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In memoriam Professor Charles E. Chidume (1947- 2021)

Alternated inertial simultaneous and semi-alternating projection algorithms for solving the split equality problem

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ABSTRACT. In this paper, we introduce the simultaneous and semi-alternating projection algorithms for solving the split equality problem by using a new choice of the step size and combining the alternated inertial technique. The weak convergence of the proposed algorithms is analyzed under standard conditions. Finally, a numerical example is presented to illustrate the efficiency and advantage of the proposed algorithms by comparing with other methods..

1. INTRODUCTION

Let H_1, H_2 and H_3 be real Hilbert spaces, let $C \subseteq H_1$ and $Q \subseteq H_2$ be two nonempty closed convex sets, and let $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be two bounded linear operators. The split equality problem (SEP) introduced by Moudafi in [12] can be described as

$$\text{Find} \quad x \in C \text{ and } y \in Q, \quad \text{such that} \quad Ax = By. \tag{1.1}$$

The split equality problem (1.1) plays an important role in phase restoration, signal restoration, image reconstruction and medical fields. In addition, the split equality problem includes the convex feasibility problem and the split feasibility problem (see, e.g., [3]) as special cases.

Denote by

$$\Gamma = \{x \in C, y \in Q : Ax = By\}$$

the solution of the SEP (1.1). Throughout this paper, we assume that the SEP (1.1) is consistent, i.e., $\Gamma \neq \emptyset$.

To solve the SEP (1.1), Moudafi [12] firstly proposed an alternating CQ algorithm. Considering that the projections onto C and Q might be difficult to calculate, Moudafi [13] proposed a relaxed alternating CQ algorithm which replaces the projections onto C and Q with those onto the half-spaces. The step sizes in these algorithms are fixed and depend on the operator norms $\|A\|$ and $\|B\|$ which may be difficult or even impossible to obtain. Even if we know the norms of A and B, the iterative algorithms with the fixed step size are generally slow.

A number of literatures on algorithms for solving SEP have been published (see [6, 10, 17–19]). For example, in [16], inspired by iterative algorithms for solving a variational inequality, Tian et al. proposed several two-step methods and relaxed two-step methods. Moudafi [14] extended the alternating CQ algorithm to the nonconvex setting

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and introduced a generalized alternating CQ algorithm by using the linearization technique.

Recently by improving the step sizes in [4], the authors [8] introduced simultaneous and semi-alternating projection algorithms which have the excellent numerical performance comparing with those in [4]. Very recently, to accelerate the convergence of the projection method in [7], the authors [9] proposed an alternated inertial projection algorithm by using the alternated inertial extrapolation, whose inertial parameters don't involve the iterative sequence.

Motivated by above work, in this paper, we introduce the alternated inertial simultaneous and semi-alternating projection algorithms by using a self-adaptive step size and applying the alternated inertia in algorithms. The structure of the paper is as follows. In the next section, we present some concepts and lemmas which will be used in the main results. In Section 3, we will present algorithms and give their weak convergence analysis. In final section, some numerical results are provided, which show the advantages of the proposed algorithms.

2. Preliminaries

Let H be a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the induced norm $\|\cdot\|$. We write $x^k \rightharpoonup x$ to indicate that the sequence $\{x^k\}_{k\in\mathbb{N}}$ converges weakly to x and $x^k \to x$ to indicate that the sequence $\{x^k\}_{k\in\mathbb{N}}$ converges strongly to x. Given a sequence $\{x^k\}_{k\in\mathbb{N}}$, denote by $\omega_w(x^k)$ it's weak ω-limit set, that is, for any $x \in \omega_w(x^k)$ there exists a subsequence $\{x^{k_j}\}_{j\in\mathbb{N}}$ of $\{x^k\}_{k\in\mathbb{N}}$ which converges weakly to x.

In this paper, an important tool of our work is the projection. Let D be a nonempty, closed and convex subset of H. Recall that the `projection` from H onto D, denoted by P_D, is defined in such a way that, for each $x \in H$, $P_D(x)$ is the unique point in D such that

$$\|x - P_D(x)\| = \min\{\|x - z\| : z \in D\}.$$

The following identity will be used for the main results (see [1, Corollary 2.15]):

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2, \tag{2.2}$$

for all $\alpha \in R$ and $(x, y) \in H \times H$.

Lemma 2.1. [1, Theorem 3.16] *Given $x \in H$ and $z \in D$. Then $z = P_D(x)$ if and only if*

$$\langle x - z, y - z\rangle \le 0, \quad \forall y \in D.$$

Lemma 2.2. [1, Propositions 4.2 and 4.16] *For any $x, y \in H$ and $z \in D$, it holds*

(i) $\|P_D(x) - P_D(y)\| \le \|x - y\|$;
(ii) $\|P_D(x) - z\|^2 \le \|x - z\|^2 - \|P_D(x) - x\|^2$.

Definition 2.1. [1, Definition 6.38] The `normal cone` of D at $v \in D$, denote by $N_D(v)$ is defined as

$$N_D(v) := \{d \in H \mid \langle d, y - v\rangle \le 0 \text{ for all } y \in D\}.$$

Definition 2.2. [1, Definition 20.2] Let $A : H \to 2^H$ be a point-to-set operator defined on a real Hilbert space H. The operator A is called a `maximal monotone operator` if A is `monotone`, i.e.,

$$\langle u - v, x - y\rangle \ge 0 \text{ for all } u \in A(x) \text{ and } v \in A(y),$$

and the graph $G(A)$ of A,

$$G(A) := \{(x, u) \in H \times H \mid u \in A(x)\},$$

is not properly contained in the graph of any other monotone operator.

It is clear ([15, Theorem 3]) that a monotone mapping A is maximal if and only if, for any $(x, u) \in H \times H$, if $\langle u - v, x - y \rangle \geq 0$ for all $(v, y) \in G(A)$, then it follows that $u \in A(x)$.

Lemma 2.3. [1, Lemma 2.47] *Let $(x^k)_{k\in\mathbb{N}}$ be a sequence in H and let D be a nonempty subset of H. Suppose that, for every $x \in D$, $(\|x^k - x\|)_{k\in\mathbb{N}}$ converges and that every weak sequential cluster point of $(x^k)_{k\in\mathbb{N}}$ belongs to D. Then $(x^k)_{k\in\mathbb{N}}$ converges weakly to a point in D.*

3. Main results

In this section, we present two classes of projection algorithms and establish their weak convergence analysis under standard conditions.

3.1. **Simultaneous projection algorithms.** Let $S = C \times Q \in H := H_1 \times H_2$. Define $K = [A, -B] : H_1 \times H_2 \to H_1 \times H_2$, and let K^* be the adjoint operator of K, then the SEP (1.1) can be rewritten as

$$\text{Find} \quad z = (x, y) \in S \quad \text{such that} \quad Kz = 0, \tag{3.3}$$

which is a split feasibility problem (see, e.g. [12]).

Note that if the solution set of (3.3) is nonempty, it equals to the following constrained minimization problem

$$\min_{z\in S} \frac{1}{2}\|K(z)\|^2, \tag{3.4}$$

which is a standard (and a simple) problem from the convex optimization point of view. There are many methods for solving the problem (3.4), such as classical projection gradient method.

Inspired by the work in [8], we propose two alternated inertial simultaneous projection algorithms by combining the self-adaptive step size and the alternated inertia with Algorithm 3.1 in [8].

Algorithm 3.1. Given constants $\mu \in (0, 1)$, and let $z^0 = (x^0, y^0) \in H = H_1 \times H_2$ be taken arbitrarily.

For $k = 0, 1, 2, \ldots$, compute

$$t^k = \begin{cases} z^k, & k = \text{even}, \\ z^k + \theta_k(z^k - z^{k-1}), & k = \text{odd}, \end{cases} \tag{3.5}$$

and

$$w^k = P_S(t^k - \beta_k K^* K(t^k)), \tag{3.6}$$

and

$$\beta_{k+1} = \begin{cases} \min\left\{ \dfrac{\mu\|t^k - w^k\|}{\|K^*K(t^k) - K^*K(w^k)\|}, \beta_k \right\}, & \text{if } \|K^*K(t^k) - K^*K(w^k)\| \neq 0, \\ \beta_k, & \text{otherwise}. \end{cases} \tag{3.7}$$

Calculate next iterates z^{k+1} via

$$z_{\mathrm{I}}^{k+1} = t^k - \gamma\rho_k d(t^k, w^k), \tag{3.8}$$

or

$$z_{\mathrm{II}}^{k+1} = P_S(t^k - \gamma\beta_k\rho_k K^*K(w^k)), \tag{3.9}$$

where $\gamma \in (0, 2)$,

$$d(t^k, w^k) := (t^k - w^k) - \beta_k(K^*K(t^k) - K^*K(w^k)), \tag{3.10}$$

and

$$\rho_k := \frac{\langle t^k - w^k, d(t^k, w^k)\rangle + \beta_k \|K(w^k)\|^2}{\|d(t^k, w^k)\|^2}. \tag{3.11}$$

Set $k = k + 1$, and continue.

Remark 3.1. Let $z = (x, y)$. Then we have (see Section 4.4.1 in [2])

$$P_S(z) = (P_C x, P_Q y).$$

It is easy to see

$$K^* K z = \begin{pmatrix} A^* A & -A^* B \\ -B^* A & B^* B \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A^*(Ax - By) \\ B^*(By - Ax) \end{pmatrix}.$$

Define the function $F : H_1 \times H_2 \to H_1$ by

$$F(x, y) = A^*(Ax - By),$$

and the function $G : H_1 \times H_2 \to H_2$ by

$$G(x, y) = B^*(By - Ax).$$

By setting $z^k = (x^k, y^k), t^k = (p^k, q^k)$ and $w^k = (u^k, v^k)$, and Algorithm 3.1 can be rewritten as follow:

For $k = 0, 1, 2, ...,$ compute

$$(p^k, q^k) = \begin{cases} (x^k, y^k), & k = \text{even}, \\ (x^k, y^k) + \theta_k[(x^k, y^k) - (x^{k-1}, y^{k-1})], & k = \text{odd}, \end{cases}$$

and

$$\begin{cases} u^k = P_C(p^k - \beta_k F(p^k, q^k)), \\ v^k = P_Q(q^k - \beta_k G(p^k, q^k)), \end{cases}$$

update

$$\beta_{k+1} = \begin{cases} \min\left\{ \dfrac{\mu(\|p^k - u^k\| + \|q^k - v^k\|)}{\|F(p^k, q^k) - F(u^k, v^k)\| + \|G(p^k, q^k) - G(u^k, v^k)\|}, \beta_k \right\}, \\ \qquad\qquad\qquad \text{if } F(p^k, q^k) - F(u^k, v^k) \neq 0 \text{ or } G(p^k, q^k) - G(u^k, v^k) \neq 0, \\ \beta_k, \qquad\qquad\quad \text{otherwise.} \end{cases}$$

Compute next iterates x^{k+1} and y^{k+1} by

$$\begin{cases} x_{\mathrm{I}}^{k+1} = p^k - \gamma \rho_k c^k, \\ y_{\mathrm{I}}^{k+1} = q^k - \gamma \rho_k d^k, \end{cases} \tag{3.12}$$

or

$$\begin{cases} x_{\mathrm{II}}^{k+1} = P_C(p^k - \gamma \beta_k \rho_k F(u^k, v^k)), \\ y_{\mathrm{II}}^{k+1} = P_Q(q^k - \gamma \beta_k \rho_k G(u^k, v^k)), \end{cases} \tag{3.13}$$

where $\gamma \in (0, 2)$,

$$\begin{cases} c_k := (p^k - u^k) - \beta_k (F(p^k, q^k) - F(u^k, v^k)), \\ d_k := (q^k - v^k) - \beta_k (G(p^k, q^k) - G(u^k, v^k)), \end{cases}$$

and

$$\rho_k := \frac{\langle p^k - u^k, c^k\rangle + \langle q^k - v^k, d^k\rangle + \beta_k \|Au^k - Bv^k\|^2}{\|c^k\|^2 + \|d^k\|^2}.$$

Set $k = k + 1$, and continue.

