Ju_\lambda + \lambda Bu_\lambda)$. The element u_λ is called the resolvent of B and it is denoted by $J^B_\lambda u$. Alternatively, $J^B_\lambda = \big(J + \lambda B)^{-1}J$, for all $\lambda > 0$. It is easy to verify that $B^{-1} = F(J^B_\lambda)$, $\forall\, \lambda > 0$, where $F(J^B_\lambda)$ denotes the set of fixed points of J^B_λ.

Lemma 2.1 ([7]). *Let C be a nonempty closed and convex subset of a smooth, strictly convex and reflexive real Banach space E. For any $x \in E$ and $y \in C$, $\tilde{x} = \Pi_C x$ if and only if $\langle \tilde{x} - y, Jx - J\tilde{x}\rangle \geq 0$, for all $y \in C$.*

Lemma 2.2 ([6]). *Let E be a reflexive strictly convex and smooth Banach space with E^* as its dual. Then,*

$$V(u, u^*) + 2\langle J^{-1}u^* - u, v^*\rangle \leq V(u, u^* + v^*), \tag{2.10}$$

for all $u \in E$ and $u^*, v^* \in E^*$, where $V(x, x^*) = \phi(x, J^{-1}x^*) = \|x\|^2 - 2\langle x, x^*\rangle + \|x^*\|^2$.

Lemma 2.3 ([10]). *Let E be a reflexive Banach space. Let $A : E \to E^*$ be a monotone, hemicontinuous and bounded mapping. Let $B : E \to 2^{E^*}$ be a maximal monotone mapping. Then $A + B$ is a maximal monotone mapping.*

Lemma 2.4 ([53]). *Let E be a 2-uniformly smooth real Banach space. Then, there exists a constant $\rho > 0$ such that $\forall\, x, y \in E$*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, Jx\rangle + \rho\|y\|^2.$$

In a real Hilbert space, $\rho = 1$.

Lemma 2.5 ([51]). *Let E be a 2-uniformly convex and smooth real Banach space. Then, there exists a positive constant μ such that*

$$\mu\|x - y\|^2 \leq \phi(x, y),\ \forall\, x, y \in E. \tag{2.11}$$

Lemma 2.6 ([27]). *Let E be a uniformly convex and smooth real Banach space, and let $\{u_n\}$ and $\{v_n\}$ be two sequences of E. If either $\{u_n\}$ or $\{v_n\}$ is bounded and $\phi(u_n, v_n) \to 0$ then $\|u_n - v_n\| \to 0$.*

Lemma 2.7 ([39]). *Let E be a uniformly smooth Banach space and $r > 0$. Then, there exists a continuous, strictly increasing, and convex function $g : [0, 2r] \to [0, 1)$ such that $g(0) = 0$ and*

$$\phi\left(u, J^{-1}[\beta Jx + (1 - \beta)Jy]\right) \leq \beta\phi(u, x) + (1 - \beta)\phi(u, y) - \beta(1 - \beta)g(\|Jx - Jy\|)$$

for all $\beta \in [0, 1],\ u \in E$ and $x, y \in B_r$.

Lemma 2.8 ([52]). *Let $\{a_n\}$ be a sequence of nonnegative numbers satisfying the condition*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n\beta_n + c_n,\ n \geq 0,$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{c_n\}$ are sequences of real numbers such that

$$(i)\ \{\alpha_n\} \subset [0, 1] \text{ and } \sum_{n=0}^{\infty} \alpha_n = \infty;\ (ii)\ \limsup_{n\to\infty} \beta_n \leq 0;\ (iii)\ c_n \geq 0,\ \sum_{n=0}^{\infty} c_n < \infty.$$

Then, $\lim\limits_{n\to\infty} a_n = 0$.

Lemma 2.9 ([37]). *Let Γ_n be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence $\{\Gamma_{n_j}\}_{j\geq 0}$ of $\{\Gamma_n\}$ which satisfies $\Gamma_{n_j} < \Gamma_{n_j+1}$ for all $j \geq 0$. Also, consider the sequence of integers $\{\tau(n)\}_{n\geq n_0}$ defined by*

$$\tau(n) = max\{k \leq n \mid \Gamma_k < \Gamma_{k+1}\}.$$

Then $\{\tau(n)\}_{n\geq n_0}$ is a nondecreasing sequence verifying $\lim_{n\to\infty} \tau(n) = \infty$ and, for all $n \geq n_0$, it holds that $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ and we have

$$\Gamma_n \leq \Gamma_{\tau(n)+1}.$$

Lemma 2.10 ([8]). *Let $\{\Gamma_n\}$,$\{\delta_n\}$ and $\{\alpha_n\}$ be sequences in $[0, \infty)$ such that*

$$\Gamma_{n+1} \leq \Gamma_n + \alpha_n(\Gamma_n - \Gamma_{n-1}) + \delta_n,$$

for all $n \geq 1$, $\sum_{n=1}^{\infty} \delta_n < +\infty$ and there exists a real number α with $0 \leq \alpha_n \leq \alpha < 1$, for all $n \in \mathbb{N}$. Then the following hold:
(i)$\sum_{n\geq 1}[\Gamma_n - \Gamma_{n-1}]_+ < +\infty$, where $[t]_+ = \max\{t, 0\}$;
(ii) there exists $\Gamma^* \in [0, \infty)$ such that $\lim_{n\to\infty} \Gamma_n = \Gamma^*$.

3. Main result

3.1. **Relaxed Halpern Tseng-type Algorithm.**

Theorem 3.5. *Let E be a 2-uniformly convex and uniformly smooth real Banach space with dual space, E^*. Let $A : E \to E^*$ be a monotone and L-Lipschitz continuous mapping, $B : E \to 2^{E^*}$ be a maximal monotone mapping and $T : E \to E$ be a relatively nonexpansive mapping. Assume the solution set $\Omega = (A+B)^{-1}0 \cap F(T) \neq \emptyset$, given $x_1 \in E$, let $\{x_n\}$ be a sequence defined by:*

$$\begin{cases} y_n = J^B_{\lambda_n} J^{-1}\big(Jx_n - \lambda_n Ax_n\big), \\ z_n = J^{-1}\big(Jy_n - \lambda_n(Ay_n - Ax_n)\big), \\ u_n = J^{-1}\big(\beta_n Jz_n + (1-\beta_n)JTz_n\big), \\ x_{n+1} = J^{-1}\big((1-\theta_n)Jx_n + \theta_n(\gamma_n Ju + (1-\gamma_n)Ju_n)\big), \end{cases} \tag{3.12}$$

where $J^B_{\lambda_n} = (J+\lambda_n B)^{-1}J$, $\{\theta_n\}, \{\beta_n\} \subset (0,1]$, $\{\gamma_n\} \subset (0,1)$ such that $\lim_{n\to\infty} \gamma_n = 0$ and $\sum_{n=1}^{\infty} \gamma_n = \infty$ and $\{\lambda_n\} \subset \big(\lambda, \frac{\sqrt{\mu}}{\sqrt{\rho}L}\big)$, $\lambda \in (0,1)$, ρ and μ are the constants appearing in Lemmas 2.4 and 2.5, respectively. Then, $\{x_n\}$ converges strongly to $x \in \Omega$.

Proof. Frist, we show that $\{x_n\}$ is bounded. Let $x \in \Omega$. Using Lemma 2.4 and D3, we have

$$\begin{aligned} \phi(x, z_n) &= \phi\big(x, J^{-1}(Jy_n - \lambda_n(Ay_n - Ax_n))\big) \\ &= \|x\|^2 - 2\langle x, Jy_n - \lambda_n(Ay_n - Ax_n)\rangle + \|Jy_n - \lambda_n(Ay_n - Ax_n)\|^2 \\ &\le \phi(x, y_n) - 2\lambda_n\langle y_n - x, Ay_n - Ax_n\rangle + \rho\|\lambda_n(Ay_n - Ax_n)\|^2 \\ &= \phi(x, x_n) + \phi(x_n, y_n) + 2\langle x_n - x, Jy_n - Jx_n\rangle \\ &\quad - 2\lambda_n\langle y_n - x, Ay_n - Ax_n\rangle + \rho\|\lambda_n(Ay_n - Ax_n)\|^2 \\ &= \phi(x, x_n) + \phi(x_n, y_n) - 2\langle y_n - x_n, Jy_n - Jx_n\rangle + 2\langle y_n - x, Jy_n - Jx_n\rangle \\ &\quad - 2\lambda_n\langle y_n - x, Ay_n - Ax_n\rangle + \rho\|\lambda_n(Ay_n - Ax_n)\|^2 \\ &= \phi(x, x_n) - \phi(y_n, x_n) - 2\langle y_n - x, Jx_n - Jy_n - \lambda_n(Ax_n - Ay_n)\rangle \\ &\quad + \rho\|\lambda_n(Ay_n - Ax_n)\|^2. \end{aligned} \tag{3.13}$$

Claim.

$$\langle y_n - x, Jx_n - Jy_n - \lambda_n(Ax_n - Ay_n)\rangle \ge 0. \tag{3.14}$$

Proof of claim. Observe that $y_n = J^B_{\lambda_n} J^{-1}\big(Jx_n - \lambda_n Ax_n\big)$ implies $(Jx_n - \lambda_n Ax_n) \in (Jy_n + \lambda_n By_n)$. Since B is maximal monotone, there exists $b_n \in By_n$ such that $Jx_n - \lambda_n Ax_n = Jy_n + \lambda_n b_n$. Thus,

$$b_n = \frac{1}{\lambda_n}(Jx_n - Jy_n - \lambda_n Ax_n). \tag{3.15}$$

Furthermore, since $0 \in (A+B)x$ and $(Ay_n + b_n) \in (A+B)y_n$, by monotonicity of $(A+B)$

$$\langle y_n - x, Ay_n + b_n\rangle \ge 0.$$

Substituting equation (3.15) into this inequality, we get

$$\langle y_n - x, Jx_n - Jy_n - \lambda_n(Ax_n - Ay_n)\rangle \ge 0,$$

which justifies our claim.

Now, substituting inequality (3.14) in (3.13) and using Lemma 2.5, we deduce that

$$\begin{aligned}\phi(x,z_n) &\le \phi(x,x_n)-\phi(y_n,x_n)+\rho\lambda_n^2\|Ax_n-Ay_n\|^2\\ &\le \phi(x,x_n)-\Big(1-\frac{\rho\lambda_n^2L^2}{\mu}\Big)\phi(y_n,x_n). \qquad (3.16)\end{aligned}$$

Since $\lambda_n\in\Big(0,\frac{\sqrt{\mu}}{\sqrt{\rho}L}\Big)$, $1-\frac{\rho\lambda_n^2L^2}{\mu}>0$. Thus,

$$\phi(x,z_n)\le\phi(x,x_n). \qquad (3.17)$$

Also, using D2 and the fact that T is relatively nonexpansive, we have

$$\begin{aligned}\phi(x,u_n) &\le \beta_n\phi(x,z_n)+(1-\beta_n)\phi(x,Tz_n)\\ &\le \beta_n\phi(x,z_n)+(1-\beta_n)\phi(z,z_n)=\phi(x,z_n). \qquad (3.18)\end{aligned}$$

Now, using D2 and, inequalities (3.18) and (3.17), we get

$$\begin{aligned}\phi(x,x_{n+1}) &= \phi\big(x,J^{-1}((1-\theta_n)Jx_n+\theta_n(\gamma_nJu+(1-\gamma_n)Ju_n))\big)\\ &\le (1-\theta_n)\phi(x,x_n)+\theta_n\gamma_n\phi(x,u)+\theta_n(1-\gamma_n)\phi(x,u_n)\\ &\le (1-\theta_n)\phi(x,x_n)+\theta_n\gamma_n\phi(x,u)+\theta_n(1-\gamma_n)\phi(x,z_n)\\ &\le (1-\theta_n)\phi(x,x_n)+\theta_n\gamma_n\phi(x,u)+\theta_n(1-\gamma_n)\phi(x,x_n)\\ &= (1-\theta_n\gamma_n)\phi(x,x_n)+\theta_n\gamma_n\phi(x,u)\\ &\le \max\{\phi(x,x_n),\phi(x,u)\}. \qquad (3.19)\end{aligned}$$

Thus, $\{\phi(x,x_n)\}$ is bounded. By D1, $\{x_n\}$ is bounded. Furthermore, $\{y_n\}$, $\{z_n\}$ and $\{u_n\}$ are bounded.

Next, we prove that $\{x_n\}$ converges to $x\in\Omega$. To achieve this, first of all we estimate $\phi(x,u_n)$ using Lemma 2.7, the fact that T is relatively nonexpansive and, inequalities (3.16) and (3.17) to get

$$\begin{aligned}\phi(x,u_n) &\le \beta_n\phi(x,z_n)+(1-\beta_n)\phi(x,Tz_n)-\beta_n(1-\beta_n)g(\|Jz_n-JTz_n\|)\\ &\le \beta_n\phi(x,z_n)+(1-\beta_n)\phi(x,z_n)-\beta_n(1-\beta_n)g(\|Jz_n-JTz_n\|)\\ &\le \beta_n\phi(x,z_n)+(1-\beta_n)\Big[\phi(x,x_n)-\Big(1-\frac{\rho\lambda_n^2L^2}{\mu}\Big)\phi(y_n,x_n)\Big]\\ &\quad -\beta_n(1-\beta_n)g(\|Jz_n-JTz_n\|)\\ &\le \phi(x,x_n)-(1-\beta_n)\Big(1-\frac{\rho\lambda_n^2L^2}{\mu}\Big)\phi(y_n,x_n)-\beta_n(1-\beta_n)g(\|Jz_n-JTz_n\|). \qquad (3.20)\end{aligned}$$

Next, we estimate $\phi(x,x_{n+1})$ using Lemma 2.7 and inequality (3.20), to get

$$\begin{aligned}\phi(x,x_{n+1}) &\le (1-\theta_n)\phi(x,x_n)+\theta_n\gamma_n\phi(x,u)+\theta_n(1-\gamma_n)\phi(x,u_n)\\ &\le (1-\theta_n)\phi(x,x_n)+\theta_n\gamma_n\phi(x,u)+\theta_n(1-\gamma_n)\Big[\phi(x,x_n)\\ &\quad -(1-\beta_n)\Big(1-\frac{\rho\lambda_n^2L^2}{\mu}\Big)\phi(y_n,x_n)-\beta_n(1-\beta_n)g(\|Jz_n-JTz_n\|)\Big]\\ &= (1-\theta_n\gamma_n)\phi(x,x_n)+\theta_n\gamma_n\phi(x,u)-\theta_n(1-\gamma_n)(1-\beta_n)\Big(1-\frac{\rho\lambda_n^2L^2}{\mu}\Big)\phi(y_n,x_n)\\ &\quad -\theta_n(1-\gamma_n)\beta_n(1-\beta_n)g(\|Jz_n-JTz_n\|). \qquad (3.21)\end{aligned}$$

Set $\eta_n = \theta_n(1-\gamma_n)(1-\beta_n)\Big(1 - \frac{\rho\lambda_n^2 L^2}{\mu}\Big)$ and $\zeta_n = \theta_n(1-\gamma_n)\beta_n(1-\beta_n)$, we deduce from inequality (3.21) that

$$\eta_n\phi(y_n,x_n) + \zeta_n g(\|Jz_n - JTz_n\|) \le \theta_n\gamma_n\big(\phi(x,u) - \phi(x,x_n)\big) + \phi(x,x_n) - \phi(x,x_{n+1}). \tag{3.22}$$

To complete the proof, we consider the following two cases:

Case 1. Assume there exits an $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$,

$$\phi(x,x_{n+1}) \le \phi(x,x_n),\ \forall n \ge n_0.$$

Then, $\{\phi(x,x_n)\}$ is convergent.

From inequality (3.22), using the fact that $\lim\limits_{n\to\infty}\gamma_n = 0$, the boundedness $\{x_n\}$ and the existence of $\lim\limits_{n\to\infty}\phi(x,x_n)$ we obtain the following:

$$\lim_{n\to\infty}\phi(y_n,x_n) = 0 \quad \text{and} \quad \lim_{n\to\infty} g(\|Jz_n - JTz_n\|) = 0.$$

This implies by Lemma 2.6 and the properties of g that

$$\lim_{n\to\infty}\|y_n - x_n\| = 0 \quad \text{and} \quad \lim_{n\to\infty}\|Jz_n - JTz_n\| = 0. \tag{3.23}$$

By the uniform continuity of J on bounded sets, this implies $\lim\limits_{n\to\infty}\|Jx_n - Jy_n\| = 0$. Also, the Lipschitz continuity of A and equation (3.23) imply that $\lim\limits_{n\to\infty}\|Ax_n - Ay_n\| = 0$. Therefore,

$$\lim_{n\to\infty}\|Jz_n - Jy_n\| = \lim_{n\to\infty}\lambda_n\|Ax_n - Ay_n\| = 0. \tag{3.24}$$

By the uniform continuity of J^{-1}, equation (3.24) implies that $\lim\limits_{n\to\infty}\|z_n - y_n\| = 0$. Thus,

$$\lim_{n\to\infty}\|x_n - z_n\| = 0. \tag{3.25}$$

Observe that

$$\begin{aligned}\|Jx_n - Jx_{n+1}\| &= \|\theta_n Jx_n - \theta_n\gamma_n Ju - \theta_n(1-\gamma_n)Ju_n\|\\ &\le \theta_n\|Jx_n - Jz_n\| + \theta_n\gamma_n\|Ju - Ju_n\| + \theta_n\|Jz_n - Ju_n\|\\ &= \theta_n\|Jx_n - Jz_n\| + \theta_n\gamma_n\|Ju - Ju_n\| + \theta_n(1-\beta_n)\|Jz_n - JTz_n\|.\end{aligned}$$

This implies that

$$\lim_{n\to\infty}\|Jx_{n+1} - Jx_n\| = 0. \tag{3.26}$$

Now, we prove that $\Omega_w(x_n) \subset \Omega$, where $\Omega_w(x_n)$ denotes the set of weak subsequential limits of $\{x_n\}$. Since $\{x_n\}$ is bounded, $\Omega_w(x_n) \ne \emptyset$. Let $x^* \in \Omega_w(x_n)$. Then, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x^*$. From equation (3.25), we have $z_{n_k} \rightharpoonup x^*$. Furthermore, from (3.23), the uniform continuity of J^{-1} on bounded sets implies $\|z_n - Tz_n\| \to 0$ and since $(I-T)$ is demiclosed at zero, $x^* \in F(T)$.

Next, we show that $x^* \in (A+B)^{-1}0$. Let $(v,w) \in G(A+B)$. Then, $(w - Av) \in Bv$. By the definition of y_n in (3.12), we have that $(J - \lambda_{n_k}A)x_{n_k} \in (J + \lambda_{n_k}B)y_{n_k}$. Thus, $\frac{1}{\lambda_{n_k}}(Jx_{n_k} - Jy_{n_k} - \lambda_{n_k}Ax_{n_k}) \in By_{n_k}$. By the monotonicity of B, we have

$$\langle v - y_{n_k}, w - Av - \frac{1}{\lambda_{n_k}}(Jx_{n_k} - Jy_{n_k} - \lambda_{n_k}Ax_{n_k})\rangle \ge 0.$$

Using the fact that A is monotone, we estimate this as follows

$$\begin{aligned}
\langle v - y_{n_k}, w\rangle &\geq \langle v - y_{n_k}, Av + \frac{1}{\lambda_{n_k}}(Jx_{n_k} - Jy_{n_k} - \lambda_{n_k} Ax_{n_k})\rangle \\
&= \langle v - y_{n_k}, Av - Ax_{n_k}\rangle + \frac{1}{\lambda_{n_k}}\langle v - y_{n_k}, Jx_{n_k} - Jy_{n_k}\rangle \\
&= \langle v - y_{n_k}, Av - Ay_{n_k}\rangle + \langle v - y_{n_k}, Ay_{n_k} - Ax_{n_k}\rangle + \frac{1}{\lambda_{n_k}}\langle v - y_{n_k}, Jx_{n_k} - Jy_{n_k}\rangle \\
&\geq \langle v - y_{n_k}, Ay_{n_k} - Ax_{n_k}\rangle + \frac{1}{\lambda_{n_k}}\langle v - y_{n_k}, Jx_{n_k} - Jy_{n_k}\rangle.
\end{aligned}$$

Since $\lim\limits_{n\to\infty}\|Ax_n - Ay_n\| = \lim\limits_{n\to\infty}\|Jy_n - Jx_n\| = 0$, $\left\{\frac{1}{\lambda_n}\right\}$ is bounded and $y_{n_k} \rightharpoonup x^*$, it follows that

$$\langle v - x^*, w\rangle \geq 0.$$

By Lemma 2.3, $A + B$ is maximal monotone. This implies that $0 \in (A + B)x^*$, i.e.; $x^* \in (A + B)^{-1}0$. Hence, $x^* \in \Omega = F(T) \cap (A + B)^{-1}0$.