For convenience, we denote the projection algorithms which use update forms (3.8) (or (3.12)) and (3.9) (or (3.13)) by Algorithm 3.1 (I) and Algorithm 3.1 (II), respectively. We suppose that the inertial parameter θ_k in Algorithm 3.1 satisfies one of the following conditions:

(A1) $-1 < \underline{\theta} \leq \theta_k \leq \overline{\theta} < \frac{2-\gamma}{\gamma}$;
(A2) $-1 < \underline{\theta} \leq \theta_k \leq \overline{\theta} < 0$.

Remark 3.2. By the definition of $d(t^k, w^k)$ in (3.10), (3.6) can be written as

$$w^k = P_S(w^k - (\beta_k K^* K(w^k) - d(t^k, w^k))).$$

So, from Lemma 2.1 we have

$$\langle z - w^k, \beta_k K^* K(w^k) - d(t^k, w^k)\rangle \geq 0, \ \ \forall z \in S. \tag{3.14}$$

Lemma 3.4. [8, Lemma 3.3] *Let $\{z^k\}_{k\in\mathbb{N}}$ and $\{w^k\}_{k\in\mathbb{N}}$ be generated by* Algorithm 3.1, *and let $d(t^k, w^k)$ be given by* [8, (21)]. *Then, for any $z^* \in \Gamma$, we have*

$$\langle z^k - z^*, d(z^k, w^k)\rangle \geq \rho_k \|d(z^k, w^k)\|^2.$$

Employing arguments which are similar to those used in the proof of [8, Lemma 3.3], we can obtain Lemma 3.5.

Lemma 3.5. *Let $\{t^k\}_{k\in\mathbb{N}}$ and $\{w^k\}_{k\in\mathbb{N}}$ be generated by* Algorithm 3.1, *and let $d(t^k, w^k)$ be given by* (3.10). *Then, for any $z^* \in \Gamma$, we have*

$$\langle t^k - z^*, d(t^k, w^k)\rangle \geq \rho_k \|d(t^k, w^k)\|^2. \tag{3.15}$$

Proof. It's easy to see that

$$\langle t^k - z^*, d(t^k, w^k)\rangle = \langle t^k - w^k, d(t^k, w^k)\rangle + \langle w^k - z^*, d(t^k, w^k)\rangle,$$

which together with Remark 3.2 and the definition of ρ_k implies

$$\langle t^k - z^*, d(t^k, w^k)\rangle \geq \langle t^k - w^k, d(t^k, w^k)\rangle + \beta_k \|K(w^k)\|^2 = \rho_k \|d(t^k, w^k)\|^2.$$

□

Lemma 3.6. *The search rule* (3.7) *is well defined. Besides, $\underline{\beta} \leq \beta_k \leq \beta_0$, where*

$$\underline{\beta} = \min\{\beta_0, \frac{\mu}{\|K\|^2}\}. \tag{3.16}$$

Furthermore, $\lim\limits_{k\to\infty} \beta_k$ exists.

Proof. Obviously, $\beta_{k+1} \leq \beta_k \leq \beta_0$. Since $\frac{\mu\|t^k - w^k\|}{\|K^*K(t^k) - K^*K(w^k)\|} \geq \frac{\mu}{\|K\|^2}$. From (3.7), we obtain (3.16). Since $\{\beta_k\}_{k\in\mathbb{N}}$ is nonincreasing, it follows from (3.16) that $\lim\limits_{k\to\infty} \beta_k$ exists. □

Lemma 3.7. *Let $\{t^k\}_{k\in\mathbb{N}}$ and $\{w^k\}_{k\in\mathbb{N}}$ be generated by* Algorithm 3.1, *and let $d(t^k, w^k)$ and ρ_k be given by* (3.10) *and* (3.11), *respectively. Then there exists a constant $\sigma_1 \in (\mu, 1)$ and a positive integer K_1 such that for any $k \geq K_1$,*

$$\rho_k \geq \frac{1 - \sigma_1}{1 + \sigma_1^2}. \tag{3.17}$$

Proof. By Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\langle t^k - w^k, d(t^k, w^k)\rangle \\
&= \|t^k - w^k\|^2 - \frac{\beta_k}{\beta_{k+1}}\beta_{k+1}\langle t^k - w^k, K^*K(t^k) - K^*K(w^k)\rangle \\
&\geq \|t^k - w^k\|^2 - \frac{\beta_k}{\beta_{k+1}}\beta_{k+1}\|t^k - w^k\|\|K^*K(t^k) - K^*K(w^k)\| \\
&\geq \|t^k - w^k\|^2 - \frac{\beta_k}{\beta_{k+1}}\mu\|t^k - w^k\|^2 \\
&= (1 - \frac{\beta_k}{\beta_{k+1}}\mu)\|t^k - w^k\|^2.
\end{aligned}$$

Due to the existence of $\lim_{k\to\infty}\beta_k$, we have $\lim_{k\to\infty}\frac{\beta_k}{\beta_{k+1}} = 1$. Therefore, there exist a constant $\sigma_1 \in (\mu, 1)$ and a positive integer K_1 such that for any $k \geq K_1$, we have $\frac{\beta_k}{\beta_{k+1}}\mu \leq \sigma_1$. So,

$$\langle t^k - w^k, d(t^k, w^k)\rangle \geq (1 - \sigma_1)\|t^k - w^k\|^2. \tag{3.18}$$

By using $\langle t^k - w^k, K^*K(t^k) - K^*K(w^k)\rangle = \langle K(t^k) - K(w^k), K(t^k) - K(w^k)\rangle = \|K(t^k) - K(w^k)\|^2$, we have

$$\begin{aligned}
\|d(t^k, w^k)\|^2 &= \|t^k - w^k\|^2 + (\frac{\beta_k}{\beta_{k+1}})^2\beta_{k+1}^2\|K^*K(t^k) - K^*K(w^k)\|^2 \\
&\quad - 2\beta_k\langle t^k - w^k, K^*K(t^k) - K^*K(w^k)\rangle \\
&\leq \|t^k - w^k\|^2 + (\frac{\beta_k}{\beta_{k+1}})^2\mu^2\|t^k - w^k\|^2 - 2\beta_k\|K(t^k) - K(w^k)\|^2 \\
&\leq \left[1 + (\frac{\beta_k}{\beta_{k+1}})^2\mu^2\right]\|t^k - w^k\|^2.
\end{aligned}$$

Similarly, we obtain

$$\|d(t^k, w^k)\|^2 \leq (1 + \sigma_1^2)\|t^k - w^k\|^2. \tag{3.19}$$

So, combining (3.18) and (3.19), we get (3.17). □

Lemma 3.8. *Take k as even and any $z^* \in \Gamma$. Let $\{z^k\}_{k\in\mathbb{N}}$ be generated by* Algorithm 3.1 (I) *and the assumption* (A1) *hold. Then, we have*

$$\begin{aligned}
\|z_{\mathrm{I}}^{k+2} - z^*\|^2 \leq &\|z^k - z^*\|^2 - \gamma(1 + \theta_{k+1})[2 - \gamma(1 + \theta_{k+1})]\rho_k^2\|d(t^k, w^k)\|^2 \\
&- \gamma(2 - \gamma)\rho_{k+1}^2\|d(t^{k+1}, w^{k+1})\|^2.
\end{aligned}$$

Proof. Then from (3.15), it follows

$$\begin{aligned}
\|z_{\mathrm{I}}^{k+2} - z^*\|^2 &= \|t^{k+1} - z^*\|^2 + \gamma^2\rho_{k+1}^2\|d(t^{k+1}, w^{k+1})\|^2 - 2\gamma\rho_{k+1}\langle t^{k+1} - z^*, d(t^{k+1}, w^{k+1})\rangle \\
&\leq \|t^{k+1} - z^*\|^2 + \gamma^2\rho_{k+1}^2\|d(t^{k+1}, w^{k+1})\|^2 - 2\gamma\rho_{k+1}^2\|d(t^{k+1}, w^{k+1})\|^2 \\
&= \|t^{k+1} - z^*\|^2 - \gamma(2 - \gamma)\rho_{k+1}^2\|d(t^{k+1}, w^{k+1})\|^2.
\end{aligned} \tag{3.20}$$

Similarly, we

$$\|z_{\mathrm{I}}^{k+1} - z^*\|^2 \leq \|t^k - z^*\|^2 - \gamma(2 - \gamma)\rho_k^2\|d(t^k, w^k)\|^2. \tag{3.21}$$

Using (2.2) and (3.5), we obtain

$$
\begin{aligned}
\|t^{k+1} - z^*\|^2 =& \|z_{\mathrm{I}}^{k+1} + \theta_{k+1}(z_{\mathrm{I}}^{k+1} - z^k) - z^*\|^2 \\
=& \|(1+\theta_{k+1})(z_{\mathrm{I}}^{k+1} - z^*) - \theta_{k+1}(z^k - z^*)\|^2 \\
=& (1+\theta_{k+1})\|z_{\mathrm{I}}^{k+1} - z^*\|^2 - \theta_{k+1}\|z^k - z^*\|^2 \\
& + \theta_{k+1}(1+\theta_{k+1})\|z_{\mathrm{I}}^{k+1} - z^k\|^2.
\end{aligned}
\tag{3.22}
$$