Now, we have all the tools to show that $\{x_n\}$ converges to $x = \Pi_\Omega u$. Since $\{x_n\}$ is bounded, there exits a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that

$$\limsup_{n\to\infty}\langle x_n - x, Ju - Jx\rangle = \lim_{k\to\infty}\langle x_{n_k} - x, Ju - Jx\rangle = \langle x^* - x, Ju - Jx\rangle \leq 0.$$

From (3.26) we also have

$$\limsup_{n\to\infty}\langle x_{n+1} - x, Ju - Jx\rangle \leq 0.$$

Next, using Lemma 2.2, D2 and inequalities (3.18) and (3.17) we have

$$\begin{aligned}
\phi(x, x_{n+1}) &= \phi\big(x, J^{-1}((1-\theta_n)x_n + \theta_n(\gamma_n Ju + (1-\gamma_n)Ju_n))\big) \\
&= V\big(x, (1-\theta_n)x_n + \theta_n(\gamma_n Ju + (1-\gamma_n)Ju_n)\big) \\
&\leq V\big(x, (1-\theta_n)x_n + \theta_n(\gamma_n Ju + (1-\gamma_n)Ju_n) - \theta_n\gamma_n(Ju - Jx)\big) \\
&\quad + 2\theta_n\gamma_n\langle x_{n+1} - x, Ju - Jx\rangle \\
&= V(x, (1-\theta_n)x_n + \theta_n(\gamma_n Jx + (1-\gamma_n)Ju_n)) + 2\theta_n\gamma_n\langle x_{n+1} - x, Ju - Jx\rangle \\
&= \phi\big(x, J^{-1}\big((1-\theta_n)x_n + \theta_n(\gamma_n Jx + (1-\gamma_n)Ju_n)\big)\big) \\
&\quad + 2\theta_n\gamma_n\langle x_{n+1} - x, Ju - Jx\rangle \\
&\leq (1-\theta_n)\phi(x, x_n) + \theta_n\big(\gamma_n\phi(x, x) + (1-\gamma_n)\phi(x, u_n)\big) \\
&\quad + 2\theta_n\gamma_n\langle x_{n+1} - x, Ju - Jx\rangle \\
&\leq (1-\theta_n\gamma_n)\phi(x, x_n) + 2\theta_n\gamma_n\langle x_{n+1} - x, Ju - Jx\rangle. \qquad (3.27)
\end{aligned}$$

By Lemma 2.8, inequality (3.27) implies that $\lim\limits_{n\to\infty}\phi(x, x_n) = 0$. Using Lemma 2.6, we obtain that $\lim\limits_{n\to\infty} x_n = x$.

Case 2. If Case 1 does not hold, then, there exists a subsequence $\{x_{m_j}\} \subset \{x_n\}$ such that

$$\phi(x, x_{m_j+1}) > \phi(x, x_{m_j}),\ \forall j \in \mathbb{N}.$$

By Lemma 2.9, there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$, such that $\lim_{k\to\infty} m_k = \infty$ and the following inequalities hold

$$\phi(x, x_{m_k}) \leq \phi(x, x_{m_k+1}) \quad \text{and} \quad \phi(x, x_k) \leq \phi(x, x_{m_k}),\ \forall\, k \in \mathbb{N}.$$

From inequality (3.22) we have

$$\begin{aligned}\eta_{m_k}\phi(y_{m_k}, x_{m_k}) + \zeta_{m_k} g(\|Jz_{m_k} - JTz_{m_k}\|) &\leq \theta_{m_k}\gamma_{m_k}\big(\phi(x,u) - \phi(x, x_{m_k})\big) \\ &\quad + \phi(x, x_{m_k}) - \phi(x, x_{m_k+1}) \\ &\leq \theta_{m_k}\gamma_{m_k}\big(\phi(x,u) - \phi(x, x_{m_k})\big).\end{aligned}$$

Following a similar argument as in Case 1, one can establish the following

$$\lim_{k\to\infty} \|y_{m_k} - x_{m_k}\| = 0 \quad \text{and} \quad \lim_{k\to\infty} \|Jz_{m_k} - JTz_{m_k}\| = 0,$$

$$\lim_{k\to\infty} \|x_{m_k+1} - x_{m_k}\| = 0 \quad \text{and} \quad \limsup_{k\to\infty}\langle x_{m_k+1} - x, Ju - Jx\rangle \leq 0.$$

From (3.27) we have

$$\phi(x, x_{m_k+1}) \leq (1 - \theta_{m_k}\gamma_{m_k})\phi(x, x_{m_k}) + 2\theta_{m_k}\gamma_{m_k}\langle x_{m_k+1} - x, Ju - Jx\rangle. \tag{3.28}$$

By Lemma 2.8, inequality (3.28) implies that $\lim_{n\to\infty} \phi(x, x_{m_k}) = 0$. Thus,

$$\limsup_{k\to\infty} \phi(x, x_k) \leq \lim_{k\to\infty} \phi(x, x_{m_k}) = 0.$$

Therefore, $\limsup_{k\to\infty} \phi(x, x_k) = 0$ and so, by Lemma 2.6, $\lim_{k\to\infty} x_k = x$. This completes the proof. □

3.2. **Relaxed Inertial Halpern Tseng-type Algorithm.**

Theorem 3.6. *Under the same setting as in Theorem 3.5, given $x_0, x_1 \in E$, let $\{x_n\}$ be a sequence defined by:*

$$\begin{cases} w_n = J^{-1}\big(Jx_n + \alpha_n(Jx_n - Jx_{n-1})\big), \\ y_n = J^B_{\lambda_n} J^{-1}\big(Jw_n - \lambda_n Aw_n\big), \\ z_n = J^{-1}\big(Jy_n - \lambda_n(Ay_n - Aw_n)\big), \\ u_n = J^{-1}\big(\beta_n Jz_n + (1-\beta_n)JTz_n\big), \\ x_{n+1} = J^{-1}\big((1-\theta_n)Jw_n + \theta_n(\gamma_n Ju + (1-\gamma_n)Ju_n)\big), \end{cases} \tag{3.29}$$

where $0 < \alpha_n \leq \bar{\alpha_n}$ and $\bar{\alpha_n} = \begin{cases} \min\left\{\alpha, \frac{\epsilon_n}{\|Jx_n - Jx_{n-1}\|^2}, \frac{\epsilon_n}{\phi(x_n, x_{n-1})}\right\}, & x_n \neq x_{n-1}, \\ \alpha, & \textit{otherwise}, \end{cases}$

$\alpha \in (0,1)$ and $\{\epsilon_n\} \subset (0,1)$ such that $\sum_{n=1}^{\infty} \epsilon_n < \infty$. The remaining parameters are the same as in Theorem 3.5. Then, $\{x_n\}$ converges strongly to $x \in \Omega$.

Proof. First, we prove that $\{x_n\}$ is bounded. Let $x \in \Omega$. Following the same argument by replacing x_n with w_n, from (3.16) and (3.17), we get

$$\phi(x, z_n) \leq \phi(x, w_n) - \Big(1 - \frac{\rho\lambda_n^2 L^2}{\mu}\Big)\phi(y_n, w_n). \tag{3.30}$$

Since $\lambda_n \in \Big(0, \frac{\sqrt{\mu}}{\sqrt{\rho}L}\Big)$, $1 - \frac{\rho\lambda_n^2 L^2}{\mu} > 0$. Thus,

$$\phi(x, z_n) \leq \phi(x, w_n). \tag{3.31}$$

Similarly, from (3.18), we have

$$\phi(x, u_n) \leq \phi(x, z_n). \tag{3.32}$$

Now, using D3, we obtain that

$$\begin{aligned}\phi(x,w_n) &= \phi(x,x_n)+\phi(x_n,w_n)+2\langle x_n-x, Jw_n-Jx_n\rangle\\ &= \phi(x,x_n)+\phi(x_n,w_n)+2\alpha_n\langle x_n-x, Jx_n-Jx_{n-1}\rangle \quad (3.33)\\ &= \phi(x,x_n)+\phi(x_n,w_n)+\alpha_n\phi(x_n,x_{n-1})+\alpha_n\phi(x,x_n)-\alpha_n\phi(x,x_{n-1}). \quad (3.34)\end{aligned}$$

Moreover, using the definition of ϕ and Lemma 2.4, we have

$$\begin{aligned}\phi(x,w_n) &= \phi\big(x, J^{-1}(Jx_n+\alpha_n(Jx_n-Jx_{n-1}))\big)\\ &= \|x\|^2+\|Jx_n+\alpha_n(Jx_n-Jx_{n-1})\|^2-2\langle x, Jx_n+\alpha_n(Jx_n-Jx_{n-1})\rangle\\ &= \|x\|^2+\|Jx_n+\alpha_n(Jx_n-Jx_{n-1})\|^2-2\langle x, Jx_n\rangle-2\alpha_n\langle x, Jx_n-Jx_{n-1}\rangle\\ &\le \phi(x,x_n)+\rho\alpha_n^2\|Jx_n-Jx_{n-1}\|^2+2\alpha_n\langle x_n-x, Jx_n-Jx_{n-1}\rangle. \quad (3.35)\end{aligned}$$

Combining (3.33) and (3.35), we have

$$\phi(x_n,w_n)\le \rho\alpha_n^2\|Jx_n-Jx_{n-1}\|^2.$$

Hence, substituting this in equation (3.34), we have

$$\begin{aligned}\phi(x,w_n)\le\ &\phi(x,x_n)+\rho\alpha_n^2\|Jx_n-Jx_{n-1}\|^2+\alpha_n\phi(x_n,x_{n-1})\\ &+\alpha_n\big(\phi(x,x_n)-\phi(x,x_{n-1})\big). \quad (3.36)\end{aligned}$$

Now, using D2, inequalities (3.32), (3.31) and (3.36), and the fact that $\{\alpha_n\}\subset(0,1)$, we get

$$\begin{aligned}\phi(x,x_{n+1}) &= \phi\big(x, J^{-1}((1-\theta_n)Jw_n+\theta_n(\gamma_n Ju+(1-\gamma_n)u_n))\big)\\ &\le (1-\theta_n)\phi(x,w_n)+\theta_n\gamma_n\phi(x,u)+\theta_n(1-\gamma_n)\phi(x,w_n)\\ &= (1-\theta_n\gamma_n)\phi(x,w_n)+\theta_n\gamma_n\phi(x,u)\\ &\le (1-\theta_n\gamma_n)\Big(\phi(x,x_n)+\alpha_n\big(\phi(x,x_n)-\phi(x,x_{n-1})\big)+\alpha_n\phi(x_n,x_{n-1})\\ &\qquad +\rho\alpha_n^2\|Jx_n-Jx_{n-1}\|^2\Big)+\theta_n\gamma_n\phi(x,u)\\ &\le \max\Big\{\phi(x,u), \phi(x,x_n)+\alpha_n\big(\phi(x,x_n)-\phi(x,x_{n-1})\big)\\ &\qquad +\rho\alpha_n\|Jx_n-Jx_{n-1}\|^2+\alpha_n\phi(x_n,x_{n-1})\Big\}. \quad (3.37)\end{aligned}$$