By $z^k = t^k$ and (3.8), we have

$$
\|z_{\mathrm{I}}^{k+1} - z^k\|^2 = \gamma^2 \rho_k^2 \|d(t^k, w^k)\|^2. \tag{3.23}
$$

Combining (3.20)-(3.23), and using $z^k = t^k$ and assumption (A1), we get

$$
\begin{aligned}
\|z_{\mathrm{I}}^{k+2} - z^*\|^2 \le& \|z^k - z^*\|^2 - \gamma(2-\gamma)(1+\theta_{k+1})\rho_k^2\|d(t^k, w^k)\|^2 \\
& + \theta_{k+1}(1+\theta_{k+1})\gamma^2\rho_k^2\|d(t^k, w^k)\|^2 \\
& - \gamma(2-\gamma)\rho_{k+1}^2\|d(t^{k+1}, w^{k+1})\|^2 \\
=& \|z^k - z^*\|^2 - \gamma(1+\theta_{k+1})[2 - \gamma(1+\theta_{k+1})]\rho_k^2\|d(t^k, w^k)\|^2 \\
& - \gamma(2-\gamma)\rho_{k+1}^2\|d(t^{k+1}, w^{k+1})\|^2.
\end{aligned}
$$

The proof is completed. □

Theorem 3.1. *Let* $\{z^k\}_{k\in\mathbb{N}}$ *be generated by* Algorithm 3.1 (I) *and the assumption* (A1) *hold. Then* $\{z^k\}_{k\in\mathbb{N}}$ *converges weakly to a solution of the SEP* (1.1).

Proof. Take arbitrarily $z^* \in \Gamma$. Since $\gamma \in (0, 2)$, and the assumption (A1) holds, Lemma 3.8 implies that the sequence $\{\|z^{2k} - z^*\|^2\}_{k\in\mathbb{N}}$ is nonincreasing and thus converges. Moreover, $\{z^{2k}\}_{k\in\mathbb{N}}$ is bounded. Using Lemma 3.8 again, we obtain

$$
\lim_{k\to\infty} \rho_k^2 \|d(t^k, w^k)\|^2 = 0. \tag{3.24}
$$

From the definition of ρ_k, (3.18), Lemmas 3.6 and 3.7, we have for any $k \ge K_1$

$$
\begin{aligned}
\rho_k^2\|d(t^k, w^k)\|^2 &= \rho_k(\langle t^k - w^k, d(t^k, w^k)\rangle + \beta_k\|K(w^k)\|^2) \\
&\ge \rho_k\left[(1-\sigma_1)\|t^k - w^k\|^2 + \beta_k\|K(w^k)\|^2\right] \\
&\ge \frac{(1-\sigma_1)^2}{1+\sigma_1^2}\|t^k - w^k\|^2 + \frac{1-\sigma_1}{1+\sigma_1^2}\underline{\beta}\|K(w^k)\|^2.
\end{aligned}
$$

Combining the above inequality and (3.24), we get

$$
\lim_{k\to\infty} \|t^k - w^k\| = 0, \tag{3.25}
$$

and

$$
\lim_{k\to\infty} \|K(w^k)\| = 0. \tag{3.26}
$$

By (3.23) and (3.24), we know that

$$
\lim_{k\to\infty} \|z_{\mathrm{I}}^{2k+1} - z_{\mathrm{I}}^{2k}\| = 0. \tag{3.27}
$$

Using (3.27) and the definition of t_k in (3.5), it holds

$$
\lim_{k\to\infty} \|t^k - z^k\| = 0. \tag{3.28}
$$

Combining (3.25) and (3.28), we have

$$
\lim_{k\to\infty} \|z^k - w^k\| = 0,
$$

which with the boundedness of K and (3.26) yields

$$\lim_{k\to\infty} \|K(z^k)\| = 0.$$

Employing arguments which are similar to those used in the proof of [8, Theorem 3.1], we obtain that the whole sequence $\{z^k\}_{k\in\mathbb{N}}$ weakly converges to a solution of SEP (1.1), which completes proof. $\square$

Lemma 3.9. *Take k as even and any $z^* \in \Gamma$. Let $\{z^k\}_{k\in\mathbb{N}}$ be generated by* Algorithm 3.1 (II). *And suppose the the assumption* (A2) *holds. Then, we have*

(3.29)
$$\begin{aligned}\|z_{\mathrm{II}}^{k+2} - z^*\|^2 \le &\|z^k - z^*\|^2 - (1+\theta_{k+1})\left[\gamma(2-\gamma)\rho_k^2\|d(t^k, w^k)\|^2 + \|z_{\mathrm{I}}^{k+1} - z_{\mathrm{II}}^{k+1}\|^2\right] \\ &- \gamma(2-\gamma)\rho_{k+1}^2\|d(t^{k+1}, w^{k+1})\|^2 - \|z_{\mathrm{I}}^{k+2} - z_{\mathrm{II}}^{k+2}\|^2.\end{aligned}$$

Proof. Due to $z^* \in \Gamma$, we have $K(z^*) = 0$. Using Lemma 2.2 (ii), we get

(3.30)
$$\begin{aligned}&\|z_{\mathrm{II}}^{k+2} - z^*\|^2 \\ =&\|P_S(t^{k+1} - \gamma\beta_{k+1}\rho_{k+1}K^*K(w^{k+1})) - z^*\|^2 \\ \le&\|t^{k+1} - \gamma\beta_{k+1}\rho_{k+1}K^*K(w^{k+1}) - z^*\|^2 - \|t^{k+1} - \gamma\beta_{k+1}\rho_{k+1}K^*K(w^{k+1}) - z_{\mathrm{II}}^{k+2}\|^2 \\ =&\|t^{k+1} - z^*\|^2 - \|t^{k+1} - z_{\mathrm{II}}^{k+2}\|^2 - 2\gamma\beta_{k+1}\rho_{k+1}\langle z_{\mathrm{II}}^{k+2} - z^*, K^*K(w^{k+1})\rangle.\end{aligned}$$

Similarly, it holds

$$\|z_{\mathrm{II}}^{k+1} - z^*\|^2 \le \|t^k - z^*\|^2 - \|t^k - z_{\mathrm{II}}^{k+1}\|^2 - 2\gamma\beta_k\rho_k\langle z_{\mathrm{II}}^{k+1} - z^*, K^*K(w^k)\rangle. \tag{3.31}$$

By setting $z = z_{\mathrm{II}}^{k+2}$ in (3.14), we get

$$\begin{aligned}&-2\gamma\beta_{k+1}\rho_{k+1}\langle z_{\mathrm{II}}^{k+2} - w^{k+1}, K^*K(w^{k+1})\rangle \\ &\le -2\gamma\rho_{k+1}\langle z_{\mathrm{II}}^{k+2} - w^{k+1}, d(t^{k+1}, w^{k+1})\rangle \\ &= -2\gamma\rho_{k+1}\langle t^{k+1} - w^{k+1}, d(t^{k+1}, w^{k+1})\rangle - 2\gamma\rho_{k+1}\langle z_{\mathrm{II}}^{k+2} - t^{k+1}, d(t^{k+1}, w^{k+1})\rangle.\end{aligned} \tag{3.32}$$

It holds

(3.33)
$$\begin{aligned}-2\gamma\rho_{k+1}\langle z_{\mathrm{II}}^{k+2} - t^{k+1}, d(t^{k+1}, w^{k+1})\rangle = &-\|t^{k+1} - z_{\mathrm{II}}^{k+2} - \gamma\rho_{k+1}d(t^{k+1}, w^{k+1})\|^2 \\ &+ \|t^{k+1} - z_{\mathrm{II}}^{k+2}\|^2 + \gamma^2\rho_{k+1}^2\|d(t^{k+1}, w^{k+1})\|^2.\end{aligned}$$

Substituting (3.33) in the right hand side of (3.32) and using $t^{k+1} - \gamma\rho_{k+1}d(t^{k+1}, w^{k+1}) = z_{\mathrm{I}}^{k+2}$, we obtain

$$\begin{aligned}&-2\gamma\beta_{k+1}\rho_{k+1}\langle z_{\mathrm{II}}^{k+2} - w^{k+1}, K^*K(w^{k+1})\rangle \\ \le& -2\gamma\rho_{k+1}\langle t^{k+1} - w^{k+1}, d(t^{k+1}, w^{k+1})\rangle \\ &- \|z_{\mathrm{I}}^{k+2} - z_{\mathrm{II}}^{k+2}\|^2 + \|t^{k+1} - z_{\mathrm{II}}^{k+2}\|^2 + \gamma^2\rho_{k+1}^2\|d(t^{k+1}, w^{k+1})\|^2.\end{aligned} \tag{3.34}$$

Also

$$-2\gamma\beta_{k+1}\rho_{k+1}\langle w^{k+1} - z^*, K^*K(w^{k+1})\rangle = -2\gamma\beta_{k+1}\rho_{k+1}\|K(w^{k+1})\|^2. \tag{3.35}$$

So, adding (3.34) and (3.35) and using the definition of ρ_k, we obtain

(3.36)
$$\begin{aligned}
&-2\gamma\beta_{k+1}\rho_{k+1}\langle z_{\mathrm{II}}^{k+2}-z^*, K^*K(w^{k+1})\rangle\\
&\le -2\gamma\rho_{k+1}\left[\langle t^{k+1}-w^{k+1}, d(t^{k+1},w^{k+1})\rangle+\beta_{k+1}\|K(w^{k+1})\|^2\right]\\
&\quad -\|z_{\mathrm{I}}^{k+2}-z_{\mathrm{II}}^{k+2}\|^2+\|t^{k+1}-z_{\mathrm{II}}^{k+2}\|^2+\gamma^2\rho_{k+1}^2\|d(t^{k+1},w^{k+1})\|^2\\
&= -2\gamma\rho_{k+1}^2\|d(t^{k+1},w^{k+1})\|^2+\gamma^2\rho_{k+1}^2\|d(t^{k+1},w^{k+1})\|^2-\|z_{\mathrm{I}}^{k+2}-z_{\mathrm{II}}^{k+2}\|^2+\|t^{k+1}-z_{\mathrm{II}}^{k+2}\|^2\\
&= -\gamma(2-\gamma)\rho_{k+1}^2\|d(t^{k+1},w^{k+1})\|^2-\|z_{\mathrm{I}}^{k+2}-z_{\mathrm{II}}^{k+2}\|^2+\|t^{k+1}-z_{\mathrm{II}}^{k+2}\|^2.
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
&-2\gamma\beta_k\rho_k\langle z_{\mathrm{II}}^{k+1}-z^*, K^*K(w^k)\rangle\\
&\le -\gamma(2-\gamma)\rho_k^2\|d(t^k,w^k)\|^2-\|z_{\mathrm{I}}^{k+1}-z_{\mathrm{II}}^{k+1}\|^2+\|t^k-z_{\mathrm{II}}^{k+1}\|^2.
\end{aligned}\tag{3.37}$$