If the maximum is $\phi(x,u)$, then $\{\phi(x,x_n)\}$ is bounded. By D1, $\{x_n\}$ is bounded. Else, there exists an $n_0\ge 1$ such that for all $n\ge n_0$, we have that

$$\phi(x,x_{n+1})\le\phi(x,x_n)+\alpha_n\big(\phi(x,x_n)-\phi(x,x_{n-1})\big)+\rho\alpha_n\|Jx_n-Jx_{n-1}\|^2+\alpha_n\phi(x_n,x_{n-1}).$$

Since $\rho\alpha_n\|Jx_n-Jx_{n-1}\|^2\le\rho\epsilon_n$, $\alpha_n\phi(x_n,x_{n-1})\le\epsilon_n$ and $\sum_{n=1}^{\infty}\epsilon_n<\infty$, by Lemma 2.10, $\{\phi(x,x_n)\}$ is convergent and thus, bounded. Furthermore, by D1, $\{x_n\}$ is bounded. This implies that $\{w_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{u_n\}$ are bounded.

Next, we proof that $\{x_n\}$ converges strongly to $x\in\Omega$. From inequality (3.20), replacing x_n with w_n and using the same argument of proof, we have

(3.38)

$$\phi(x,u_n)\le\phi(x,w_n)-(1-\beta_n)\Big(1-\frac{\rho\lambda_n^2L^2}{\mu}\Big)\phi(y_n,w_n)-\beta_n(1-\beta_n)g(\|Jz_n-JTz_n\|).$$

Similarly, from (3.21), we have

$$\phi(x,x_{n+1}) \le (1-\theta_n\gamma_n)\phi(x,w_n) + \theta_n\gamma_n\phi(x,u) - \theta_n(1-\gamma_n)(1-\beta_n)\Big(1-\frac{\rho\lambda_n^2L^2}{\mu}\Big)\phi(y_n,w_n)$$
$$- \theta_n(1-\gamma_n)\beta_n(1-\beta_n)g(\|Jz_n - JTz_n\|). \tag{3.39}$$

Set $\eta_n = \theta_n(1-\gamma_n)(1-\beta_n)\Big(1-\frac{\rho\lambda_n^2L^2}{\mu}\Big)$ and $\zeta_n = \theta_n(1-\gamma_n)\beta_n(1-\beta_n)$. Using inequality (3.34), we deduce from inequality (3.39) that

$$\begin{aligned}\eta_n\phi(y_n,w_n) + \zeta_n g(\|Jz_n - JTz_n\|) &\le \theta_n\gamma_n\big(\phi(x,u)-\phi(x,w_n)\big) + \phi(x,w_n) - \phi(x,x_{n+1})\\ &\le \theta_n\gamma_n\big(\phi(x,u)-\phi(x,w_n)\big) + \phi(x,x_n)\\ &\quad + \rho\alpha_n\|Jx_n - Jx_{n-1}\|^2 + \alpha_n\phi(x_n,x_{n-1})\\ &\quad + \alpha_n\big(\phi(x,x_n)-\phi(x,x_{n-1})\big) - \phi(x,x_{n+1})\\ &= \theta_n\gamma_n\big(\phi(x,u)-\phi(x,w_n)\big) + \phi(x,x_n) - \phi(x,x_{n+1})\\ &\quad + \alpha_n\phi(x_n,x_{n-1})\\ &\quad + \rho\alpha_n\|Jx_n - Jx_{n-1}\|^2 + \alpha_n\big(\phi(x,x_n)-\phi(x,x_{n-1})\big).\end{aligned} \tag{3.40}$$

To complete the proof, we consider the following two cases:

Case 1. Assume there exits an $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$,

$$\phi(x,x_{n+1}) \le \phi(x,x_n),\ \forall n \ge n_0.$$

Then, $\{\phi(x,x_n)\}$ is convergent.

From inequality (3.40), using the fact that $\lim\limits_{n\to\infty}\gamma_n = 0$, the boundedness $\{w_n\}$, the existence of $\lim\limits_{n\to\infty}\phi(x,x_n)$, the fact that $\lim\limits_{n\to\infty}\rho\alpha_n\|Jx_n - Jx_{n-1}\|^2 = 0 = \lim\limits_{n\to\infty}\alpha_n\phi(x_n,x_{n-1})$, we obtain the following:

$$\lim_{n\to\infty}\phi(y_n,w_n) = 0 \quad \text{and} \quad \lim_{n\to\infty} g(\|Jz_n - JTz_n\|) = 0.$$

By Lemma 2.6 and the properties of g we get

$$\lim_{n\to\infty}\|y_n - w_n\| = 0 \quad \text{and} \quad \lim_{n\to\infty}\|Jz_n - JTz_n\| = 0. \tag{3.41}$$

Furthermore, since

$$\|Jx_n - Jw_n\| = \alpha_n\|Jx_n - Jx_{n-1}\|, \quad \lim_{n\to\infty}\|Jx_n - Jw_n\| = 0.$$

Moreover, by the uniform continuity of J^{-1} on bounded sets, $\lim\limits_{n\to\infty}\|x_n - w_n\| = 0$. This and equation (3.41) imply that $\lim\limits_{n\to\infty}\|x_n - y_n\| = 0$. By the uniform continuity of J on bounded sets, this implies $\lim\limits_{n\to\infty}\|Jx_n - Jy_n\| = 0$. Also, the Lipschitz continuity of A and equation (3.41) imply that
$\lim\limits_{n\to\infty}\|Aw_n - Ay_n\| = 0$. Therefore,

$$\lim_{n\to\infty}\|Jz_n - Jy_n\| = \lim_{n\to\infty}\lambda_n\|Aw_n - Ay_n\| = 0. \tag{3.42}$$

By the uniform continuity of J^{-1}, equation (3.42) implies that $\lim\limits_{n\to\infty}\|z_n - y_n\| = 0$. Thus,

$$\lim_{n\to\infty}\|x_n - z_n\| = 0. \tag{3.43}$$

Now, observe that

$$\begin{aligned}\|Jx_n - Jx_{n+1}\| &\leq \|Jx_n - Jw_n\| + \|\theta_n Jw_n - \theta_n\gamma_n Ju - \theta_n(1-\gamma_n)Ju_n\| \\ &\leq \|Jx_n - Jw_n\| + \theta_n\|Jw_n - Jz_n\| + \theta_n\gamma_n\|Ju - Ju_n\| + \theta_n\|Jz_n - Ju_n\| \\ &= \|Jx_n - Jw_n\| + \theta_n\|Jw_n - Jz_n\| + \theta_n\gamma_n\|Ju - Ju_n\| + \theta_n(1-\beta_n)\|Jz_n - JTz_n\|.\end{aligned}$$

This implies that

$$\lim_{n\to\infty}\|Jx_{n+1} - Jx_n\| = 0. \tag{3.44}$$

Next, the prove that $\Omega_w(x_n) \subset \Omega$ follows similarly as in the proof of Theorem 3.5.

Finally, we show that $\{x_n\}$ converges to $x = \Pi_\Omega u$. Since $\{x_n\}$ is bounded, there exits a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that

$$\limsup_{n\to\infty}\langle x_n - x, Ju - Jx\rangle = \lim_{k\to\infty}\langle x_{n_k} - x, Ju - Jx\rangle = \langle x^* - x, Ju - Jx\rangle \leq 0.$$

From (3.44) we also have

$$\limsup_{n\to\infty}\langle x_{n+1} - x, Ju - Jx\rangle \leq 0.$$

Next, using Lemma 2.2, D2 and inequalities (3.32), (3.31) and (3.36), we have

$$\begin{aligned}\phi(x, x_{n+1}) &= \phi\big(x, J^{-1}((1-\theta_n)w_n + \theta_n(\gamma_n Ju + (1-\gamma_n)Ju_n))\big) \\ &= V\big(x, (1-\theta_n)w_n + \theta_n(\gamma_n Ju + (1-\gamma_n)Ju_n)\big) \\ &\leq V\big(x, (1-\theta_n)w_n + \theta_n(\gamma_n Ju + (1-\gamma_n)Ju_n) - \theta_n\gamma_n(Ju - Jx)\big) \\ &\quad + 2\theta_n\gamma_n\langle x_{n+1} - x, Ju - Jx\rangle \\ &= \phi(x, J^{-1}((1-\theta_n)w_n + \theta_n(\gamma_n Jx + (1-\gamma_n)Ju_n))) \\ &\quad + 2\theta_n\gamma_n\langle x_{n+1} - x, Ju - Jx\rangle \\ &\leq (1-\theta_n)\phi(x, w_n) + \theta_n\gamma_n\phi(x, x) + \theta_n(1-\gamma_n)\phi(x, u_n) \\ &\quad + 2\theta_n\gamma_n\langle x_{n+1} - x, Ju - Jx\rangle \\ &\leq (1-\theta_n\gamma_n)\phi(x, w_n) + 2\theta_n\gamma_n\langle x_{n+1} - x, Ju - Jx\rangle \\ &\leq (1-\theta_n\gamma_n)\Big(\phi(x, x_n) + \rho\alpha_n^2\|Jx_n - Jx_{n-1}\|^2 + \alpha_n\phi(x_n, x_{n-1}) \\ &\quad + \alpha_n\big(\phi(x, x_n) - \phi(x, x_{n-1})\big)\Big) + 2\theta_n\gamma_n\langle x_{n+1} - x, Ju - Jx\rangle \qquad (3.45) \\ &\leq (1-\theta_n\gamma_n)\phi(x, x_n) + 2\theta_n\gamma_n\langle x_{n+1} - x, Ju - Jx\rangle + \rho\alpha_n\|Jx_n - Jx_{n-1}\|^2 \\ &\quad + \alpha_n\phi(x_n, x_{n-1}). \qquad (3.46)\end{aligned}$$

By Lemma 2.8, inequality (3.46) implies that $\lim_{n\to\infty}\phi(x, x_n) = 0$. Using Lemma 2.6, we obtain that $\lim_{n\to\infty} x_n = x$.