Therefore, by substituting (3.36) into (3.30), (3.37) into (3.31), respectively, we have

$$\|z_{\mathrm{II}}^{k+2}-z^*\|^2 \le\|t^{k+1}-z^*\|^2-\gamma(2-\gamma)\rho_{k+1}^2\|d(t^{k+1},w^{k+1})\|^2-\|z_{\mathrm{I}}^{k+2}-z_{\mathrm{II}}^{k+2}\|^2,\tag{3.38}$$

and

$$\|z_{\mathrm{II}}^{k+1}-z^*\|^2 \le\|t^k-z^*\|^2-\gamma(2-\gamma)\rho_k^2\|d(t^k,w^k)\|^2-\|z_{\mathrm{I}}^{k+1}-z_{\mathrm{II}}^{k+1}\|^2.\tag{3.39}$$

Combing (3.22), (3.38), (3.39), and taking into account of the facts $\|z_{\mathrm{II}}^{k+1}-z^k\|\le\gamma\beta_k\rho_k\|K^*K(w^k)\|$, $z^k=t^k$, we obtain

$$\begin{aligned}
&\|z_{\mathrm{II}}^{k+2}-z^*\|^2\\
\le&(1+\theta_{k+1})\|z_{\mathrm{II}}^{k+1}-z^*\|^2-\theta_{k+1}\|z^k-z^*\|^2+\theta_{k+1}(1+\theta_{k+1})\|z_{\mathrm{II}}^{k+1}-z^k\|^2\\
&-\gamma(2-\gamma)\rho_{k+1}^2\|d(t^{k+1},w^{k+1})\|^2-\|z_{\mathrm{I}}^{k+2}-z_{\mathrm{II}}^{k+2}\|^2\\
\le&\|z^k-z^*\|^2-(1+\theta_{k+1})\left[\gamma(2-\gamma)\rho_k^2\|d(t^k,w^k)\|^2+\|z_{\mathrm{I}}^{k+1}-z_{\mathrm{II}}^{k+1}\|^2\right]\\
&+\theta_{k+1}(1+\theta_{k+1})\gamma^2\beta_k^2\rho_k^2\|K^*K(w^k)\|^2\\
&-\gamma(2-\gamma)\rho_{k+1}^2\|d(t^{k+1},w^{k+1})\|^2-\|z_{\mathrm{I}}^{k+2}-z_{\mathrm{II}}^{k+2}\|^2.
\end{aligned}$$

Since the assumption (A2) holds and $\gamma\in(0,2)$, we obtain (3.29). □

Theorem 3.2. *Let $\{z^k\}_{k\in\mathbb{N}}$ be generated by* Algorithm 3.1 (II) *and the assumption* (A2) *hold. Then $\{z^k\}_{k\in\mathbb{N}}$ converge weakly to a solution of the SEP* (1.1).

Proof. Employing arguments which are similar to those used in the proof of Theorem 3.1, we obtain that the whole sequence $\{z^k\}_{k\in\mathbb{N}}$ converge weakly to a solution of the SEP (1.1), which completes proof. □

3.2. **Semi-alternating projection algorithms.** Based on Algorithm 3.1, we present two semi-alternating projection algorithms, whose name comes from an alternating technique taken in the second step.

Algorithm 3.2. Given constants $\mu\in(0,1)$ and let $x^0\in H_1$ and $y^0\in H_2$ be taken arbitrary.
For $k=0,1,2,\ldots$, compute Compute

$$(p^k,q^k)=\begin{cases}(x^k,y^k), & k=\text{even},\\ (x^k,y^k)+\theta_k[(x^k,y^k)-(x^{k-1},y^{k-1})], & k=\text{odd},\end{cases}\tag{3.40}$$

and

$$\begin{cases}u^k=P_C(p^k-\beta_kF(p^k,q^k)),\\ v^k=P_Q(q^k-\beta_kG(u^k,q^k)).\end{cases}\tag{3.41}$$

Update

$$\beta_{k+1}=\begin{cases}\min\left\{\mu\left(\frac{(\|p^k-u^k\|^2+\|q^k-v^k\|^2)}{\|F(p^k,q^k)-F(u^k,v^k)\|^2+\|G(u^k,q^k)-G(u^k,v^k)\|^2}\right)^{\frac{1}{2}},\beta_k\right\},\\ \qquad\qquad\text{if } F(p^k,q^k)-F(u^k,v^k)\neq 0 \text{ or } G(u^k,q^k)-G(u^k,v^k)\neq 0,\\ \beta_k, \qquad\qquad \text{otherwise.}\end{cases}$$

Compute next iterates x^{k+1} and y^{k+1} by

$$\begin{cases}x_{\mathrm{I}}^{k+1}=p^k-\gamma\rho_k c_k,\\ y_{\mathrm{I}}^{k+1}=q^k-\gamma\rho_k d_k,\end{cases}\tag{3.42}$$

or

$$\begin{cases}x_{\mathrm{II}}^{k+1}=P_C(p^k-\gamma\beta_k\rho_k F(u^k,v^k)),\\ y_{\mathrm{II}}^{k+1}=P_Q(q^k-\gamma\beta_k\rho_k G(u^k,v^k)),\end{cases}\tag{3.43}$$

where $\gamma\in(0,2)$,

$$\begin{cases}c^k:=(p^k-u^k)-\beta_k(F(p^k,q^k)-F(u^k,v^k)),\\ d^k:=(q^k-v^k)-\beta_k(G(u^k,q^k)-G(u^k,v^k)),\end{cases}\tag{3.44}$$

and

$$\rho_k:=\frac{\langle p^k-u^k,c^k\rangle+\langle q^k-v^k,d^k\rangle+\beta_k\|Au^k-Bv^k\|^2}{\|c^k\|^2+\|d^k\|^2}.\tag{3.45}$$

Set $k=k+1$, and continue.

For convenience, we call the projection algorithms which use update forms (3.42) and (3.43) Algorithm 3.2 (I) and Algorithm 3.2 (II), respectively.

Remark 3.3. By the definitions of c^k and d^k in (3.44), the projection equation (3.41) can be written as

$$\begin{cases}u^k=P_C(u^k-(\beta_k F(u^k,v^k)-c^k)),\\ v^k=P_Q(v^k-(\beta_k G(u^k,v^k)-d^k)).\end{cases}$$

So, from Lemma 2.1 we have

$$\begin{cases}\langle x-u^k,\beta_k F(u^k,v^k)-c^k\rangle\geq 0, & \forall x\in C,\\ \langle y-v^k,\beta_k G(u^k,v^k)-d^k\rangle\geq 0, & \forall y\in Q.\end{cases}\tag{3.46}$$

Lemma 3.10. [8, Lemma 3.6] *Let (x^k,y^k) and (u^k,v^k) be generated by* Algorithm 3.2, *and let c^k and d^k be given by* [8, (58)]. *Then, for all $(x^*,y^*)\in\Gamma$, we have*

$$\langle x^k-x^*,c^k\rangle+\langle y^k-y^*,d^k\rangle\geq\rho_k(\|c^k\|^2+\|d^k\|^2).$$

Lemma 3.11. *Let (x^k,y^k) and (u^k,v^k) be generated by* Algorithm 3.2, *and let c^k and d^k be given by* (3.44). *Then, for all $(x^*,y^*)\in\Gamma$, we have*

$$\langle p^k-x^*,c^k\rangle+\langle q^k-y^*,d^k\rangle\geq\rho_k(\|c^k\|^2+\|d^k\|^2).$$

Proof. Employing arguments which are similar to those used in the proof of [8, Lemma 3.6], we can obtain Lemma 3.11. $\square$

Lemma 3.12. *The search rule β_k is well defined. Besides, $\beta^* \le \beta_k \le \beta_0$, where*

$$\beta^* = \min\{\beta_0, \frac{\mu}{\sqrt{2}\|A\|^2}, \frac{\mu}{\|B\|\sqrt{(2\|A\|^2 + \|B\|^2)}}\}. \tag{3.47}$$

Furthermore, $\lim_{k\to\infty} \beta_k$ exists.

Proof. Obviously, $\beta_{k+1} \le \beta_k \le \beta_0$. In the latter case, we know

$$\begin{aligned}
&\|F(p^k, q^k) - F(u^k, v^k)\|^2 + \|G(u^k, q^k) - G(u^k, v^k)\|^2 \\
=&\|A^*(Ap^k - Bq^k) - A^*(Au^k - Bv^k)\|^2 + \|B^*(Bq^k - Au^k) - B^*(Bv^k - Au^k)\|^2 \\
\le&\|A\|^2(\|Ap^k - Au^k\| + \|Bq^k - Bv^k\|)^2 + \|B\|^4\|q^k - v^k\|^2 \\
\le&2\|A\|^2(\|A\|^2\|p^k - u^k\|^2 + \|B\|^2\|q^k - v^k\|^2) + \|B\|^4\|q^k - v^k\|^2 \\
\le&2\|A\|^4\|p^k - u^k\|^2 + \|B\|^2(2\|A\|^2 + \|B\|^2)\|q^k - v^k\|^2 \\
\le&\max\{2\|A\|^4, \|B\|^2(2\|A\|^2 + \|B\|^2)\}(\|p^k - u^k\|^2 + \|q^k - v^k\|^2)
\end{aligned}$$

So, we get (3.47). Since $\{\beta_k\}_{k\in\mathbb{N}}$ is nonincreasing, it follows from (3.47) that $\lim_{k\to\infty} \beta_k$ exists. □

Lemma 3.13. *Let (x^k, y^k) and (u^k, v^k) be generated by* Algorithm 3.2 *, and let c^k , d^k and ρ_k be given by* (3.44) *and* (3.45)*, respectively. Then there exists a constant $\sigma_2 \in (\mu, 1)$ and a positive integer K_2 such that for any $k \ge K_2$,*

$$\rho_k \ge \frac{1 - \sigma_2}{1 + \sigma_2^2}.$$

Proof. By Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\langle p^k - u^k, c^k\rangle + \langle q^k - v^k, d^k\rangle \\
=&\|p^k - u^k\|^2 + \|q^k - v^k\|^2 - \beta_k\langle p^k - u^k, F(p^k, q^k) - F(u^k, v^k)\rangle \\
&- \beta_k\langle q^k - v^k, G(u^k, q^k) - G(u^k, v^k)\rangle \\
\ge&\|p^k - u^k\|^2 + \|q^k - v^k\|^2 \\
&- \beta_k(\|p^k - u^k\|\|F(p^k, q^k) - F(u^k, v^k)\| + \|q^k - v^k\|\|G(u^k, q^k) - G(u^k, v^k)\|).
\end{aligned} \tag{3.48}$$

For the term $\beta_k(\|p^k - u^k\|\|F(p^k, q^k) - F(u^k, v^k)\| + \|q^k - v^k\|\|G(u^k, q^k) - G(u^k, v^k)\|)$ in (3.48), using the inequality

$$(ab + cd)^2 \le (a^2 + c^2)(b^2 + d^2),$$

we get

$$\begin{aligned}
&\beta_k^2(\|p^k - u^k\|\|F(p^k, q^k) - F(u^k, v^k)\| + \|q^k - v^k\|\|G(u^k, q^k) - G(u^k, v^k)\|)^2 \\
\le&\beta_k^2(\|p^k - u^k\|^2 + \|q^k - v^k\|^2)(\|F(p^k, q^k) - F(u^k, v^k)\|^2 + \|G(u^k, q^k) - G(u^k, v^k)\|^2) \\
\le&\frac{\beta_k^2}{\beta_{k+1}^2}\mu^2(\|p^k - u^k\|^2 + \|q^k - v^k\|^2)^2,
\end{aligned} \tag{3.49}$$

where the second inequality comes from the definition of β_k. Combining (3.48) and (3.49), we obtain

$$\langle p^k - u^k, c^k\rangle + \langle q^k - v^k, d^k\rangle \ge (1 - \frac{\beta_k}{\beta_{k+1}}\mu)(\|p^k - u^k\|^2 + \|q^k - v^k\|^2).$$

Due to the existence of $\lim_{k\to\infty}\beta_k$, we have $\lim_{k\to\infty}\frac{\beta_k}{\beta_{k+1}}=1$. Therefore, there exist a constant $\sigma_2\in(\mu,1)$ and a positive integer K_2 such that for any $k\geq K_2$, we have $\frac{\beta_k}{\beta_{k+1}}\mu\leq\sigma_2$ and

$$\langle p^k-u^k,c^k\rangle+\langle q^k-v^k,d^k\rangle\geq(1-\sigma_2)(\|p^k-u^k\|^2+\|q^k-v^k\|^2). \tag{3.50}$$

From the definitions of F,G and β_k, it follows

$$\begin{aligned}
&\|c^k\|^2+\|d^k\|^2\\
=&\|p^k-u^k\|^2+\|q^k-v^k\|^2\\
&+\beta_k^2(\|F(p^k,q^k)-F(u^k,v^k)\|^2+\|G(u^k,q^k)-G(u^k,v^k)\|^2)\\
&-2\beta_k(\langle p^k-u^k,F(p^k,q^k)-F(u^k,v^k)\rangle+\langle q^k-v^k,G(u^k,q^k)-G(u^k,v^k)\rangle)\\
\leq&\|p^k-u^k\|^2+\|q^k-v^k\|^2+\frac{\beta_k^2}{\beta_{k+1}^2}\mu^2(\|p^k-u^k\|^2+\|q^k-v^k\|^2) \qquad (3.51)\\
&-2\beta_k[\langle A(p^k-u^k),A(p^k-u^k)-B(q^k-v^k)\rangle+\langle B(q^k-v^k),B(q^k-v^k)\rangle]\\
=&(1+\frac{\beta_k^2}{\beta_{k+1}^2}\mu^2)(\|p^k-u^k\|^2+\|q^k-v^k\|^2)\\
&-2\beta_k[\|A(p^k-u^k)\|^2-\langle A(p^k-u^k),B(q^k-v^k)\rangle+\|B(q^k-v^k)\|^2].
\end{aligned}$$

Since

$$\begin{aligned}
&-2\beta_k(\|A(p^k-u^k)\|^2-\langle A(p^k-u^k),B(q^k-v^k)\rangle+\|B(q^k-v^k)\|^2)\\
\leq&-2\beta_k(\|A(p^k-u^k)\|^2-\|A(p^k-u^k)\|\|B(q^k-v^k)\|+\|B(q^k-v^k)\|^2)\\
\leq&-2\beta_k[\|A(p^k-u^k)\|^2-\frac{1}{2}(\|A(p^k-u^k)\|^2+\|B(q^k-v^k)\|^2)+\|B(q^k-v^k)\|^2]\\
\leq&-\beta_k(\|A(p^k-u^k)\|^2+\|B(q^k-v^k)\|^2),
\end{aligned}$$

by (3.51), we get

$$\begin{aligned}
&\|c^k\|^2+\|d^k\|^2\\
\leq&(1+\frac{\beta_k^2}{\beta_{k+1}^2}\mu^2)(\|p^k-u^k\|^2+\|q^k-v^k\|^2)-\beta_k(\|A(p^k-u^k)\|^2+\|B(q^k-v^k)\|^2)\\
\leq&(1+\frac{\beta_k^2}{\beta_{k+1}^2}\mu^2)(\|p^k-u^k\|^2+\|q^k-v^k\|^2).
\end{aligned}$$

Therefore, we have

$$\|c^k\|^2+\|d^k\|^2\leq(1+{\sigma_2}^2)(\|p^k-u^k\|^2+\|q^k-v^k\|^2),\quad\forall k\geq K_2. \tag{3.52}$$

Combining (3.50) and (3.52), we complete proof. □

Lemma 3.14. *Take k as even and any $(x^*,y^*)\in\Gamma$. Let $\{(x^k,y^k)\}_{k\in\mathbb{N}}$ be generated by* Algorithm 3.2 (I). *Then we have*

$$\begin{aligned}
\|x_{\mathrm{I}}^{k+2}-x^*\|^2+\|y_{\mathrm{I}}^{k+2}-y^*\|^2\leq&\|x^k-x^*\|^2+\|y^k-y^*\|^2-(2-\gamma)\gamma\rho_{k+1}^2(\|c^{k+1}\|^2+\|d^{k+1}\|^2)\\
&-(2-\gamma(1+\theta_{k+1}))(1+\theta_{k+1})\gamma\rho_k^2(\|c^k\|^2+\|d^k\|^2).
\end{aligned}$$

Proof. From (3.42), we have

$$\begin{aligned}
\|x_{\mathrm{I}}^{k+2}-x^*\|^2&=\|p^{k+1}-\gamma\rho_{k+1}c^{k+1}-x^*\|^2\\
&=\|p^{k+1}-x^*\|^2+\gamma^2\rho_{k+1}^2\|c^{k+1}\|^2-2\gamma\rho_{k+1}\langle p^{k+1}-x^*,c^{k+1}\rangle,
\end{aligned}$$

Similarly,

$$\|y_{\mathrm{I}}^{k+2} - y^*\|^2 = \|q^{k+1} - y^*\|^2 + \gamma^2\rho_{k+1}^2\|d^{k+1}\|^2 - 2\gamma\rho_{k+1}\langle q^{k+1} - y^*, d^{k+1}\rangle,$$

Adding the above inequalities and using Lemma 3.11, we obtain

$$\begin{aligned} &\|x_{\mathrm{I}}^{k+2} - x^*\|^2 + \|y_{\mathrm{I}}^{k+2} - y^*\|^2 \\ =&\|p^{k+1} - x^*\|^2 + \|q^{k+1} - y^*\|^2 + \gamma^2\rho_{k+1}^2(\|c^{k+1}\|^2 + \|d^{k+1}\|^2) \\ &- 2\gamma\rho_{k+1}\langle p^{k+1} - x^*, c^{k+1}\rangle - 2\gamma\rho_{k+1}\langle q^{k+1} - y^*, d^{k+1}\rangle \\ \le&\|p^{k+1} - x^*\|^2 + \|q^{k+1} - y^*\|^2 - (2-\gamma)\gamma\rho_{k+1}^2(\|c^{k+1}\|^2 + \|d^{k+1}\|^2). \end{aligned} \tag{3.53}$$

Similarly,

$$\begin{aligned} &\|x_{\mathrm{I}}^{k+1} - x^*\|^2 + \|y_{\mathrm{I}}^{k+1} - y^*\|^2 \\ \le&\|p^k - x^*\|^2 + \|q^k - y^*\|^2 - (2-\gamma)\gamma\rho_k^2(\|c^k\|^2 + \|d^k\|^2). \end{aligned} \tag{3.54}$$

By (2.2) and (3.40), we get

$$\begin{aligned} \|p^{k+1} - x^*\|^2 =&\|(1+\theta_{k+1})(x^{k+1} - x^*) - \theta_{k+1}(x^k - x^*)\|^2 \\ =&(1+\theta_{k+1})\|x^{k+1} - x^*\|^2 - \theta_{k+1}\|x^k - x^*\|^2 + \theta_{k+1}(1+\theta_{k+1})\|x^{k+1} - x^k\|^2, \end{aligned} \tag{3.55}$$

which with $\|x_{\mathrm{I}}^{k+1} - x^k\|^2 = \|p^k - \gamma\rho_k c^k - x^k\|^2$ and $x^k = p^k$, we have

$$\|p^{k+1} - x^*\|^2 =(1+\theta_{k+1})\|x^{k+1} - x^*\|^2 - \theta_{k+1}\|x^k - x^*\|^2 + \theta_{k+1}(1+\theta_{k+1})\gamma^2\rho_k^2\|c^k\|^2. \tag{3.56}$$

Similarly,

$$\|q^{k+1} - y^*\|^2 =(1+\theta_{k+1})\|y^{k+1} - y^*\|^2 - \theta_{k+1}\|y^k - y^*\|^2 + \theta_{k+1}(1+\theta_{k+1})\gamma^2\rho_k^2\|y^{k+1} - y^k\|^2. \tag{3.57}$$

$$\|q^{k+1} - y^*\|^2 =(1+\theta_{k+1})\|y^{k+1} - y^*\|^2 - \theta_{k+1}\|y^k - y^*\|^2 + \theta_{k+1}(1+\theta_{k+1})\gamma^2\rho_k^2\|d^k\|^2. \tag{3.58}$$

Combining (3.53), (3.54), (3.56) and (3.58), and using $x^k = p^k, y^k = q^k$, we obtain

$$\begin{aligned} &\|x_{\mathrm{I}}^{k+2} - x^*\|^2 + \|y_{\mathrm{I}}^{k+2} - y^*\|^2 \\ \le&\|x^k - x^*\|^2 + \|y^k - y^*\|^2 - (1+\theta_{k+1})(2-\gamma)\gamma\rho_k^2(\|c^k\|^2 + \|d^k\|^2) \\ &+ \theta_{k+1}(1+\theta_{k+1})\gamma^2\rho_k^2(\|c^k\|^2 + \|d^k\|^2) - (2-\gamma)\gamma\rho_{k+1}^2(\|c^{k+1}\|^2 + \|d^{k+1}\|^2) \\ =&\|x^k - x^*\|^2 + \|y^k - y^*\|^2 - [2 - \gamma(1+\theta_{k+1})](1+\theta_{k+1})\gamma\rho_k^2(\|c^k\|^2 + \|d^k\|^2) \\ &- (2-\gamma)\gamma\rho_{k+1}^2(\|c^{k+1}\|^2 + \|d^{k+1}\|^2). \end{aligned}$$

The proof completes. □

Theorem 3.3. *Let the assumption* (A1) *hold. Then the sequence* $\{(x^k, y^k)\}_{k\in\mathbb{N}}$ *be generated by* Algorithm 3.2 (I) *converges weakly to a solution of the SEP* (1.1).