Case 2. If Case 1 does not hold, then, there exists a subsequence $\{x_{m_k}\} \subset \{x_n\}$ such that

$$\phi(x, x_{m_k+1}) > \phi(x, x_{m_k}),\ \forall k \in \mathbb{N}.$$

By Lemma 2.9, there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$, such that $\lim_{k\to\infty} m_k = \infty$ and the following inequalities hold

$$\phi(x, x_{m_k}) \leq \phi(x, x_{m_k+1}) \quad \text{and} \quad \phi(x, x_k) \leq \phi(x, x_{m_k}),\ \forall\, k \in \mathbb{N}.$$

From inequality (3.40) we have

$$\begin{aligned}\eta_{m_k}\phi(y_{m_k}, x_{m_k}) + \zeta_{m_k} g(\|Jz_{m_k} - JTz_{m_k}\|) &\leq \theta_{m_k}\gamma_{m_k}\big(\phi(x,u) - \phi(x, w_{m_k})\big)\\ &\quad + \phi(x, x_{m_k}) - \phi(x, x_{m_k+1})\\ &\quad + \alpha_{m_k}\phi(x_{m_k}, x_{m_k+1})\\ &\quad + \rho\alpha_{m_k}\|Jx_{m_k} - Jx_{m_k-1}\|^2\\ &\quad + \alpha_{m_k}\big(\phi(x, x_{m_k}) - \phi(x, x_{m_k-1})\big)\\ &\leq \theta_{m_k}\gamma_{m_k}\big(\phi(x,u) - \phi(x, w_{m_k})\big)\\ &\quad + \alpha_{m_k}\phi(x_{m_k}, x_{m_k+1})\\ &\quad + \rho\alpha_{m_k}\|Jx_{m_k} - Jx_{m_k-1}\|^2.\end{aligned}$$

Following a similar argument as in Case 1, one can establish the following

$$\lim_{k\to\infty}\|y_{m_k} - x_{m_k}\| = 0 \quad \text{and} \quad \lim_{k\to\infty}\|Jz_{m_k} - JTz_{m_k}\| = 0,$$

$$\lim_{k\to\infty}\|x_{m_k+1} - x_{m_k}\| = 0 \quad \text{and} \quad \limsup_{k\to\infty}\langle x_{m_k+1} - x, Ju - Jx\rangle \leq 0.$$

From (3.45) we have

$$\begin{aligned}\phi(x, x_{m_k+1}) &\leq (1 - \theta_{m_k}\gamma_{m_k})\Big(\phi(x, x_{m_k}) + \rho\alpha_{m_k}^2\|Jx_{m_k} - Jx_{m_k-1}\|^2 + \alpha_{m_k}\phi(x_{m_k}, x_{m_k-1})\\ &\quad + \alpha_{m_k}\big(\phi(x, x_{m_k}) - \phi(x, x_{m_k-1})\big)\Big) + 2\theta_{m_k}\gamma_{m_k}\langle x_{m_k+1} - x, Ju - Jx\rangle\\ &\leq (1 - \theta_{m_k}\gamma_{m_k})\phi(x, x_{m_k}) + 2\theta_{m_k}\gamma_{m_k}\langle x_{m_k+1} - x, Ju - Jx\rangle\\ &\quad + \rho\alpha_{m_k}\|Jx_{m_k} - Jx_{m_k-1}\|^2 + \alpha_{m_k}\phi(x_{m_k}, x_{m_k-1})\\ &\quad + \big(\phi(x, x_{m_k}) - \phi(x, x_{m_k-1})\big).\end{aligned} \tag{3.47}$$

By Lemma 2.8, inequality (3.47) implies that $\lim_{n\to\infty}\phi(x, x_{m_k}) = 0$. Thus,

$$\limsup_{k\to\infty}\phi(x, x_k) \leq \lim_{k\to\infty}\phi(x, x_{m_k}) = 0.$$

Therefore, $\limsup_{k\to\infty}\phi(x, x_k) = 0$ and so, by Lemma 2.6, $\lim_{k\to\infty} x_k = x$. This completes the proof. □

3.3. **Corollaries.** Setting $T = I$ in Theorems 3.5 and 3.6, where I is the identity map on E, we obtain the extended versions of the relaxed and relaxed inertial algorithms of Cholamjiak et al. [24] in Banach spaces.

Setting $\theta_n = 1$ and $T = I$ in Theorem 3.6, we obtain the inertial Halpern Tseng-type algorithm for approximating solutions of the VIP (1.1).

4. Applications and numerical illustrations

4.1. Application to J-fixed point.

Definition 4.4. Let $T : E \to 2^{E^*}$ be any map. A point $u \in E$ is called a *J-fixed point of* T if $Ju \in Tu$, where $J : E \to E^*$ is the single valued normalized duality map on E. We shall denote the set of J-fixed point of T by $F_J(T) := \{x \in E : \; Tx = Jx\}$.

This notion has been defined by Zegeye [54] who called it called *semi-fixed point*. Also, Liu [35] called it *duality fixed point*. In 2016, Chidume and Idu [17], coined the name J-fixed point. They gave motivations and interesting results concerning J-fixed point for maps from a space, say E, to its dual space E^*. An intriguing property of J-fixed point is its connection with optimization problems see, e.g.; [17] for the connection. Currently, there is a growing interest in the study of J-fixed point (see, e.g.; [14, 20, 40], for some interesting results concerning J-fixed point).

Recently, the notion of *relatively J-nonexpansive maps* was introduced and discussed by Chidume *et al.* [19] in a uniformly smooth and uniformly convex real Banach spaces. They gave the following definitions:

Definition 4.5. Let $T : E \to E^*$ be a map. A point $x^* \in E$ is called an *asymptotic J-fixed point of* T if there exists a sequence $\{x_n\} \subset E$ such that $x_n \rightharpoonup x^*$ and $\|Jx_n - Tx_n\| \to 0$, as $n \to \infty$. Let $\widehat{F}_J(T)$ be the set of asymptotic J-fixed points of T.

Definition 4.6. A map $T : E \to E^*$ is said to be *relatively J-nonexpansive* if

(i) $\widehat{F}_J(T) = F_J(T) \neq \emptyset$,
(ii) $\phi(u, J^{-1}Tx) \leq \phi(u, x), \; \forall\, x \in E, \; u \in F_J(T)$.

Remark 4.5. See Chidume *et al.* [19] for a nontrivial example of relatively J-nonexpansive mapping. One can easily verify from the definition above, that if an operator T is relatively J-nonexpansive then the operator $J^{-1}T$ is relatively nonexpansive in the usual sense and vice versa. Furthermore, $x^* \in F_J(T) \Leftrightarrow x^* \in F(J^{-1}T)$.

We shall explore this notion of relatively J-nonexpansive mapping and introduce a hybrid algorithm for approximating solutions of the inclusion problem (1.1) which are also J-fixed points of relatively J-nonexpansive mapping.

Theorem 4.7. *Let E be a 2-uniformly convex and uniformly smooth real Banach space with dual space, E^*. Let $A : E \to E^*$ be a monotone and L-Lipschitz continuous mapping, $B : E \to 2^{E^*}$ be a maximal monotone mapping and $T : E \to E^*$ be a relatively J-nonexpansive mapping. Assume the solution set $\Omega = (A + B)^{-1}0 \cap F_J(T) \neq \emptyset$, given $x_1 \in E$, let $\{x_n\}$ be a sequence defined by:*

$$
\begin{cases}
y_n = J^B_{\lambda_n} J^{-1}\big(Jx_n - \lambda_n Ax_n\big), \\
z_n = J^{-1}\big(Jy_n - \lambda_n(Ay_n - Ax_n)\big), \\
u_n = J^{-1}\big(\beta_n Jz_n + (1 - \beta_n)Tz_n\big), \\
x_{n+1} = J^{-1}\big((1 - \theta_n)Jx_n + \theta_n(\gamma_n Ju + (1 - \gamma_n)Ju_n)\big),
\end{cases}
\tag{4.48}
$$

where $\{\theta_n\} \subset (0, 1]$ $\{\beta_n\} \subset (0, 1)$, $\{\gamma_n\} \subset (0, 1)$ such that $\lim\limits_{n\to\infty} \gamma_n = 0$ and $\sum\limits_{n=1}^{\infty} \gamma_n = \infty$ and $\{\lambda_n\} \subset \big(\lambda, \frac{\sqrt{\mu}}{\sqrt{\rho}L}\big)$, where $\lambda \in (0, 1)$, ρ and μ are the constants appearing in Lemmas 2.4 and 2.5, respectively. Then, $\{x_n\}$ converges strongly to $x \in \Omega$.

Proof. By Remark 4.5, $J^{-1}T$ is relatively nonexpansive. The conclusion follows from Theorem 3.5. □

Theorem 4.8. *Under the same setting as in Theorem 4.7, given $x_0, x_1 \in E$, let $\{x_n\}$ be a sequence defined by:*

$$\begin{cases} w_n = J^{-1}\big(Jx_n + \alpha_n(Jx_n - Jx_{n-1})\big), \\ y_n = J^B_{\lambda_n}J^{-1}\big(Jw_n - \lambda_n Aw_n\big), \\ z_n = J^{-1}\big(Jy_n - \lambda_n(Ay_n - Aw_n)\big), \\ u_n = J^{-1}\big(\beta_n Jz_n + (1-\beta_n)Tz_n\big), \\ x_{n+1} = J^{-1}\big((1-\theta_n)Jw_n + \theta_n(\gamma_n Ju + (1-\gamma_n)Ju_n)\big), \end{cases} \tag{4.49}$$

where the parameters are the same as in Theorem 4.8. Then, $\{x_n\}$ converges strongly to $x \in \Omega$.

4.2. **Application to convex minimization problem.** In this section, we give application of our Theorems to the structured nonsmooth convex minimization problem:

$$\min_{x\in H}\{f(x) + g(x)\}, \tag{4.50}$$

where H is a real Hilbert space, $f : H \to \mathbb{R}\cup\{+\infty\}$ is proper, convex and lower semi-continuous (lsc) and $g : H \to \mathbb{R}$ is smooth and convex with gradient ∇g which is L-Lipschitz continuous. As we shall see in subsections 4.3 and 4.4 problem (4.50) is suitable for modeling problems coming from image deblurring and denoising, and compressed sensing.