Proof. Due to $\gamma \in (0,2)$, Lemma 3.14 implies that the sequence $\{\|x^{2k} - x^*\|^2 + \|y^{2k} - y^*\|^2\}_{k\in\mathbb{N}}$ is nonincreasing for $\forall\,(x^*, y^*) \in \Gamma$ and thus converges. Moreover, $\{x^{2k}\}_{k\in\mathbb{N}}$ and $\{y^{2k}\}_{k\in\mathbb{N}}$ are bounded. Lemma 3.14 implies that for $\forall k \in \mathbb{N}$

$$\lim_{k\to\infty} \rho_k^2(\|c^k\|^2 + \|d^k\|^2) = 0. \tag{3.59}$$

From definition of ρ_k, (3.50), Lemmas 3.12 and 3.13, for any $k \geq K_2$, we have

$$
\begin{aligned}
&\rho_k^2(\|c^k\|^2 + \|d^k\|^2) \\
=&\rho_k(\langle p^k - u^k, c^k\rangle + \langle q^k - v^k, d^k\rangle + \beta_k\|Au^k - Bv^k\|^2) \\
\geq&\rho_k\left[(1-\sigma_2)(\|p^k - u^k\|^2 + \|q^k - v^k\|^2) + \beta_k\|Au^k - Bv^k\|^2\right] \\
\geq&\frac{(1-\sigma_2)^2}{1+\sigma_2^2}(\|p^k - u^k\|^2 + \|q^k - v^k\|^2) + \frac{1-\sigma_2}{1+\sigma_2^2}\beta^*\|Au^k - Bv^k\|^2,
\end{aligned}
$$

which together with (3.59), we get

$$\lim_{k\to\infty}\|p^k - u^k\| = 0, \qquad \lim_{k\to\infty}\|q^k - v^k\| = 0, \tag{3.60}$$

and

$$\lim_{k\to\infty}\|Au^k - Bv^k\| = 0.$$

Furthermore, we have

$$\lim_{k\to\infty}\|Ap^k - Bq^k\| = 0, \qquad \text{and} \qquad \lim_{k\to\infty}\|Bq^k - Au^k\| = 0.$$

Next, we show $\lim_{k\to\infty}\|Ax^k - By^k\| = 0$. Using the above equality and $p^{2k} = x^{2k}, q^{2k} = y^{2k}$, we get

$$\lim_{k\to\infty}\|Ax^{2k} - By^{2k}\| = 0. \tag{3.61}$$

Using $\|x_{\mathrm{I}}^{2k+1} - x_{\mathrm{I}}^{2k}\| = \gamma\rho_{2k}\|c^{2k}\|$ and (3.59), we obtain

$$\lim_{k\to\infty}\|x_{\mathrm{I}}^{2k+1} - x_{\mathrm{I}}^{2k}\| = 0, \tag{3.62}$$

which together with $\|Ax^{2k+1} - Ax^{2k}\| \leq \|A\|\|x_{\mathrm{I}}^{2k+1} - x_{\mathrm{I}}^{2k}\|$ implies

$$\lim_{k\to\infty}\|Ax^{2k+1} - Ax^{2k}\| = 0. \tag{3.63}$$

Similarly, we know that

$$\lim_{k\to\infty}\|y_{\mathrm{I}}^{2k+1} - y_{\mathrm{I}}^{2k}\| = 0, \qquad \lim_{k\to\infty}\|By^{2k+1} - By^{2k}\| = 0. \tag{3.64}$$

Due to

$$\|Ax^{2k+1} - By^{2k+1}\| \leq \|Ax^{2k+1} - Ax^{2k}\| + \|By^{2k+1} - By^{2k}\| + \|Ax^{2k} - By^{2k}\|. \tag{3.65}$$

So, from (3.61), (3.63), (3.64) and (3.65), we get

$$\lim_{k\to\infty}\|Ax^{2k+1} - By^{2k+1}\| = 0. \tag{3.66}$$

Combining (3.61) and (3.66), we have $\lim_{k\to\infty}\|Ax^k - By^k\| = 0$.

Then we illustrate that $\lim_{k\to\infty}\|x^k - u^k\| = 0$, and $\lim_{k\to\infty}\|y^k - v^k\| = 0$. Using (3.60) and $x^{2k} = p^{2k}$, we know that

$$\lim_{k\to\infty}\|x_{\mathrm{I}}^{2k} - u^{2k}\| = 0. \tag{3.67}$$

By the second inequality of (3.40), we have

$$
\begin{aligned}
\|x_{\mathrm{I}}^{2k+1} - u^{2k+1}\| \leq&\|x_{\mathrm{I}}^{2k+1} - p^{2k+1}\| + \|p^{2k+1} - u^{2k+1}\| \\
\leq&\theta_{2k+1}\|x_{\mathrm{I}}^{2k+1} - x_{\mathrm{I}}^{2k}\| + \|p^{2k+1} - u^{2k+1}\|.
\end{aligned} \tag{3.68}
$$

From (3.60), (3.62) and (3.68), we get

$$\lim_{k\to\infty}\|x_{\mathrm{I}}^{2k+1} - u^{2k+1}\| = 0,$$

which with (3.67) shows

$$\lim_{k\to\infty}\|x^k-u^k\|=0.$$

Similarly, we also obtain

$$\lim_{k\to\infty}\|y^k-v^k\|=0.$$

Employing arguments which are similar to those used in the proof of [8, Theorem 3.3], we obtain that the whole sequence $\{(x^k,y^k)\}_{k\in\mathbb{N}}$ converges weakly to a solution of SEP (1.1), which completes proof. □

Lemma 3.15. *Take k as even and any* $(x^*,y^*)\in\Gamma$. *Let* $\{(x^k,y^k)\}_{k\in\mathbb{N}}$ *be generated by* Algorithm 3.2 (II). *Suppose that* (A2) *holds, then we have*

$$\begin{aligned}
&\|x_{\mathrm{II}}^{k+2}-x^*\|^2+\|y_{\mathrm{II}}^{k+2}-y^*\|^2\\
\le&\|x^k-x^*\|^2+\|y^k-y^*\|^2\\
&-(1+\theta_{k+1})\left[\gamma(2-\gamma)\rho_k^2(\|c^k\|^2+\|d^k\|^2)+(\|x_{\mathrm{I}}^{k+1}-x_{\mathrm{II}}^{k+1}\|^2+\|y_{\mathrm{I}}^{k+1}-y_{\mathrm{II}}^{k+1}\|^2)\right]\\
&-\gamma(2-\gamma)\rho_{k+1}^2(\|c^{k+1}\|^2+\|d^{k+1}\|^2)-(\|x_{\mathrm{I}}^{k+2}-x_{\mathrm{II}}^{k+2}\|^2+\|y_{\mathrm{I}}^{k+2}-y_{\mathrm{II}}^{k+2}\|^2).
\end{aligned}\tag{3.69}$$

Proof. By Lemma 2.2(ii), we have

$$\begin{aligned}
&\|x_{\mathrm{II}}^{k+2}-x^*\|^2\\
\le&\|p^{k+1}-\gamma\beta_{k+1}\rho_{k+1}F(u^{k+1},v^{k+1})-x^*\|^2-\|p^{k+1}-\gamma\beta_{k+1}\rho_{k+1}F(u^{k+1},v^{k+1})-x_{\mathrm{II}}^{k+2}\|^2\\
=&\|p^{k+1}-x^*\|^2-\|p^{k+1}-x_{\mathrm{II}}^{k+2}\|^2-2\gamma\beta_{k+1}\rho_{k+1}\langle x_{\mathrm{II}}^{k+2}-x^*,F(u^{k+1},v^{k+1})\rangle.
\end{aligned}$$

Similarly, we have

$$\|y_{\mathrm{II}}^{k+2}-y^*\|^2\le\|q^{k+1}-y^*\|^2-\|q^{k+1}-y_{\mathrm{II}}^{k+2}\|^2-2\gamma\beta_{k+1}\rho_{k+1}\langle y_{\mathrm{II}}^{k+2}-y^*,G(u^{k+1},v^{k+1})\rangle.$$

Adding the above inequalities, we obtain

$$\begin{aligned}
&\|x_{\mathrm{II}}^{k+2}-x^*\|^2+\|y_{\mathrm{II}}^{k+2}-y^*\|^2\\
\le&\|p^{k+1}-x^*\|^2+\|q^{k+1}-y^*\|^2-\|p^{k+1}-x_{\mathrm{II}}^{k+2}\|^2-\|q^{k+1}-y_{\mathrm{II}}^{k+2}\|^2\\
&-2\gamma\beta_{k+1}\rho_{k+1}(\langle x_{\mathrm{II}}^{k+2}-x^*,F(u^{k+1},v^{k+1})\rangle+\langle y_{\mathrm{II}}^{k+2}-y^*,G(u^{k+1},v^{k+1})\rangle).
\end{aligned}\tag{3.70}$$

By setting $(x,y)=(x_{\mathrm{II}}^{k+2},y_{\mathrm{II}}^{k+2})$ in (3.46), we get

$$\begin{aligned}
&-2\gamma\beta_{k+1}\rho_{k+1}\langle x_{\mathrm{II}}^{k+2}-u^{k+1},F(u^{k+1},v^{k+1})\rangle-2\gamma\beta_{k+1}\rho_{k+1}\langle y_{\mathrm{II}}^{k+2}-v^{k+1},G(u^{k+1},v^{k+1})\rangle\\
\le&-2\gamma\rho_{k+1}\langle x_{\mathrm{II}}^{k+2}-u^{k+1},c^{k+1}\rangle-2\gamma\rho_{k+1}\langle y_{\mathrm{II}}^{k+2}-v^{k+1},d^{k+1}\rangle\\
=&-2\gamma\rho_{k+1}(\langle p^{k+1}-u^{k+1},c^{k+1}\rangle+\langle q^{k+1}-v^{k+1},d^{k+1}\rangle)\\
&-2\gamma\rho_{k+1}(\langle x_{\mathrm{II}}^{k+2}-p^{k+1},c^{k+1}\rangle+\langle y_{\mathrm{II}}^{k+2}-q^{k+1},d^{k+1}\rangle).
\end{aligned}\tag{3.71}$$

It holds

$$\begin{aligned}
&-2\gamma\rho_{k+1}\langle x_{\mathrm{II}}^{k+2}-p^{k+1},c^{k+1}\rangle\\
=&-\|p^{k+1}-x_{\mathrm{II}}^{k+2}-\gamma\rho_{k+1}c^{k+1}\|^2+\|p^{k+1}-x_{\mathrm{II}}^{k+2}\|^2+\gamma^2\rho_{k+1}^2\|c^{k+1}\|^2.
\end{aligned}\tag{3.72}$$

Similarly, we get

$$\begin{aligned}
&-2\gamma\rho_{k+1}\langle y_{\mathrm{II}}^{k+2}-q^{k+1},d^{k+1}\rangle\\
=&-\|q^{k+1}-y_{\mathrm{II}}^{k+2}-\gamma\rho_{k+1}d^{k+1}\|^2+\|q^{k+1}-y_{\mathrm{II}}^{k+2}\|^2+\gamma^2\rho_{k+1}^2\|d^{k+1}\|^2.
\end{aligned}\tag{3.73}$$

Substituting (3.72) and (3.73) in the right side of (3.71) and using $p^{k+1}-\gamma\rho_{k+1}c^{k+1}=x_{\mathrm{I}}^{k+2}$ and $q^{k+1}-\gamma\rho_{k+1}d^{k+1}=y_{\mathrm{I}}^{k+2}$, we obtain

(3.74)

$$\begin{aligned}&-2\gamma\beta_{k+1}\rho_{k+1}\langle x_{\mathrm{II}}^{k+2}-u^{k+1},F(u^{k+1},v^{k+1})\rangle-2\gamma\beta_{k+1}\rho_{k+1}\langle y_{\mathrm{II}}^{k+2}-v^{k+1},G(u^{k+1},v^{k+1})\rangle\\ \le&-2\gamma\rho_{k+1}(\langle p^{k+1}-u^{k+1},c^{k+1}\rangle+\langle q^{k+1}-v^{k+1},d^{k+1}\rangle)\\ &-\|x_{\mathrm{I}}^{k+2}-x_{\mathrm{II}}^{k+2}\|^2+\|p^{k+1}-x_{\mathrm{II}}^{k+2}\|^2+\gamma^2\rho_{k+1}^2\|c^{k+1}\|^2\\ &-\|y_{\mathrm{I}}^{k+2}-y_{\mathrm{II}}^{k+2}\|^2+\|q^{k+1}-y_{\mathrm{II}}^{k+2}\|^2+\gamma^2\rho_{k+1}^2\|d^{k+1}\|^2.\end{aligned}$$

Also

(3.75)

$$\begin{aligned}&-2\gamma\beta_{k+1}\rho_{k+1}\langle u^{k+1}-x^*,F(u^{k+1},v^{k+1})\rangle-2\gamma\beta_{k+1}\rho_{k+1}\langle v^{k+1}-y^*,G(u^{k+1},v^{k+1})\rangle\\ =&-2\gamma\beta_{k+1}\rho_{k+1}\|Au^{k+1}-Bv^{k+1}\|^2.\end{aligned}$$

So, using (3.74), (3.75) and the definition of ρ_k, we obtain

(3.76)

$$\begin{aligned}&-2\gamma\beta_{k+1}\rho_{k+1}\langle x_{\mathrm{II}}^{k+2}-x^*,F(u^{k+1},v^{k+1})\rangle-2\gamma\beta_{k+1}\rho_{k+1}\langle y_{\mathrm{II}}^{k+2}-y^*,G(u^{k+1},v^{k+1})\rangle\\ \le&-2\gamma\rho_{k+1}(\langle p^{k+1}-u^{k+1},c^{k+1}\rangle+\langle q^{k+1}-v^{k+1},d^{k+1}\rangle+\beta_{k+1}\|Au^{k+1}-Bv^{k+1}\|^2)\\ &-\|x_{\mathrm{I}}^{k+2}-x_{\mathrm{II}}^{k+2}\|^2+\|p^{k+1}-x_{\mathrm{II}}^{k+2}\|^2+\gamma^2\rho_{k+1}^2\|c^{k+1}\|^2\\ &-\|y_{\mathrm{I}}^{k+2}-y_{\mathrm{II}}^{k+2}\|^2+\|q^{k+1}-y_{\mathrm{II}}^{k+2}\|^2+\gamma^2\rho_{k+1}^2\|d^{k+1}\|^2\\ =&-2\gamma\rho_{k+1}^2(\|c^{k+1}\|^2+\|d^{k+1}\|^2)+\gamma^2\rho_{k+1}^2(\|c^{k+1}\|^2+\|d^{k+1}\|^2)\\ &-\|x_{\mathrm{I}}^{k+2}-x_{\mathrm{II}}^{k+2}\|^2-\|y_{\mathrm{I}}^{k+2}-y_{\mathrm{II}}^{k+2}\|^2+\|p^{k+1}-x_{\mathrm{II}}^{k+2}\|^2+\|q^{k+1}-y_{\mathrm{II}}^{k+2}\|^2\\ =&-\gamma(2-\gamma)\rho_{k+1}^2(\|c^{k+1}\|^2+\|d^{k+1}\|^2)\\ &-\|x_{\mathrm{I}}^{k+2}-x_{\mathrm{II}}^{k+2}\|^2-\|y_{\mathrm{I}}^{k+2}-y_{\mathrm{II}}^{k+2}\|^2+\|p^{k+1}-x_{\mathrm{II}}^{k+2}\|^2+\|q^{k+1}-y_{\mathrm{II}}^{k+2}\|^2.\end{aligned}$$

Adding (3.70) and (3.76), we obtain

$$\begin{aligned}&\|x_{\mathrm{II}}^{k+2}-x^*\|^2+\|y_{\mathrm{II}}^{k+2}-y^*\|^2\\ \le&\|p^{k+1}-x^*\|^2+\|q^{k+1}-y^*\|^2-\gamma(2-\gamma)\rho_{k+1}^2(\|c^{k+1}\|^2+\|d^{k+1}\|^2)\\ &-\|x_{\mathrm{I}}^{k+2}-x_{\mathrm{II}}^{k+2}\|^2-\|y_{\mathrm{I}}^{k+2}-y_{\mathrm{II}}^{k+2}\|^2.\end{aligned}\tag{3.77}$$

Using similar arguments in obtaining (3.77), one can show

$$\begin{aligned}&\|x_{\mathrm{II}}^{k+1}-x^*\|^2+\|y_{\mathrm{II}}^{k+1}-y^*\|^2\\ \le&\|p^{k}-x^*\|^2+\|q^{k}-y^*\|^2-\gamma(2-\gamma)\rho_{k}^2(\|c^{k}\|^2+\|d^{k}\|^2)\\ &-\|x_{\mathrm{I}}^{k+1}-x_{\mathrm{II}}^{k+1}\|^2-\|y_{\mathrm{I}}^{k+1}-y_{\mathrm{II}}^{k+1}\|^2.\end{aligned}\tag{3.78}$$

Using (3.43) and $x^k=p^k$, we know

$$\|x_{\mathrm{II}}^{k+1}-x^k\|=\|P_C(p^k-\gamma\beta_k\rho_kF(u^k,v^k))-x^k\|\le\gamma\beta_k\rho_k\|F(u^k,v^k)\|.\tag{3.79}$$

Similarly, we also have

$$\|y_{\mathrm{II}}^{k+1}-y^k\|\le\gamma\beta_k\rho_k\|G(u^k,v^k)\|.\tag{3.80}$$

So, by combining (3.55), (3.57), (3.77)-(3.80), and taking into account the facts $x^k = p^k, y^k = q^k$, we get

$$\begin{aligned}
&\|x_{\mathrm{II}}^{k+2} - x^*\|^2 + \|y_{\mathrm{II}}^{k+2} - y^*\|^2 \\
\leq &\|x^k - x^*\|^2 + \|y^k - y^*\|^2 \\
&- (1+\theta_{k+1})\gamma(2-\gamma)\rho_k^2(\|c^k\|^2 + \|d^k\|^2) - (1+\theta_{k+1})(\|x_{\mathrm{I}}^{k+1} - x_{\mathrm{II}}^{k+1}\|^2 + \|y_{\mathrm{I}}^{k+1} - y_{\mathrm{II}}^{k+1}\|^2) \\
&+ \theta_{k+1}(1+\theta_{k+1})\gamma^2\beta_k^2\rho_k^2(\|F(u^k, v^k)\|^2 + \|G(u^k, v^k)\|^2) \\
&- \gamma(2-\gamma)\rho_{k+1}^2(\|c^{k+1}\|^2 + \|d^{k+1}\|^2) - \|x_{\mathrm{I}}^{k+2} - x_{\mathrm{II}}^{k+2}\|^2 - \|y_{\mathrm{I}}^{k+2} - y_{\mathrm{II}}^{k+2}\|^2.
\end{aligned}$$

Using the assumption (A2), we obtain (3.69). This completes proof. □

Theorem 3.4. *Suppose that* (A2) *is satisfied. Then the sequence* $\{(x^k, y^k)\}_{k\in\mathbb{N}}$ *be generated by* Algorithm 3.2 (II) *converges weakly to a solution of the SEP* (1.1).

Proof. Employing arguments which are similar to those used in the proof of Theorem 3.3, we obtain that the whole sequence $\{(x^k, y^k)\}_{k\in\mathbb{N}}$ weakly converges to a solution of the SEP (1.1), which completes proof. □

4. Numerical examples

In this section, we use the numerical example in [5] to demonstrate the efficiency and advantage of Algorithms 3.1 and 3.2 by comparing them with Algorithms 3.1, 3.2 in [8].

We denote the vector with all elements 0 by e_0, and the vector with all elements 1 by e_1 in what follows. In the numerical results listed in the following table, 'Iter.' and 'Sec.' denote the number of iterations and the cpu time in seconds, respectively.