Problem (4.50) can be recast as the following inclusion problem:

$$\text{find } \; u \in H \; \text{ such that } \; 0 \in (\partial f(u) + \nabla g(u)), \tag{4.51}$$

which the FBA (1.2) can be used to solve it. Just as in the case of arbitrary monotone operators, acceleration process has been an active topic of nonsmooth convex minimization. In the context of convex minimization, the inertial extrapolation technique of Ployak [43] has been employed as an acceleration process. A particular case of the inertial FBA introduced independently by Moudafi and Oliny [38] and, Lorenz and Pock [36] is the *fast iterative shrinkage-thresholding algorithm* (FISTA) developed by Beck and Teboulle [11] which has captured the interest of many authors. The algorithm is defined by:

$$\begin{cases} t_n = \frac{1+\sqrt{1+4t_{k-1}^2}}{2}, \; a_n = \frac{t_{n-1}-1}{t_n}, \\ y_n = x_n + a_n(x_n - x_{n-1}), \\ x_{n+1} = (I + \lambda\partial f)^{-1}(y_n - \lambda\nabla g(y_n)), \end{cases} \tag{4.52}$$

where $t_0 = 1,\ \lambda = \frac{1}{L},\ x_0 = x_1 \in H$ and, f and g are as defined in problem (4.50). Beck and Teboulle [11] proved weak convergence of the sequence generated by (4.52) in the setting of real Hilbert space H.

Remark 4.6. The literature on the modifications of the sequence $\{t_n\}$ in FISTA to take care of the oscillatory behavior of the scheme abound. Interested readers may see, for example [33].

Now, in our proposed algorithms in Theorems 3.5 and 3.6, setting $A = \nabla g$ and $B = \partial f$, one gets algorithms for solving problem (4.50).

4.3. **Application to image restoration problems.** Observed images are often distorted by the process of acquisition. The aim of image restoration techniques is to restore the original image from a noisy observation of it. Basically, there are two ways to go about image restoration problems: Model-based approach (see, e.g.; [25]) and learning based approach (see, e.g.; [55]). In this section, we are interested in the model based approach which are often formulated as optimization problems. Our model of interest is:

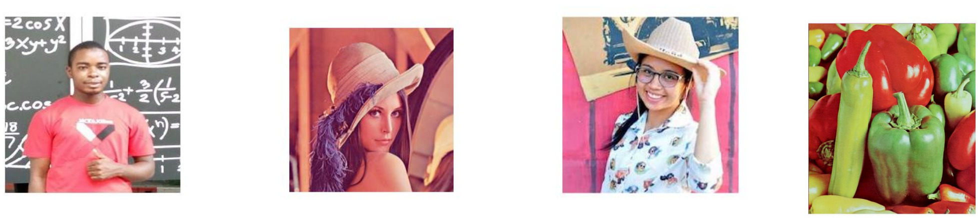
$$b = Lx + y,$$

where b is an observed image, x is an unknown image, y is a noise and L is a linear map. It is well known that regularization methods are used in image restoration problems. The l_1-regularization is a powerful tool in image denoising and is given by:

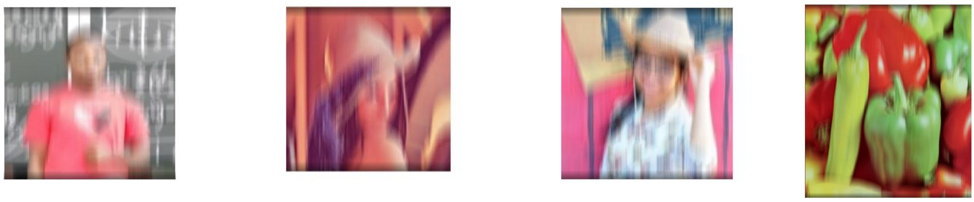
$$\min_x \frac{1}{2}\|Lx - b\|^2 + \lambda\|x\|_1, \tag{4.53}$$

where $\|\cdot\|$ denotes the Euclidean norm, λ is a positive regularization parameter and $\|\cdot\|_1$ is the l_1-regularization term. Set $Ax := \nabla\big(\frac{1}{2}\|Lx - b\|_2^2\big) = L^T(Lx - y)$ and $Bx := \partial(\lambda\|x\|_1)$. It is well-known that A is $\|L\|^2$-Lipschitz continuous and monotone (see, e.g.; [13]). Moreover, B is maximal monotone (see [45]).

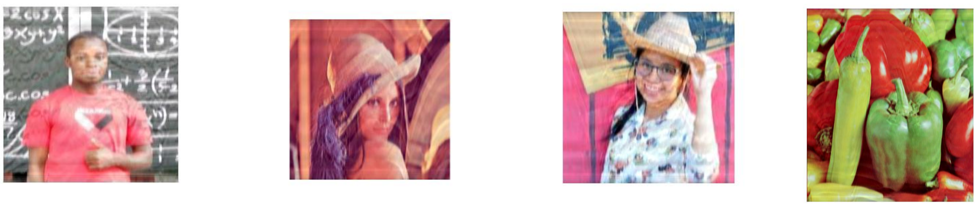
In our numerical simulations, we discretize the three layers of the colored test images (Red, Green and Blue) into 255 pixels and the test images were degraded using the MATLAB blur function "P=fspecial('motion',20,30)" and random noise. We initialize the vectors x_0 and u to be zeros and x_1 to be ones in R^N. In algorithms (1.5) and (1.6) of Cholamjiak *et al.* [24] we choose the same parameters as used in section 6 of their paper. In the FISTA (4.52) we choose $t_0 = 1$ and $\lambda = 0.03$ and in our proposed algorithms (3.12) and (3.29) we choose $\alpha = 0.95$, $\beta = 1$, $\gamma_n = \frac{1}{(n+1)^2}$ and $\theta_n = \frac{n+2}{n+1}$, and we set $Tx = \frac{n}{n+1}x$. Finally, we used a tolerance of 10^{-4} and maximum number of iterations (n) to be 200, for all the algorithms. The results are presented in Figures 1 and 2, and Table 1 below.



(A) original test images

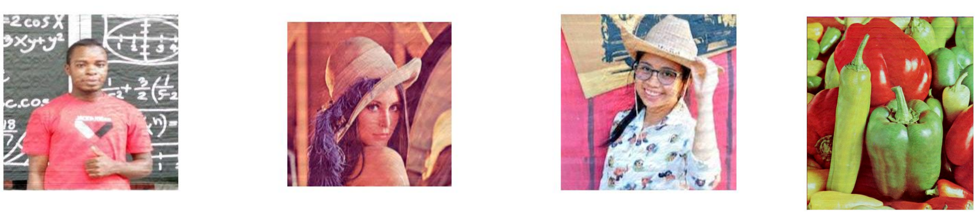


(B) test images degraded by motion blur and random noise



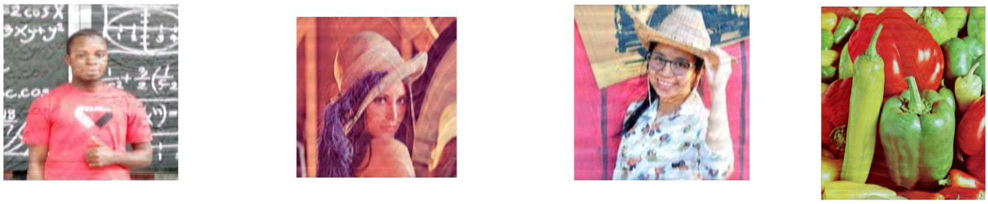
(C) restored images with algorithm (1.5)

(D) restored image with algorithm (1.6)

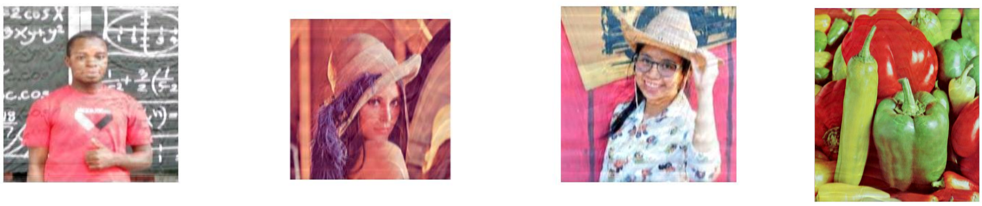


(E) restored images with algorithm (4.52)

FIGURE 1. Restoration process via algorithms (1.5), (1.6) and (4.52)6



(A) restored images with our algorithm (3.12)



(B) restored image with our algorithm (3.29)

FIGURE 2. Restoration process via algorithms (4.52), (3.12) and (3.29)

The signal to noise ratio (SNR) and the improvement in signal to noise ratio (ISNR) are performance metrics used to measure the performance algorithms in the restoration of degraded images. They are defined as:

$$\text{SNR} := 10 \log \frac{\|x\|^2}{\|x - x_n\|} \quad \text{and} \quad \text{ISNR} := 10 \log \frac{\|x - b\|^2}{\|x - x_n\|},$$

where x, b and x_n are the original, observed and estimated image at iteration n, respectively. Using these performance metrics the higher the SNR value or the ISNR value for a restored image, the better the restoration process via the algorithm.

TABLE 1. SNR and ISNR values of the images in Figure 1

Image	Algorithm (1.5)		Algorithm (1.6)		Algorithm (4.52)		Algorithm (3.12)		Algorithm (3.29)	
	SNR	ISNR	SNR	ISNR	SNR	ISNR	SNR	ISNR	SNR	ISNR
Abubakar	35.75	5.565	36.42	5.902	46.03	10.70	38.46	6.921	49.27	12.32
Barbra	43.40	6.388	44.01	6.687	50.39	9.881	45.99	7.681	53.88	11.62
Kitkuan	42.36	6.011	42.98	6.317	52.19	10.92	45.08	7.369	55.41	12.53
Peppers	46.72	6.995	47.35	7.307	55.66	11.46	49.37	8.319	56.58	11.92

4.4. **Application to Signal Processing.** In signal processing, compressed sensing can be inferred as a method of reconstructing a *sparse signal* from a measured data (see, e.g [32], [49]). Just like for the case of image restoration, to model this problem, the observed data

$y \in \mathbb{R}^M$ is related to the original signal $x \in \mathbb{R}^N$ by the equation equation

$$y = Ax + b, \tag{4.54}$$

where $A \in \mathbb{R}^{M\times N}$ is the measurement or sensing matrix and b is noise. We will always assume that $M < N$. Thus, the interest becomes how to recover the sparse signal $x \in \mathbb{R}^N$ by solving (4.54). Due to the nature of A, the system (4.54) is an undetermined system of linear equations which can be solve via the approach of (4.53).

In our simulations, the sparse vector $x \in \mathbb{R}^N$ with m nonzero entries is constructed from a uniform distribution on the interval $[-1, 1]$, the observation y is constructed using Gaussian noise distributed normally with mean 0 and variance 10^{-4} and the sensing matrix $A \in \mathbb{R}^{M\times N}$ is constructed from a normal distribution with mean zero and variance one. For algorithms (1.5) and (1.6) of Cholamjiak *et al.* [24] we choose the same parameters as used by the authors in their paper, for the FISTA (4.52) we choose $t_0 = 1$ and $\lambda = 0.3$ and in our proposed algorithms (3.12) and (3.29) we choose $\alpha = 0.93$, $\beta = 1$, $\gamma_n = \frac{1}{(n+1)^2}$ and $\theta_n = \frac{n+2}{n+1}$, and we set $Tx = \frac{n}{n+1}x$. Furthermore, we use the mean square error (MSE) defined by:

$$MSE = \frac{1}{N}\|x^* - x\|^2 < 10^{-5} \tag{4.55}$$

as stopping criterion, where x^* is an approximated signal of x.

We consider the following cases (N is the length of the original signal and M is the number of observations made):

Case 1 : $N = 4096,\ M = 2048$, and $m = 50$.
Case 2 : $N = 4096,\ M = 2048$, and $m = 100$.

The numerical results are presented in Figures 3, 4, 5, 6, 7 and 8.

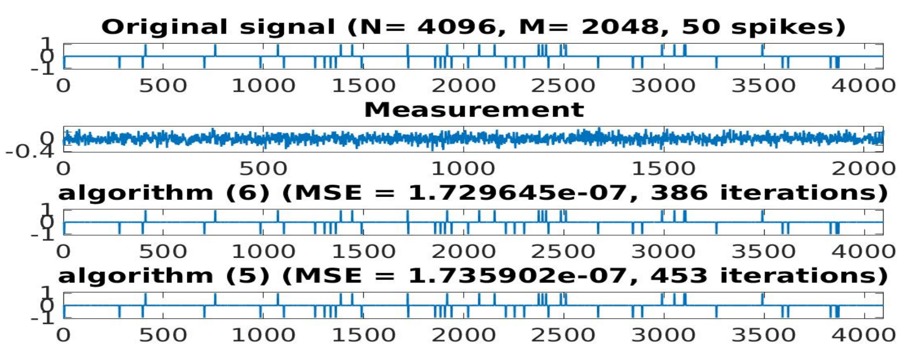


FIGURE 3. Restored signal via algorithms (1.5) and (1.6) of Cholamjiak *et al.* [24] for Case 1

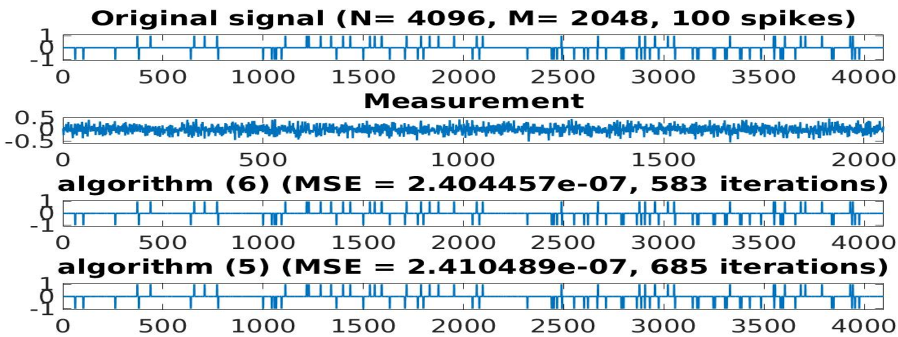


FIGURE 4. Restored signal via algorithms (1.5) and (1.6) of Cholamjiak *et al.* [24] for Case 2

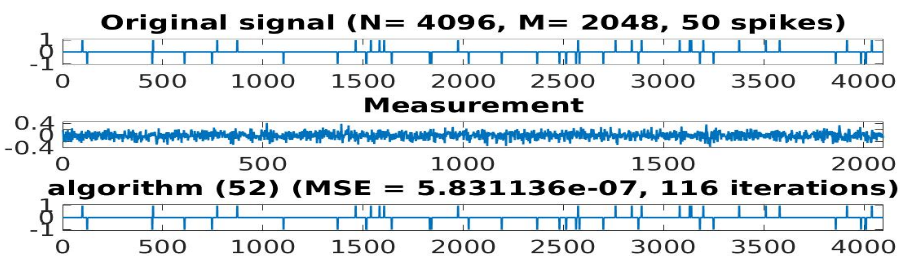


FIGURE 5. Restored signal via FISTA (4.52) shown for Case 1

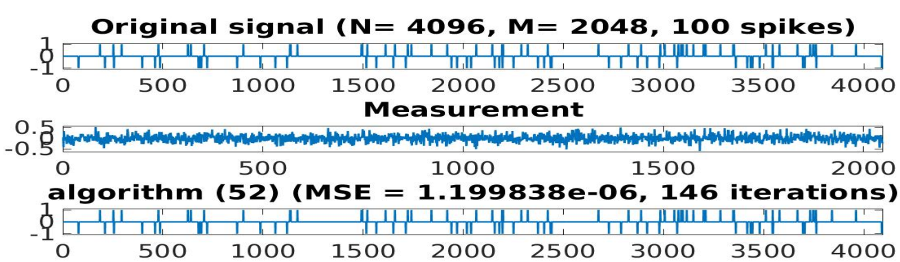


FIGURE 6. Restored signal via FISTA (4.52) for Case 2

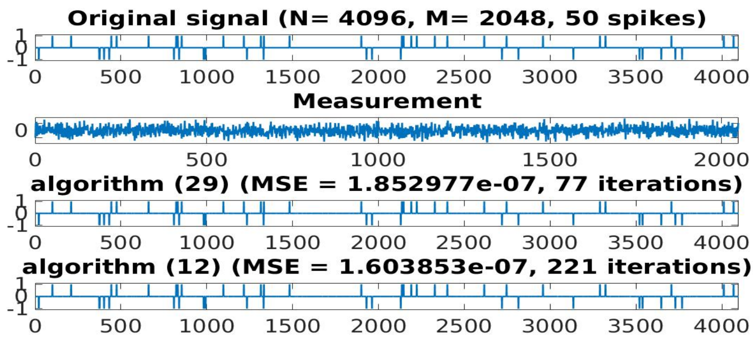


FIGURE 7. Restored signal via algorithms (3.29) and (3.12) shown for Case 1

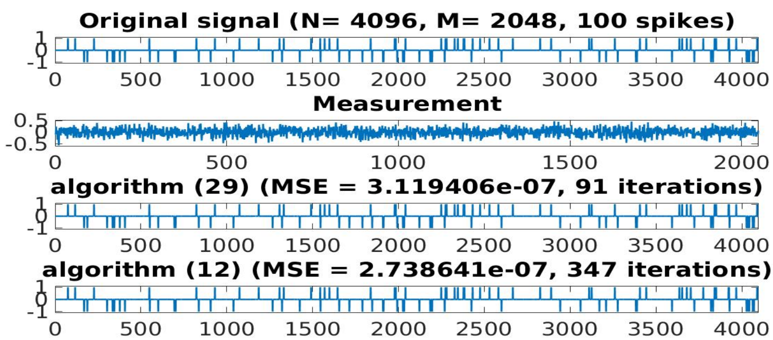


FIGURE 8. Restored signal via algorithms (3.29) and (3.12) shown for Case 2

4.5. **Discussion of the Numerical Simulations and Conclusion.**

Discussion.

- For the image restoration problem, while all the methods did not satisfy the stopping criterion before the maximum number of iterations was exhausted, the relaxed inertial version of our proposed algorithm, algorithm (3.29) has higher SNR and ISNR values than algorithms (1.5) and (1.6) of Cholamjiak et al. [24] and the FISTA (4.52), meaning, the quality of the restored images via our proposed algorithm (3.29) is better than that of algorithms (1.5), (1.6) and (4.52).
- For the compressed sensing problem, we observed that even as we vary the number of spikes (the number of nonzero entries) the relaxed version of our proposed method, algorithm (3.29) requires less number of iterations to restore the signal compared to algorithms (1.5), (1.6) and (4.52).

Conclusion. This paper presents a modified relaxed and relaxed inertial versions of Tseng's algorithm which approximate solutions of the inclusion problem (1.8). Strong convergence theorems and some interesting applications of the theorems to J-fixed points, image restoration and compressed sensing problems are presented. Finally, some interesting numerical simulations of our proposed method are presented and from the experiments, our proposed methods appear to competitive and promising.

5. DECLARATIONS

5.1. **Competing interest.** The authors declare that they have no conflict of interest.

5.2. **Funding.** This project was funded by National Research Council of Thailand (NRCT) under Research Grants for Talented Mid-Career Researchers (Contract no. N41A640089).

Acknowledgements. The first and second author acknowledge with thanks, the Center of Excellence in Theoretical and Computational Science (TaCS-CoE) for their financial support.

REFERENCES

[1] Abubakar, J.; Kumam, P.; Hassan Ibrahim, A.; Padcharoen, A. Relaxed inertial Tseng's type method for solving the inclusion problem with application to image restoration. *Mathematics* **8** (2020), no. 5, 818, https://doi.org/10.3390/math8050818

[2] Adamu, A.; Adam, A. A. Approximation of solutions of split equality fixed point problems with applications. *Carpathian J. Math.* **37** (2021), no. 3, 381–392.

[3] Adamu, A.; Deepho, J.; Ibrahim, A. H.; Abubakar, A. B. Approximation of zeros of sum of monotone mappings with applications to variational inequality problem and image processing. *Nonlinear Funct. Anal. Appl.* **26** (2021), no. 2, 411–432.

[4] Adamu, A.; Kitkuan, D.; Kumam, P.; Padcharoen, A.; Seangwattana, T. Approximation method for monotone inclusion problems in real Banach spaces with applications. *J. Inequal. Appl.* 2022, Paper No. 70, 20 pp.

[5] Adamu, A.; Kitkuan, D.; Padcharoen, A.; Chidume, C.; Kumam, P. Inertial viscosity-type iterative method for solving inclusion problems with applications. *Math. Comput. Simulation* 194 (2022), 445–459.

[6] Alber, Y.; Ryazantseva, I. *Nonlinear ill posed problems of monotone type.* Springer, Netherlands, 2006.