Example 4.1. The SEP with $A = (a_{ij})_{J\times N}$, $B = (b_{ij})_{J\times M}$, $C = \{x \in R^N\} | \|x\| \leq 0.25\}$, $Q = \{y \in R^M | e_0 \leq y \leq u\}$, where $a_{ij} \in [0,1]$, $b_{ij} \in [0,1]$ and $u \in [e_1, 2e_1]$ are all generated uniform randomly.

In the implementations, we take $\|Ax - By\| < \varepsilon = 10^{-4}$ as the stopping criterion. Take the initial value $x_0 = 10e_1$, $y_0 = -10e_1$.

We make comparison of Algorithms 3.1, 3.2 and 3.1, 3.2 in [8] with different J, N, M, and report the results in Tables 1, 2, 3 and Figures 1-6. In Algorithms 3.1, and 3.2, we choose $\mu = 0.9, \gamma = 1, \beta_0 = 0.8, \theta_k = 0.9$ for Algorithms 3.1(I), $\mu = 0.9, \gamma = 1.1, \beta_0 = 0.7, \theta_k = -0.1$ for Algorithms 3.1(II), $\mu = 0.9, \gamma = 1, \beta_0 = 0.7, \theta_k = 0.9$ for Algorithms 3.2(I), $\mu = 0.9, \gamma = 1.4, \beta_0 = 0.4, \theta_k = -0.3$ for Algorithms 3.2(II). We take $\gamma = 0.8, \theta = 0.99, \sigma = 50, \rho = 0.1$ and $\alpha = 0.1$ for Algorithms 3.1, and 3.2 in [8].

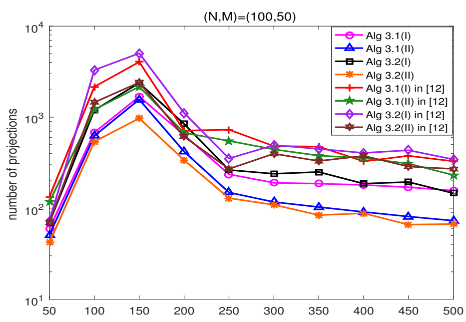

FIGURE 1. Numbers of projections with $(N, M) = (100, 50)$

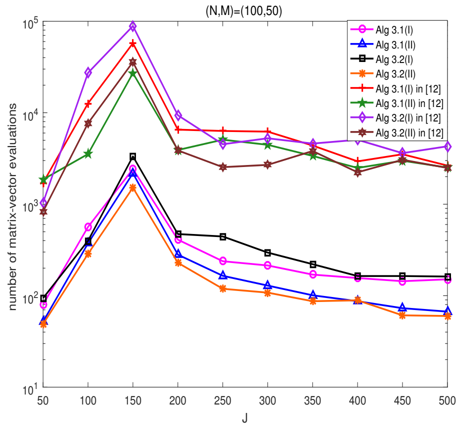

FIGURE 2. Numbers of matrix-vector evaluations with $(N, M) = (100, 50)$

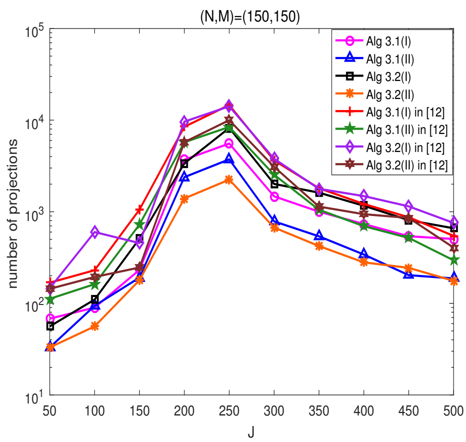

FIGURE 3. Numbers of projections with $(N, M) = (150, 150)$

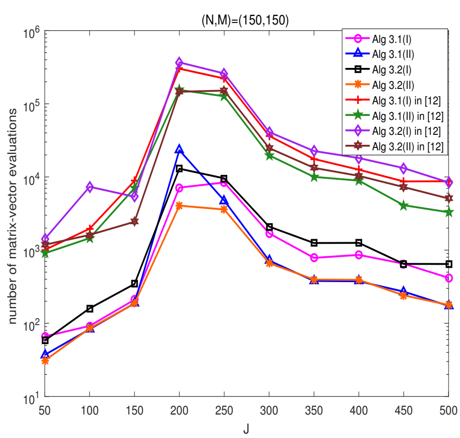

FIGURE 4. Numbers of matrix-vector evaluations with $(N, M) = (150, 150)$

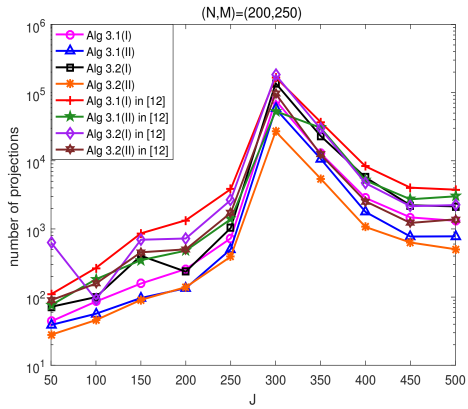

FIGURE 5. Numbers of projections with $(N, M) = (200, 250)$

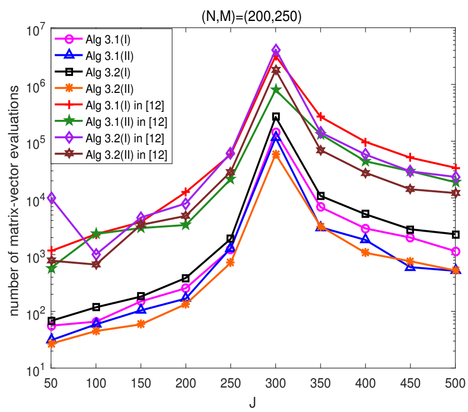

FIGURE 6. Numbers of matrix-vector evaluations with $(N, M) = (200, 250)$

TABLE 1. Computational results for Example 4.1 with $(N, M) = (100, 50)$

J		50	100	150	200	250
Alg 3.1(I)	Iter.	60	585	2144	338	350
	Sec.	0.012	0.394	1.125	0.281	0.282
Alg 3.1(II)	Iter.	**45**	**412**	1807	313	179
	Sec.	**0.009**	**0.375**	1.031	0.281	0.094
Alg 3.2(I)	Iter.	102	869	2656	478	394
	Sec.	0.011	0.656	1.969	0.291	0.281
Alg 3.2(II)	Iter.	47	769	**1296**	**213**	**145**
	Sec.	0.009	0.469	**0.750**	**0.094**	**0.085**
Alg 3.2(I) in [12]	Iter.	119	1821	5830	1179	672
	InIt.	329	3054	8007	1365	792
	Sec.	0.031	1.078	5.156	1.031	0.469

J		50	100	150	200	250
Alg 3.2(II) in [12]	Iter.	80	1101	4365	881	521
	InIt.	90	1179	4461	941	569
	Sec.	0.013	0.563	3.281	0.563	0.462
Alg 3.3(I) in [12]	Iter.	200	2075	6584	1321	887
	InIt.	265	3252	8651	1543	1019
	Sec.	0.016	1.313	4.500	1.422	0.750
Alg 3.3(II) in [12]	Iter.	94	1509	3413	741	678
	InIt.	99	1641	3533	807	702
	Sec.	0.016	1.031	2.906	0.750	0.656

TABLE 2. Computational results for Example 4.1 with $(N, M) = (150, 150)$

J		50	100	150	200	250
Alg 3.1(I)	Iter.	54	100	211	7686	5013
	Sec.	0.006	0.094	0.109	6.563	4.219
Alg 3.1(II)	Iter.	31	**65**	283	7059	4081
	Sec.	0.004	**0.037**	0.203	5.438	3.656
Alg 3.2(I)	Iter.	52	154	566	7862	7558
	Sec.	0.006	0.094	0.469	6.094	6.938
Alg 3.2(II)	Iter.	**29**	77	**196**	**2508**	**2114**
	Sec.	**0.002**	0.089	**0.094**	**2.438**	**2.063**
Alg 3.2(II) in [12]	Iter.	120	394	774	7523	4118
	InIt.	162	418	816	5613	4196
	Sec.	0.203	0.313	0.656	5.969	4.125
Alg 3.3(I) in [12]	Iter.	125	385	1378	22734	11409
	InIt.	173	445	1516	25542	13197
	Sec.	0.289	0.375	1.594	23.344	14.250
Alg 3.3(II) in [12]	Iter.	87	217	936	6561	6200
	InIt.	135	223	966	6693	6272
	Sec.	0.016	0.203	1.234	7.406	7.594

TABLE 3. Computational results for Example 4.1 with $(N, M) = (200, 250)$

J		50	100	150	200	250
Alg 3.1(I)	Iter.	48	62	104	213	1557
	Sec.	0.094	0.098	0.103	0.281	1.500
Alg 3.1(II)	Iter.	31	49	96	225	1181
	Sec.	0.056	0.078	0.094	0.281	1.406
Alg 3.2(I)	Iter.	94	126	120	380	3136
	Sec.	0.074	0.098	0.094	0.469	2.906

J		50	100	150	200	250
Alg 3.2(II)	Iter.	**29**	**44**	**63**	**198**	**880**
	Sec.	**0.021**	**0.077**	**0.059**	**0.188**	**1.031**
Alg 3.2(I) in [12]	Iter.	209	214	318	1187	5987
	InIt.	227	268	378	9184	46324
	Sec.	0.188	0.297	0.391	4.406	22.875
Alg 3.2(II) in [12]	Iter.	136	179	189	740	3710
	InIt.	160	191	207	5817	29392
	Sec.	0.188	0.188	0.291	2.938	15.094
Alg 3.3(I) in [12]	Iter.	161	187	627	380	2665
	InIt.	72	193	717	2353	3899
	Sec.	0.094	0.203	0.688	2.328	4.125
Alg 3.3(II) in [12]	Iter.	68	303	218	616	1085
	InIt.	84	221	230	628	1136
	Sec.	0.109	0.281	0.281	0.750	1.219

The numerical results are listed in Tables 1, 2, 3 and Figures 1-6, from which we can get some conclusions:

(i) In the Figures 1-6, we see that the numbers of projections and matrix-vector evaluations that Algorithms 3.1 and 3.2 need are less than those of Algorithms 3.1 and 3.2 in [8].

(ii) In the Tables 1-3, it's easy to observe that in the number and time of iterations, Algorithms 3.1 and 3.2 are superior to Algorithms 3.1 and 3.2 in [8].

(iii) Among Algorithms 3.1 and 3.2, Algorithm 3.2(II) has better performance than Algorithms 3.1 and 3.2(I) for most cases while Algorithm 3.1(II) behaves best for the other few cases.

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