[7] Alber, Y. Metric and generalized projection operators in Banach spaces: properties and applications. *Theory and applications of nonlinear operators of accretive and monotone type*, 15–50, Lecture Notes in Pure and Appl. Math., 178, Dekker, New York, 1996.

[8] Alvarez, F. Weak convergence of a relaxed and inertial hybrid projection-proximal point algorithm for maximal monotone operators in Hilbert space. *SIAM J. Optim.* **14** (2004), no. 3, 773–782.

[9] Attouch, H.; Cabot, A. Convergence of a relaxed inertial proximal algorithm for maximally monotone operators. *Math. Program.* **184** (2020), no. 1, 243–287.

[10] Barbu, V. *Nonlinear semigroups and differential equations in Banach spaces.* Springer, Dordrecht, 1976.

[11] Beck, A.; Teboulle, M. A fast iterative shrinkage-thresholding algorithm for linear inverse problems. *SIAM J. Imaging Sci.* **2** (2009), no. 1, 183–202.

[12] Bello, A.; Okeke, C.; Isyaku, M.; Omojola, M. Forward-reflected-backward splitting method without cocoercivity for the sum of maximal monotone operators in Banach spaces. *Optimization* 1–22.

[13] Byrne, C. A unified treatment of some iterative algorithms in signal processing and image reconstruction. *Inverse Problems* **20** (2004), no. 1, 103–120.

[14] Cheng, Q.; Su, Y.; Zhang, J.; Duality fixed point and zero point theorems and applications. *Abstr. Appl. Anal.* volume 2012. Hindawi.

[15] Chidume, C.; Adamu, A.; Kumam, P.; Kitkuan, D. Generalized hybrid viscosity-type forward-backward splitting method with application to convex minimization and image restoration problems. *Numer. Funct. Anal. Optim.* (2021), 1–22.

[16] Chidume, C. E.; Adamu, A.; Nnakwe, M. O. Strong convergence of an inertial algorithm for maximal monotone inclusions with applications. *Fixed Point Theory Appl.* 2020, Paper No. 13, 22 pp.

[17] Chidume, C.; Idu, K. Approximation of zeros of bounded maximal monotone maps, solutions of Hammerstein integral equations and convex minimization problems. *Fixed Point Theory Appl.* 2016, 1–22.

[18] Chidume, C. E.; Adamu, A.; Nnakwe, M. O. An inertial algorithm for solving Hammerstein equations. *Symmetry* **13** (2021), no. 3, 376.

[19] Chidume, C. E.; Adamu, A.; Okereke, L. C. Strong convergence theorem for some nonexpansive-type mappings in certain Banach spaces. *Thai J. Math.* **18** (2020), no. 3, 1537–1548.

[20] Chidume, C. E.; Kumam, P.; Adamu, A. A hybrid inertial algorithm for approximating solution of convex feasibility problems with applications. *Fixed Point Theory Appl.* **2020**, Paper No. 12, 17 pp.

[21] Cholamjiak, P.; Kankam, K.; Srinet, P.; Pholasa, N. A double forward-backward algorithm using linesearches for minimization problem. *Thai J. Math.* **18** (2020), no. 1, 63–76.

[22] Cholamjiak, P.; Shehu, Y. Inertial forward-backward splitting method in Banach spaces with application to compressed sensing. *Appl. Math.* **64** (2019), no. 4, 409–435.

[23] Cholamjiak, P.; Sunthrayuth, P.; Singta, A.; Muangchoo, K. Iterative methods for solving the monotone inclusion problem and the fixed point problem in Banach spaces. *Thai J. Math.* **18** (2020), no. 3, 1225–1246.

[24] Cholamjiak, P.; Van Hieu, D.; Cho, Y. J. Relaxed forward-backward splitting methods for solving variational inclusions and applications. *J. Sci. Comput.* **88** (2021), no. 3, Paper No. 85, 23 pp.

[25] Elad, M.; Aharon, M. Image denoising via sparse and redundant representations over learned dictionaries. *IEEE Trans. Image Process.* **15** (2006), no. 12, 3736–3745.

[26] Iutzeler, F.; Hendrickx, J. M. A generic online acceleration scheme for optimization algorithms via relaxation and inertia. *Optim. Methods Softw.* **34** (2019), no. 2, 383–405.

[27] Kamimura, S.; Takahashi, W. Strong convergence of a proximal-type algorithm in a Banach space. *SIAM J. Optim.* **13** (2002), no. 3,938–945.

[28] Kimura, Y.; Nakajo, K. Strong convergence for a modified forward-backward splitting method in Banach spaces. *J. Nonlinear Var. Anal.* **3** (2019), no. 1, 5–18.

[29] Kitkuan, D.; Kumam, P.; and Martínez-Moreno, J. Generalized Halpern-type forward-backward splitting methods for convex minimization problems with application to image restoration problems. *Optimization* (2020), 1–25.

[30] Kitkuan, D.; Kumam, P.; Padcharoen, A.; Kumam, W.; and Thounthong, P. Algorithms for zeros of two accretive operators for solving convex minimization problems and its application to image restoration problems. *J. Comput. Appl. Math.* **354** (2019), 471–495.

[31] Korpelevich, G. M. The extragradient method for finding saddle points and other problems. *Matecon* **12** (2010), 747–756.

[32] Kutyniok, G. Theory and applications of compressed sensing. *GAMM-Mitteilungen* **36** (2013), no. 1, 79–101.

[33] Liang, J.; Luo, T.; Schönlieb, C.-B. Improving " fast iterative shrinkage-thresholding algorithm": Faster, smarter and greedier. *arXiv preprint arXiv:1811.01430*, 2018.

[34] Lions, P. L.; Mercier, B. Splitting algorithms for the sum of two nonlinear operators. *SIAM J. Numer. Anal.* **16** (1979), no. 6, 964–979.

[35] Liu, B. Fixed point of strong duality pseudocontractive mappings and applications. *Abstr. Appl. Anal.* vol. **2012**. Hindawi.

[36] Lorenz, D. A.; Pock, T. An inertial forward-backward algorithm for monotone inclusions. *J. Math. Imaging Vision* **51** (2015), no. 2, 311–325, 2015.

[37] Maingé, P.-E. The viscosity approximation process for quasi-nonexpansive mappings in Hilbert spaces. *Comput. Math. Appl.* **59** (2010), no. 1. 74–79.

[38] Moudafi, A.; Oliny, M. Convergence of a splitting inertial proximal method for monotone operators. *J. Comput. Appl. Math.* **155** (2003), no. 2, 447–454.

[39] Nilsrakoo, W.; Saejung, S. Strong convergence theorems by Halpern–mann iterations for relatively nonexpansive mappings in Banach spaces. *Appl. Math. Comput.* **217** (2011), no. 14, 6577–6586.

[40] Nnakwe, M. O.; Okeke, C. C. A common solution of generalized equilibrium problems and fixed points of pseudo-contractive-type maps. *J. Appl. Math. Comput.* **66** (2021), no. 1, 701–716.

[41] Padcharoen, A.; Kitkuan, D.; Kumam, W.; Kumam, P. Tseng methods with inertial for solving inclusion problems and application to image deblurring and image recovery problems. *Comput. Math. Methods* **3** (2021), no. 3, Paper No. e1088, 14 pp.

[42] Passty, G. B. Ergodic convergence to a zero of the sum of monotone operators in Hilbert space. *J. Math. Anal. Appl.* **72** (1979), no. 2, 383–390.

[43] Polyak, B. T. Some methods of speeding up the convergence of iteration methods. *USSR Comput. Math. Math. Phys.* **4** no. 5, 1–17.

[44] Qin, X.; Cho, S. Y.; Yao, J.-C. Weak and strong convergence of splitting algorithms in Banach spaces. *Optimization.* **69** (2020), no. 2, 243–267.

[45] Rockafellar, R. On the maximal monotonicity of subdifferential mappings. *Pacific J. Math.* **33** (1970), no. 1, 209–216.

[46] Shehu, Y. Convergence results of forward-backward algorithms for sum of monotone operators in Banach spaces. *Results Math.* **74** (2019), no. 4, 1–24.

[47] Takahashi, S.; Takahashi, W.; Toyoda, M. Strong convergence theorems for maximal monotone operators with nonlinear mappings in Hilbert spaces. *J. Optim. Theory Appl.* **147** (2010), no. 1, 27–41.

[48] Thong, D. V. and Cholamjiak, P. Strong convergence of a forward-backward splitting method with a new step size for solving monotone inclusions. *Comput. Appl. Math.* **38** (2019), no. 2, Paper No. 94, 16 pp.

[49] Tropp, J. A. A mathematical introduction to compressive sensing. *Bull. Amer. Math. Soc.* **54** (2017), no. 1, 151–165.

[50] Tseng, P. A modified forward-backward splitting method for maximal monotone mappings. *SIAM J. Control Optim.* **38** (2000), no. 2, 431–446.

[51] Xu, H.-K. Inequalities in banach spaces with applications. *Nonlinear Anal. Theory, Methods & Appl.* **16** (1991), no. 12, 1127–1138.

[52] Xu, H.-K. Iterative algorithms for nonlinear operators. *J. Lond. Math. Soc.* **66** (2002), no. 1, 240–256.

[53] Xu, Z.-B.; Roach, G. F. Characteristic inequalities of uniformly convex and uniformly smooth Banach spaces. *J. Math. Anal. Appl.* **157** (1991), no. 1, 189–210.

[54] Zegeye, H. Strong convergence theorems for maximal monotone mappings in Banach spaces. *J. Math. Anal. Appl.* **343** (2008), no. 2, 663–671.

[55] Zhang, K.; Zuo, W.; Gu, S.; Zhang, L. Learning deep cnn denoiser prior for image restoration. *Proc. IEEE Conf. Comput. Vision Pattern Recognit.* (2017), 3929–3938.

[1]Mathematics Institute, African University of Science and Technology, Abuja, Nigeria

[2]Center of Excellence in Theoretical and Computational Science (TaCS-CoE)
Science Laboratory Building, Faculty of Science
King Mongkut's University of Technology Thonburi (KMUTT), 126 Pracha-Uthit Road
Bang Mod, Thung Khru, Bangkok 10140, Thailand
Email address: aadamu@aust.edu.ng
Email address: poom.kum@kmutt.ac.th

[3]Department of Medical Research, China Medical University Hospital
China Medical University, Taichung 40402, Taiwan
Email address: poom.kum@kmutt.ac.th

[4] Department of Mathematics, Faculty of Science and Technology
Rambhai Barni Rajabhat University, Chanthaburi 22000, Thailand
Email address: duangkamon.k@rbru.ac.th and anantachai.p@rbru.ac.th